Oscar Randal-Williams minicourse II

Need more 'explicit' description of $Q(A)$.

$$(*)\quad \begin{tikzcd} A \arrow[r, "\varepsilon"] \arrow[d, "\varepsilon"'] & \Bbbk \arrow[d, "\varepsilon"] \\ \Bbbk \arrow[r] & \mathrm{Bar}\,A \end{tikzcd}$$

pushout in E_∞-alg.

$\Bbbk \otimes_A \Bbbk$ can be computed many ways. One is $\left| [p] \mapsto A^{\otimes p} \right|$

Taking (ordinary) fibres in the square $(*)$ we get a map

$$\overline{A} \longrightarrow \Sigma^{-1}\,\overline{\mathrm{Bar}(A)}$$

Thm $Q(A) \simeq \operatorname{colim}_r \Sigma^{-r}\,\overline{\mathrm{Bar}^{(r)}(A)}$

[to get E_r-indecomposable, iterate r times]

Proof sketch. Think about what happens in $\mathrm{Ch}^{\mathbb{N}}$.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} E_\infty^+(X) \arrow[r] \arrow[d] & \Bbbk \arrow[d] \\ \Bbbk \arrow[r] & E_\infty^+(\Sigma X) = \mathrm{Bar}(E_\infty^+ X) \end{tikzcd}$$

$\therefore$ on free algebras, $\mathrm{Bar}(-)$ returns free object on suspension of generators.

$$\bar{A} \longrightarrow \Sigma^{-1}\overline{\mathrm{Bar}\, A}$$

$$\wr$$

$$\bigoplus_{k=1}^{\infty} E_*\Sigma_k \underset{\Sigma_k}{\otimes} X^{\otimes k} \longrightarrow \Sigma^{-1} \bigoplus_{k=1}^{\infty} E_*\Sigma_k \underset{\Sigma_k}{\otimes} (\Sigma X)^{\otimes k} \qquad (**)$$

only reasonable description of $(**)$ could be: identity on $k=1$ component, zero outside.

$\rightsquigarrow$ on colimit get $X = Q E_\infty^+(X)$.

End of pf sketch.

Want to apply this to $C_*(\mathcal{G})$.

Consider $r: \mathcal{G} \longrightarrow \mathbb{N}$ symm. monoidal functor.

Induces $Ch^{\mathbb{N}} \underset{r^*}{\overset{r_*}{\rightleftarrows}} Ch^{\mathcal{G}}$

left adjoint r_* is Kan extension, $(r_* X)(n) = EGL_n(F) \underset{GL_n F}{\otimes} X(n) \simeq C_*(GL_n F; X(n))$

let $\underline{k} \in Ch^{\mathcal{G}}$ constant.

$C_*(\mathcal{G}) = r_*(\underline{k})$.

In fact $\underline{k}$ is an E_∞-alg. Need monoidal structure on $Ch^{\mathcal{G}}$.

Day convolution $(A \otimes B)(n) = \bigoplus_{a+b=n} \mathrm{Ind}_{GL_a F \times GL_b F}^{GL_n F} A(a) \otimes B(b)$

<u>Theorem</u> $\mathrm{Bar}(\underline{k})(n)$ is the double suspension of the reduced k-chains of the semi simplicial set

$$[p] \longmapsto \{V_0 \oplus \cdots \oplus V_{p+1} = F^n \mid V_i \neq 0\}$$

Proof $\mathrm{Bar}(\underline{\mathbb{k}})(n) = \left| [p] \longmapsto \underline{\mathbb{k}}^{\otimes p}(n) \right|$

$$= \left| [p] \longmapsto \bigoplus_{n_1+\cdots+n_p=n} \underbrace{\mathrm{Ind}^{GL_n}_{GL_{n_1}\times\cdots\times GL_{n_p}} (\mathbb{k}\otimes\cdots\otimes\mathbb{k})}_{} \right|$$

$$* \quad = \mathbb{k}\left[GL_n / GL_{n_1}\times\cdots\times GL_{n_p}\right]$$

$$= \left| [p] \longmapsto \mathbb{k}\{V_1 \oplus \cdots \oplus V_p = F^n\} \right|$$

[two differences: • V_i can be zero
• wrong number of summands]

Face maps are: d_0 forgets V_1, get 0 if $V_1 \neq 0$
d_p " V_p " $V_p \neq 0$
d_i merges adjacent entries.

Collapsing degenerate simplices = do not allow any $V_i = 0$.
This makes d_0, d_p zero which results in a double suspension.

$\boxtimes$

Def $\tilde{T}(F^n) = \left| [p] \longmapsto V_0 \oplus \cdots \oplus V_{p+1} = F^n \right|$

$T(F^n) = \left| [p] \longmapsto 0 \subset W_0 \subset \cdots \subset W_{p+1} = F^n \right|$ Tits building

$\exists$ map $\tilde{T} \longrightarrow T$

Solomon–Tits: $T(F^n) \simeq \bigvee S^{n-2}$
Charney : $\tilde{T}(F^n) \simeq \bigvee S^{n-2}$

$St(F^n) = \tilde{H}_{n-2}(T(F^n))$

$St^{E_1}(F^n) = \tilde{H}_{n-2}(\tilde{T}(F^n))$

$$\therefore \quad H_d(\mathrm{Bar}(\underline{\mathbb{k}})(n)) = \begin{cases} 0 & d \neq n \\ St^{E_1}(n) & d = n \end{cases}$$

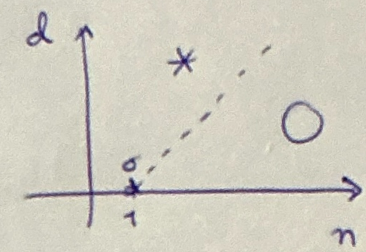
$$\Rightarrow \quad H_d\Big(\mathrm{Bar}\big(\bigoplus_n C_*(GL_n F)\big)\Big) = \bigoplus_n H_{d-n}\big(GL_n F \,;\, St^{E_1}(n)\big)$$

vanishes for $d<n$.

$\Sigma^{-1}\overline{\mathrm{Bar}(-)}$ does not decrease connectivity

$$\Rightarrow \quad H^{E_\infty}_{n,d}\big(\bigoplus_n C_*(GL_n F)\big) = 0 \quad \text{for } d<n-1$$

[get slope $\frac{1}{2}$ stability for $GL_n F$
from slope $\frac{1}{2}$ stability for Symm. grps.]



Thm (GKRW) If F is an infinite field, then

$$H^{E_\infty}_{n,d}\big(\bigoplus_n C_*(GL_n F)\big) = 0 \quad \text{for } d<2(n-1).$$

Moreover,

$$H^{E_\infty}_{n,2(n-1)}(\;\text{"}\;) \cong \begin{cases} k & n=1 \\ k/pk & n=p^r \text{ prime power} \\ 0 & \text{otherwise} \end{cases}$$

Cor (Independently discovered by G, K, RW)

$$H_d(GL_n F \,;\, St(F^n)) = 0 \quad \text{for } d<n-1, \quad F \text{ inf. field.}$$