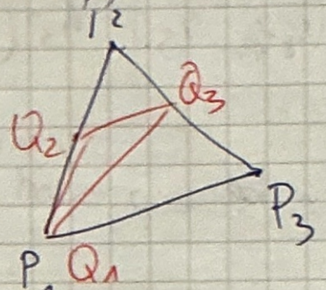
... list
as a whole?

(3) Special pairs P - proj. simplex

Def: (P,Q) - special pair if $Q_1 = P_1$, $Q_i \in \langle P_{i-1}, P_i \rangle$

generic " " " $=$ $Q_i \neq P_i$ for $i = 2, \dots, d$

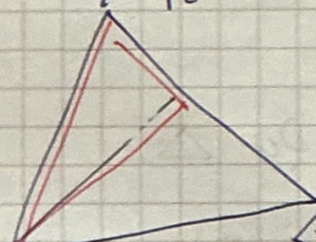


generic special pairs (P,Q)

$\updownarrow$

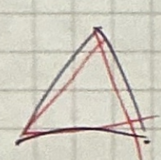
$P_i := \langle v_d + \dots + v_{d-i+1} \rangle$, $i = 1, \dots, d$

$Q_i = \langle v_{d-i+1} \rangle$ some basis $v_1, \dots, v_d$



$St^H(\langle v_1, v_2 \rangle) \cdot St^H(\langle v_3 \rangle)$

$\longrightarrow L[v_1, v_2] \cdot L[v_3]$



$L[v_1]\, L[v_2]\, L[v_3]$

Duality $P \longrightarrow P^\vee$ $(P_1, \dots, P_d) \longrightarrow (H_1, \dots, H_d) \in \mathbb{P}(V)^d$

$H_i := \langle P_1, \dots, \hat{P}_i, \dots, P_d \rangle$

$S(P) = [P_1, \dots, P_d]$, ~~$St(V)$~~

$[P] \to [P^\vee]$ is well-def! map $St(V) \longrightarrow St(V^\vee)$

$(P,Q) \xmapsto{D} (Q^\vee, P^\vee)$

Claim: D respects ~~maps~~ special pairs $\overset{?}{\supseteq}$ ~~D~~

Sanders antipode?

$L[v_1, \dots, v_d] \mapsto (-1)^d\, I[v^d, \dots, v^1]$

$\in St^{\mathcal{H}}(V)$

$I[w_1, \dots, w_d] = [w_d, w_{d-1}, \dots, w_1] \otimes [w_d, w_{d-1} - w_d, \dots, w_1 - w_2]$

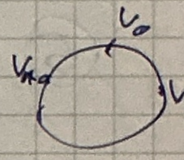
ϱ must be drawn

$\cdot\, L[v_1,\dots,v_k]\cdot L[v_{k+1},\dots,v_{k+\ell}] = \sum_{\sigma\in\Sigma_{k\ell}} L[v_{\sigma(1)},\dots,v_{\sigma(k+\ell)}] \longleftrightarrow$ Li quasi shuffle mod lower dp

$I[v_1,\dots,v_k]\cdot I[\dots] = \sum(\dots \mid L\to I)$

Cor: $St^{\mathcal{L}} := \mathrm{Coker}\,(St^{\mathcal{H}} \otimes_{\text{Day}} St^{\mathcal{H}} \to St^{\mathcal{H}})$

On $\sum v_i = 0$



$(v_0,\dots,v_n) \xrightarrow{\varphi} L^{\mathcal{L}}[v_1,\dots,v_n]$ – dih. symm in $\mathbb{Z}/(n+1)\mathbb{Z}$

$\varphi(v_0,\dots,v_n) = \varphi(v_1,\dots,v_n,v_0)$
$= \varphi(v_n, v_{n-1},\dots,v_0)$

$V^{n+1}/\Delta V$:

$(v_0,\dots,v_n) \longmapsto I^{\mathcal{L}}[v_1-v_0,\dots,v_n-v_0]$

④ $H_n(T^d) \xrightarrow{\sim} St^{\mathcal{H}}(V)\otimes \mathrm{Sym}^{n-d}(V)$

surjectivity $\rightsquigarrow$ $St^{\mathcal{H}}(V)$ is gen^d by $L[v_1,\dots,v_d]$ $v_i\in V$, $v_1,\dots,v_d$ – basis

