Oscar Randal-Williams III

Consequences for $H_*(GL_n F)$.
For simplicity take $\mathbb{k} = \mathbb{Q}$.

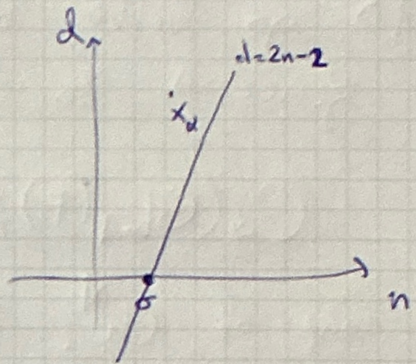
Then $\mathrm{Alg}^{E_\infty}_{\mathrm{aug}}(\mathrm{Ch}_{\mathbb{Q}}^{\mathbb{N}}) \simeq$ diff. bigraded comm. dg alg.

let BGL be "the" bigraded diff. algebra corr. to $C_*(\mathcal{C}_\delta)$ under this equiv.

$$H_{n,d}(BGL) = H_d(GL_n F; \mathbb{Q}).$$

The last lecture says in this case that BGL can be taken to be quasi-free with generators σ and x_α of bideg (n,d) w/

$d > 0$ and $d > 2n-2$



Consider $BGL/\sigma = \left(BGL[\bar\sigma], \, d\bar\sigma = \sigma\right)$ $\quad |\bar\sigma| = (1,1)$

have cofiber seq

$$BGL \to BGL/\sigma \to \Sigma^{1,1} BGL$$

$\to$ LES on homology

$$H_d(GL_nF) \to H_{n,d}(BGL/\sigma) \to H_{d-1}(GL_{n-1}F) \xrightarrow{\partial} H_{d-1}(GL_nF)$$

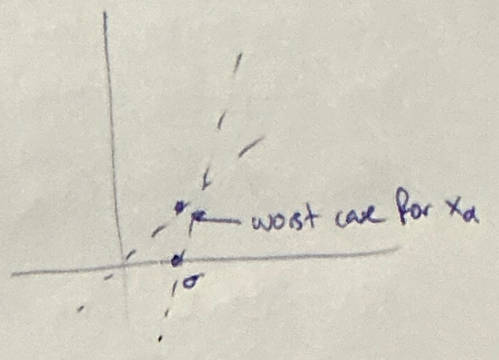
$$H_{n,d}(BGL/\sigma) = H_d(GL_nF, GL_{n-1}F; \mathbb{Q})$$

mult by σ

$BGL/\sigma \simeq (S^*[\sigma, \bar{\sigma}, x_\alpha \dots], d)$ not minimal, $d\bar{\sigma} = \sigma$

$\simeq (S^*[x_\alpha, \dots], d)$ minimal

all have slope ≥ 1



<u>Cor</u> $H_{n,d}(BGL/\sigma) = 0$ for $d < n$.

i.e. $H_d(GL_{n-1}F; \mathbb{Q}) \to H_d(GL_n F; \mathbb{Q})$ epi $d < n$, iso $d < n-1$

To go further, need to understand generators x_α better.

$$C_*(GL_1(F); \mathbb{Q}) \to \bigoplus_n C_*(GL_n(F); \mathbb{Q})$$

map of $Ch^{\mathbb{N}}$. Extends to map from free E_∞-alg on source.

$E_\infty^+(C_*(GL_1 F; \mathbb{Q})) = \bigoplus_n C_*(GM_n(F)) = BGM$

where $GM_n(F)$ = group of monomial matrices $= \Sigma_n \ltimes GL_1(F)^n$

i) $H_{1,d}(BGL) = \Lambda^d F^\times_{\mathbb{Q}}$

ii) Hurewicz principle: lowest nonzero relative homology equals lowest nonzero rel. E_∞-homology

$$H_{2,3}(BGL, BGM) \xrightarrow{\sim} H^{E_\infty}_{2,3}(BGL, BGM)$$
$$\uparrow \sim$$
$$H^{E_\infty}_{2,3}(BGL)$$

as BGM only has E_∞-hom. in grading 1

And $H_{2,3}(BGL, BGM) = H_3(GL_2(F), GM_2(F); \mathbb{Q})$.

