Oscar Randal-Williams minicourse I.

Goal Exploit the multiplicative structure on $\bigoplus_n H_*(GL_n(F))$ (*)

F ring, usually field/PID, some things not true otherwise (?!)

$\mathbb{k}$ coefficient field (implicit in notation)

Block-sum: $GL_n(F) \times GL_m(F) \to GL_{n+m}(F)$

gives (*) structure of bigraded ass. algebra.

$\sigma \in H_0(GL_1(F))$, generator.

hom. stab $\longleftrightarrow$ $\sigma \cdot$) iso in range.

Consider groupoid $\mathcal{G} = \coprod_n GL_n(F)$

$(\mathcal{G}, \oplus)$ symmetric monoidal.

in particular

$$\begin{tikzcd} \mathcal{G}^k \arrow[r] \arrow[d, "\text{this is really a zig-zag}"'] & \mathcal{G} \\ \mathcal{G}^k \times_{\Sigma_k} E\Sigma_k \arrow[ur, dotted] & \end{tikzcd}$$

Taking nerves, cellular chains, gives

$$\underbrace{\bigoplus_k E_*(\Sigma_k) \otimes_{\Sigma_k} C_*(\mathcal{G})^{\otimes k}}_{=: E_\infty^+(C_*(\mathcal{G}))} \to C_*(\mathcal{G})$$

and $\{E_*(\Sigma_k)\}$ is the Barratt-Eccles operad.

$\longrightarrow$ E_∞^+ is a monad, whose algebras are E_∞-algebras.

($+$: means unital)

Basic consequences:

$$C_*(\mathcal{G})^{\otimes 2} \xleftarrow[\sim]{} C_*(\mathcal{G})^{\otimes 2} \otimes E\Sigma_2 \to C_*(\mathcal{G})^{\otimes 2} \underset{\Sigma_2}{\otimes} E\Sigma_2 \to C_*(\mathcal{G})$$

gives multiplication on homology (block-s-m)
the fact that it extends on $E\Sigma_2 \to$ graded commutative.

Less basic consequence.

let $[x] \in H_d(\mathcal{G}; \mathbb{F}_p)$, corr. to $\mathbb{F}_p[d] \to C_*(\mathcal{G})$.

$$(*) \quad E\Sigma_p \underset{\Sigma_p}{\otimes} \mathbb{F}_p[d]^{\otimes p} \longrightarrow E\Sigma_p \underset{\Sigma_p}{\otimes} C_*(\mathcal{G})^{\otimes p} \to C_*(\mathcal{G})$$

homology of $(*)$ is $H_*(\Sigma_p; \mathbb{F}_p^{\pm 1})[pd]$ $\quad \mathbb{F}_p^{\pm 1} = \begin{cases} \mathbb{F}_p\langle triv \rangle & d \text{ even} \\ \mathbb{F}_p \langle sgn \rangle & d \text{ odd} \end{cases}$

and this gives homology operations $Q^1, Q^2, \dots$
(Dyer-Lashof)

Example if $p=2$, $Q^1(\sigma) \in H_1(GL_2 F; \mathbb{F}_2)$
is the class of $\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$.

Further structure. $C_*(\mathcal{G})$ $\mathbb{N}$-graded E_∞-alg,
augmentation $\varepsilon : C_*(\mathcal{G}) \to \mathbb{k}[0,0]$,
connected.

[From now on, will work "synthetically ∞-categorically"...]

Some constructions.

$$\mathrm{Alg}^{\mathrm{aug}}_{E_\infty^+}(\mathrm{Ch}(\Bbbk)^{\mathbb{N}}) \xrightarrow{U} \mathrm{Ch}(\Bbbk)^{\mathbb{N}}$$

$$A \longmapsto \bar{A} = \mathrm{fib}(\varepsilon) \qquad \varepsilon \text{ augmentation.}$$

$$\mathrm{Ch}(\Bbbk)^{\mathbb{N}} \xrightarrow{Z} \mathrm{Alg}^{\mathrm{aug}}_{E_\infty^+}(\mathrm{Ch}(\Bbbk)^{\mathbb{N}})$$

$$X \longmapsto \Bbbk \oplus X \qquad \text{square-zero ring.}$$

<u>check</u> U, Z preserve all limits.

$\longrightarrow$ have left adjoints F, Q.

free   indecomposables

<u>observe</u> $U \circ Z = \mathrm{id} \implies Q \circ F = \mathrm{id}$.

If A is an $\mathbb{N}$-graded E_∞^+-alg, and $[x] \in H_{n,d}(\bar{A})$ then

$$\Bbbk[n,d] \xrightarrow{x} \bar{A} = U(A)$$

gives

$$\begin{array}{ccc} E_\infty^+ \Bbbk[n,d] & \xrightarrow{x} & A \\ \downarrow{\scriptstyle \varepsilon} & & \\ \Bbbk & & \end{array}$$

Form a pushout. Denote it $A \underset{x}{\overset{E_\infty}{\cup}} D^{n,d+1}$.

Apply $Q(-)$ to the pushout square. (It preserves colimits)

$$\rightsquigarrow \quad \begin{tikzcd} \Bbbk[n,d] \arrow[r] \arrow[d] & QA \arrow[d] \\ 0 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Q(A \underset{x}{\overset{E_\infty}{\cup}} D^{n,d+1}) \end{tikzcd}$$

$\therefore$ attaching an E_∞-alg. cell to A

$\Updownarrow$ ($\Downarrow$ clear, $\Uparrow$ is the following theorem)

attaching a chain complex cell to QA.

Thm (GKRW) Letting $H^{E_\infty}_{n,d}(A) := H_{n,d}(QA)$,

A can be built out of $\Bbbk$ by attaching <u>exactly</u> $\dim_{\Bbbk} H^{E_\infty}_{n,d}(A)$ (n,d)-E_∞-cells.

Pf Prove a version of Hurewicz theorem.

In this world, $H_{n,d}(-) \leftrightarrow$ homotopy

$\downarrow$

$H^{E_\infty}_{n,d}(-1) \leftrightarrow$ homology

So for a given A, understanding $H^{E_\infty}_{n,d}(A)$ tells us (in principle) how to build A using E_∞-cells.

Apply this to $C_*(\mathcal{G}) = \bigoplus_{n \geq 0} C_*(GL_n(F); k)$.

Need <u>another</u> way of computing QA.