

Analytic Langlands correspondence

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Let G is a connected reductive group and X a smooth irreducible projective curve, both defined over a field F . The goal of *the global Langlands program over function fields* is to do harmonic analysis on the moduli space (stack) $\mathrm{Bun}_G(X)$ of principal G -bundles on X , for various fields F . This means that we want to find the spectral decomposition for a family of commuting Hecke operators (or functors) acting on a suitable space of functions (or category of sheaves) on $\mathrm{Bun}_G(X)$. More precisely, the Langlands program aims to parametrize this spectrum by data related to the *Langlands dual group* $G^\vee$, namely appropriate $G^\vee$ -local systems on X . The corresponding eigenvectors (or eigensheaves) are called *automorphic functions (sheaves)*. There is also a ramified generalizations of this theory, in which one considers G -bundles with level structure at a finite set of points of X .

In the most classical setup, developed by Langlands himself around 1970 (the *arithmetic Langlands program*), F is a finite field $\mathbb{F}_q$, so $\mathrm{Bun}_G(X)(F)$ is an infinite countable set, and one considers Hecke operators $H_{x,\lambda}$ acting on $L^2(\mathrm{Bun}_G(X)(\mathbb{F}_q))$ (a family of bounded, commuting normal operators). In this case, the Langlands conjecture (currently known in many cases) states that the spectrum of Hecke operators is parametrized by local systems on X with values in $G^\vee$ (or representations of the absolute Galois group of the field $\mathbb{F}_q(X)$), understood in a suitable sense.

In 1990s Beilinson and Drinfeld proposed a geometric generalization of this theory (the *geometric Langlands program*). In this generalization, $F = \mathbb{C}$, and the role of the space $L^2(\mathrm{Bun}_G(X)(\mathbb{F}_q))$ is played by the category of (*critically twisted*) D -modules (de Rham version) or *constructible sheaves* of $\mathbb{C}$ -vector spaces (Betti version) on the stack $\mathrm{Bun}_G(X)$; the two versions are related by the Riemann-Hilbert correspondence. On this category there is an action of a family of commuting *Hecke functors*, and the goal of the geometric Langlands program is to “diagonalize” these functors by finding and parametrizing their eigensheaves. The geometric Langlands conjecture of Beilinson and Drinfeld was that such eigensheaves are parametrized by $G^\vee$ -local systems on the Riemann surface $X(\mathbb{C})$, i.e., in the de Rham version, holomorphic $G^\vee$ -bundles on X with a connection, and in the Betti version, finite dimensional complex representations of its fundamental group. This conjecture has recently been proved in general by D. Gaitsgory and his collaborators. Also, Kapustin and Witten showed that the geometric Langlands program is related to topologically twisted 4-dimensional $N = 4$ supersymmetric gauge theory.

There is also a version of this theory in which F is any field and the constructible sheaves are ℓ -adic (i.e., of $\overline{\mathbb{Q}}_\ell$ -vector spaces). This version, with $F = \mathbb{F}_q$ where $(\ell, q) = 1$, connects to the arithmetic version via the *sheaf-function correspondence* (i.e., taking the trace of Frobenius in the étale cohomology of the sheaf to get a function). This gives ℓ -adic, rather than

complex, functions, but it does not matter since automorphic functions take values in $\overline{\mathbb{Q}}$.

In the de Rham version, the automorphic D-module M_L attached to a $G^\vee$ -local system L is easiest to describe when L is an *oper*, i.e., a connection on a special *oper* $G^\vee$ -bundle E , induced from the oper bundle for the principal SL_2 of $G^\vee$, which is (for $\text{genus}(X) \neq 1$) the non-trivial extension of $K_X^{-1/2}$ by $K_X^{1/2}$. The space of opers, $\text{Op}_{G^\vee}(X)$, is an affine space whose underlying vector space is the *Hitchin base* $\mathcal{B} = \bigoplus_{i=1}^r H^0(X, K_X^{d_i})$, where d_i are the degrees of basic invariants for G and r is its rank. Beilinson and Drinfeld showed that the classical Hitchin system $\pi : T^*\text{Bun}_G(X) \rightarrow \mathcal{B}$ admits a quantization, which gives an inclusion (in fact, an isomorphism) $D : \mathbb{C}[\text{Op}_{G^\vee}(X)] \rightarrow \mathcal{D}(\text{Bun}_G(X), K^{1/2})$ (differential operators on half-forms, i.e., critically twisted ones). The twisted D-module M_L is then defined by the system of *quantum Hitchin differential equations*