~~Thm~~ Define $S(P,Q) = [P]\otimes[Q] \in St^{\mathcal{H}}(V)$

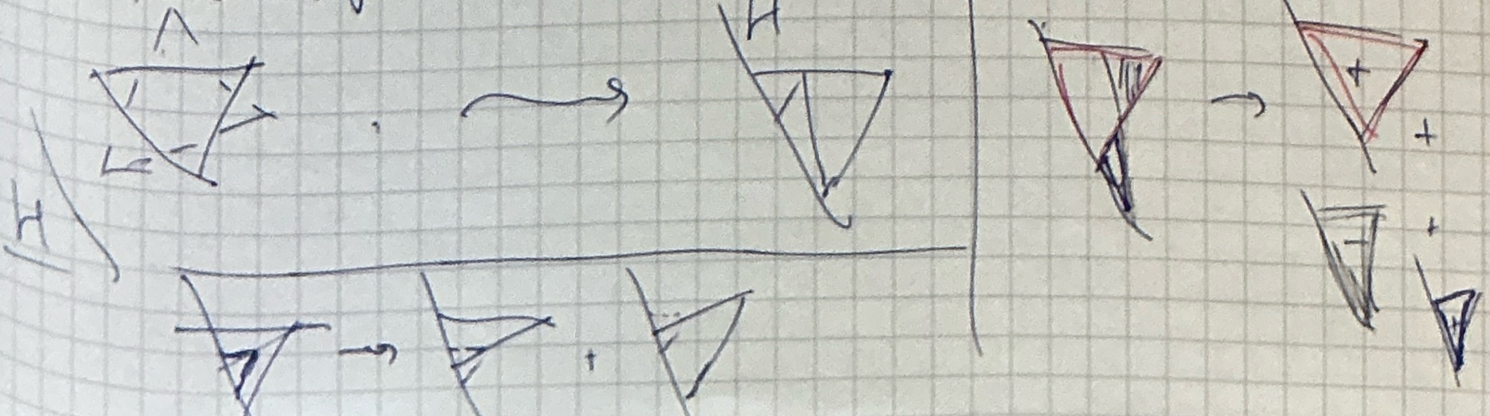
<u>Thm</u>

$H\subset V$ hyperplane fixed, then
$\{S(P,Q) \mid (P,Q)$ – sp. pairs with $\langle P_1,\dots,P_{d-1}\rangle = H\}$
– generates $St^{\mathcal{H}}(V)$ $/\mathbb{Q}$

<u>Thm$^\vee$</u>

$L\subset V$ line
$\{S(P,Q) \mid (P,Q)$ – special $P_1 = Q_1 = L\}$
generate $St^{\mathcal{H}}(V)$

Reduction steps:



tac/ Dum?

④ $\cancel{H_d(GL_d(\mathbb{Z}), St^{\mathbb{Z}})}$ $\mathbb{Q}$?

Thm: (GKRW) $H_0(GL_d(\mathbb{Q}), St^{\mathcal{H}}(\mathbb{Q}^d)) = \mathbb{Q}$

Thm$^{\vee}$ $\Rightarrow St^{\mathcal{H}}(F^d)$ is spanned by $L[v_1,\dots,v_d]$

$S: St^{\mathcal{H}} \longrightarrow B_d\, St(V) \xrightarrow{\text{augm}} \mathbb{Q}$

⑤ $H_1(GL_d(F), St^{\mathcal{L}}(F^d)) \twoheadrightarrow \mathcal{L}_d(F)$ combinatorial version

Kupers
Rudenko
Sierra

mod prod's $I \to \mathcal{L}_d^{\mathbb{R}}(T^d) \longrightarrow St^{\mathcal{L}}(\mathbb{Q}^d) \twoheadrightarrow$ index St mod, (no symm power)

$GL_d(\mathbb{Q}) \cdot \mathcal{L}^{\mathbb{R}}_{\dots}(x)$

$$(I)_{GL_d(\mathbb{Q})} \to \mathcal{L}_d(\mathbb{Q})$$

$$I^F[v_1,\dots,v_d] \mapsto I^{\mathcal{L}}(\infty; \varphi(v_1),\dots,\varphi(v_d), 0) \in \mathcal{L}_d(\mathbb{Q}),\ \varphi \in V^{\vee}$$

$$\sum c_i\, I^F[v_1^{(i)},\dots,v_d^{(i)}] \in I \quad , \quad (v_1^{(i)},\dots,v_d^{(i)}) = A_i$$

$$\sum c_i\, L_{1,\dots,1,A_i}(x_1,\dots,x_d) = 0 \quad \text{in} \quad \mathcal{L}_d(\overline{\mathbb{Q}}(T^d))$$

$\varphi(e_i) := a_i \qquad x_1 = t^{a_1},\dots, x_d = t^{a_d},\ t \to 1$

$$\lim_{t\to 1} I^{\mathcal{L}}(\infty, t^{a_1},\dots,t^{a_d}, 0) = I^{\mathcal{L}}(\infty; \overset{\varphi(e_1)}{a_1},\dots,\overset{\varphi(e_d)}{a_d}, 0)$$

$$\lim_{t\to 1} I^{\mathcal{L}}(\infty, \zeta_1 t^{a_1} \dots \zeta_d t^{a_d}, 0) = I^{\mathcal{L}}(\infty, \zeta_1,\dots,\zeta_d, 0)$$