Jenny Wilson

V fin dim v.sp. $/F$ field

<u>Def</u> $T(V)$ – Tits building of V — simplicial cx

whose vertices are subspaces of V, nonzero, proper,
simplices = flags.

<u>Ex</u> $T(F^2)$: set of lines in F^2

<u>Ex</u> $T(F^3)$: bipartite graph whose vertices are $\{\text{lines}\} \sqcup \{\text{planes}\}$ in F^3, edges = containment

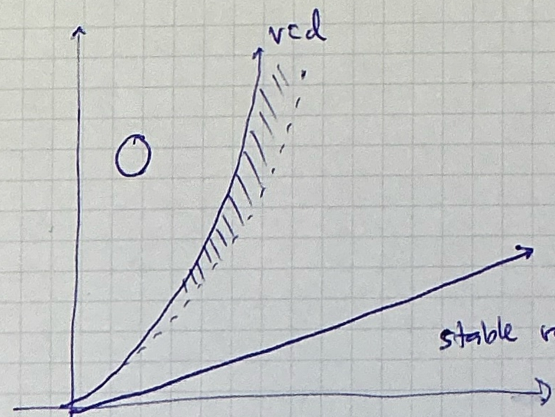
<u>Thm</u> (Solomon–Tits) $T(V) \simeq \bigvee S^{\dim(V)-2}$

<u>Def</u> $St(V) = \tilde{H}_{n-2}(T(V))$.

Today (to make life easier) we take $\mathbb{Q}$-coefficients.

Borel-Serre duality for $SL_n(\mathbb{Z})$:

<u>Thm</u> $\operatorname{vcd} SL_n(\mathbb{Z}) = \binom{n}{2}$. $H^{\binom{n}{2}-i}(SL_n\mathbb{Z};\mathbb{Q}) \cong H_i(SL_n\mathbb{Z}; St(\mathbb{Q}^n))$



$H^*(SL_n(\mathbb{Z});\mathbb{Q})$

<u>Conjecture</u> (Church-Farb-Putman) $H^{\binom{n}{2}-i}(SL_n(\mathbb{Z});\mathbb{Q}) = 0 \quad \forall n \geq i+2$.

(shaded region in diagram)

<u>Thm</u> ($i=2$: Brück-Miller-Patzt-Sroka-W.) Conj holds for $i=0,1,2$

($i=1$: Bykovskii, Church-Putman, Charlton-Rudchenko-Rudenko simpler proof)

($i=0$: Lee-Szczarba)

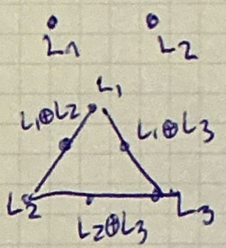
An approach: take a resolution of $St(\mathbb{Q}^n)$ by flat $SL_n\mathbb{Z}$-modules. Hope to find explicit "small" resolution st coinvariants vanish / are small

Fix a frame of $\mathbb{Q}^n$, ie $\mathbb{Q}^n = L_1 \oplus \cdots \oplus L_n$

<u>apartment</u> $A(L_1, \dots, L_n) \subseteq T(\mathbb{Q}^n)$ full subcx spanned by vertices given by sums of the L_i.

$n=2$

$n=3$



in general barycentric subdivision of simplex.

<u>Def</u> $[L_1, \dots, L_n] \in H_{n-2}(St(\mathbb{Q}^n))$ <u>apartment class</u>.

<u>Thm</u> $St(\mathbb{Q}^n)$ generated by apartment classes.

<u>NB</u> $SL_n\mathbb{Q}$ acts transitively on frames, $SL_n\mathbb{Z}$ does not

e.g. $\underset{\det 1}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$ different orbit than $\underset{\det 2}{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}$

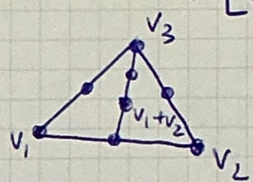
Thm (Ash-Rudolph) $St_n(\mathbb{Q})$ generated by $SL_n(\mathbb{Z})\cdot[\mathrm{id}]$
the "integral apartment classes"

Thm (Bykovskii) $St_n(\mathbb{Q})$ has the presentation

$$\mathbb{Q}\{[L_1,\dots,L_n] \mid \text{integral apartment class}\}/\sim$$

where the relations are

- $[L_1,\dots,L_n] = \mathrm{sgn}(\sigma)\,[L_{\sigma(1)},\dots,L_{\sigma(n)}]$
- $[\mathbb{Q}v_1,\dots,\mathbb{Q}v_n] = [\mathbb{Q}v_1,\mathbb{Q}(v_1+v_2),\mathbb{Q}v_3,\dots,\mathbb{Q}v_n] + [\mathbb{Q}(v_1+v_2),\mathbb{Q}v_2,\dots,\mathbb{Q}v_n]$

[Ex


]

CFP conj in codim ≤ 1 follows from Bykovskii presentation.

Pf in codim 0.

Enough to show $(St(\mathbb{Q}^n))_{SL_n\mathbb{Z}}$ vanishes.

generated as $SL_n\mathbb{Z}$-module by $[\mathrm{id}]$ (Ash-Rudolph)

but observe $\underbrace{\left[\begin{array}{cc|c} 0 & 1 & \\ -1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & \mathrm{id} \end{array}\right]}_{\in SL_n\mathbb{Z}}\cdot[\mathrm{id}] = -[\mathrm{id}]$

$\Rightarrow$ vanishes on coinvariants. $\boxtimes$

Proof of integral generation for $n=2$.

$$St(\mathbb{Q}^2) = \left\langle \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle \quad \gcd(a,b)=1$$

WTS : $\begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$ lin. comb. of integral classes.

① extend to basis of $\mathbb{Z}^2$ $\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} c' \\ d' \end{bmatrix}$

② $\exists q$ st $|d' - qb| < |b|$ (wlog $b \neq 0$)

$$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} c' - qa \\ d' - qb \end{bmatrix} \text{ is a basis w/ } \begin{bmatrix} a \\ b \end{bmatrix}$$

iterate until 2^{nd} coordinate is zero.

Interpretation X Farey graph

vertices primitive vectors in $\mathbb{Z}^2$ mod ± 1.

edges $L_1 - L_2$ iff det $= \pm 1$

gen. by int. class $\Leftrightarrow$ X connected.

Proof constructs path from $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.