Sushin: $H_3(GL_2 F, GM_2 F) \cong B_2(F)$

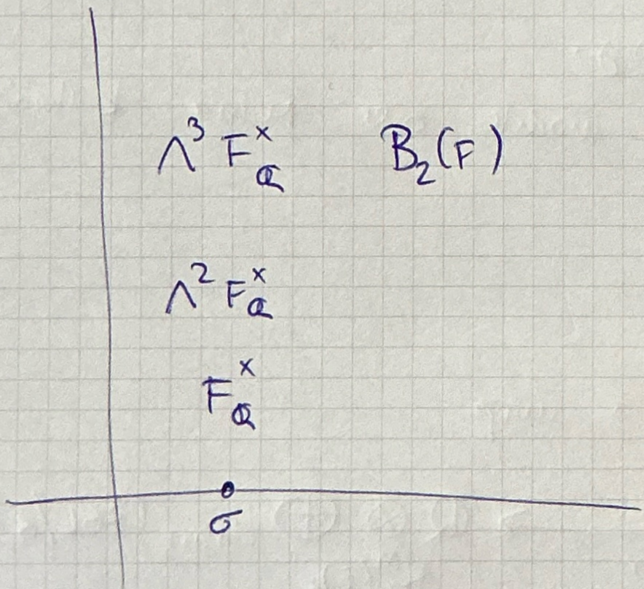
$$\downarrow \partial$$

$$H_2(GM_2 F) \simeq \Lambda^2 F^\times_{\mathbb{Q}} \oplus \Lambda^2 F^\times_{\mathbb{Q}}$$

$\sigma \cdot H_2(GM_1)$

invariant part of $H_2(F^\times \times F^\times) \supset H_1(F^\times)^{\otimes 2}$

and $\partial [x] = (x \wedge (1-x), -x \wedge (1-x))$



In a model for BGL/σ, ∂ tells us how "$B_2(F)$-cells" are attached to kill classes in bideg $(2,2)$ (products of $F^\times_{\mathbb{Q}}$)

We can now compute $H_*(\mathcal{E}_\gamma)$ along line of slope 1 through origin.

$$\bigoplus_n H_n(GL_n F, GL_{n-1} F; \mathbb{Q}) = \bigoplus_n H_{n,n}(BGL/\sigma)$$

$$= \Lambda^* F^\times_{\mathbb{Q}} \Big/ (x \wedge (1-x))$$

which by definition is Milnor K-theory.

(true also integrally w/ little more work)

Theorem of Suslin-Nesterenko.

As BGL/σ is still a CDGA, get more structure.

Vanishing along diag.

Diag. is Milnor K-theory.

Every other diagonal is module over Milnor K-theory.

<u>Thm</u> (GKRW) There is a map

$$\bigoplus_n \mathrm{Harr}_3(K^M_*(F)_{\mathbb{Q}})_n \longrightarrow \mathbb{Q} \otimes_{K^M_*(F)_{\mathbb{Q}}} \bigoplus H_{n,n+1}(BGL/\sigma)$$

which is an isom. in degs $n \geq 4$.

<u>Cor</u> If $K^M_*(F)_{\mathbb{Q}}$ is Koszul then LHS is supported at $n=3$, $\Rightarrow$ RHS vanishes for $n \geq 4$.

<u>Example</u> F number field

$$\Rightarrow K_2 F_{\mathbb{Q}} = 0 \Rightarrow K_2^M F_{\mathbb{Q}} = 0 \Rightarrow \text{Koszul}.$$

But when $K_2 = 0$ something much better is true.

<u>Thm</u> (GKRW) F ∞ field with $K_2(F)_{\mathbb{Q}} = 0$

then $H_d(GL_n F, GL_{n-1} F; \mathbb{Q}) = 0$ for $d < \frac{4n-1}{3}$.