$$D(f)\psi = f(L)\psi, \quad f \in \mathbb{C}[\text{Op}_{G^\vee}(X)].$$

Finally, in the last 6 years, jointly with E. Frenkel and D. Kazhdan, we developed an *analytic version* of the Langlands program. This program develops the earlier ideas and results of Braverman-Kazhdan, Kontsevich, Langlands and Teschner. In this version, F is a local field, so there are two cases: archimedean ($F = \mathbb{R}, \mathbb{C}$) and non-archimedean (F is a finite extension of $\mathbb{Q}_p$ or $F = \mathbb{F}_q((t))$). In both cases, we are interested in the spectrum of Hecke operators $H_{x,\lambda}$ acting on the Hilbert space $L^2(\text{Bun}_G(X)(F))$, where $\text{Bun}_G(X)(F)$ is now generically an analytic F -variety, and L^2 stands for the space of square integrable half-densities. The operators $H_{x,\lambda}$ are densely defined on the Hilbert space, and conjecturally extend to bounded (in fact, compact) commuting normal operators; this conjecture has been proved in the ramified setting for genus 0 and 1 for $G = PGL_2$.

This raises the question what parametrizes the spectrum of the Hecke operators: can we describe it in terms of the Langlands dual group? In the non-archimedean case, we don't even have a conjecture how to do this, even though it has been shown by Braverman, Kazhdan and Polishchuk that automorphic functions of arithmetic Langlands, i.e., over $\mathbb{F}_q$ (the cuspidal ones, i.e. from discrete spectrum of Hecke operators) can be lifted to ones over a local field F with residue field $\mathbb{F}_q$ if we fix a lift of X over its ring of integers $\mathcal{O}_F$.

The situation is better in the archimedean case, however. In this case, a prominent role is played by quantum Hitchin operators, which (conjecturally) act on $L^2(\text{Bun}_G(X)(F))$ as commuting unbounded normal operators and commute with Hecke operators, hence have the same spectral decomposition. This connects the story to geometric Langlands, showing that automorphic functions of analytic Langlands theory should be parametrized by a certain (discrete) family of $G^\vee$ -opers. And indeed, for $F = \mathbb{C}$, these (conjecturally) turn out to be the *opers with real monodromy*, i.e., ones

whose monodromy representation can be conjugated into the split real form $G^\vee(\mathbb{R})$. For such opers, the automorphic function ψ on $\text{Bun}_G(X)(\mathbb{C})$ is simply a single valued analytic joint eigenfunction of $D(f)$ and $\overline{D(f)}$, which is unique up to scaling:

$$D(f)\psi_L = f(L)\psi_L, \quad \overline{D(f)}\psi_L = \overline{f(L)}\psi_L.$$

Gaiotto and Witten showed that this conjecture is compatible to the structure of branes in the Kapustin-Witten theory. Also, this conjecture is proved in genus zero and (almost) in genus 1 in the ramified case, for $G = PGL_2$; the last steps in genus zero were made in a recent paper of Ambrosino and Teschner using Sklyanin's separation of variables transform. If $F = \mathbb{R}$, the situation is more complicated (as it depends on the structure of real ovals on X , real forms of G , etc.) but still in many cases one can formulate the condition on the oper (always a topological one, i.e., in the Betti realization) which makes it a point of the spectrum.

In the simplest case of PGL_2 and genus zero with tame ramification at 4 points, $\text{Bun}_G(X)$ is 1-dimensional and the eigenfunctions ψ_L have the form

$$\psi_L(z, \bar{z}) = \eta_1(z)\overline{\eta_2(z)} + \eta_2(z)\overline{\eta_1(z)},$$

where η_1, η_2 is a basis of solutions of a Heun equation (in the most basic case, Lamé equation).

In the mini-course I will review this theory and consider various examples of arising spectral decompositions.

References

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