D. Radchenko minicourse II

Recap/clarification

$$\mathcal{H}^{MTM}_\bullet(F) \quad \supset \quad \mathcal{C}^{MTM}_\bullet(F)$$

$\uparrow$

functions on the motivic Galois group of $MTM(F)$

In $\mathcal{H}^{MTM}_\bullet(F)$ look at sub Hopf algebra generated by motivic polylogs $Li^{\mathcal{M}}$ (previously $Li^{\mathcal{H}}$) resp $Li^{\mathcal{C}}$ in $\mathcal{C}^{MTM}$

call them $\mathcal{H}_\bullet(F)$, $\mathcal{C}_\bullet(F)$

here Δ, m, δ can be described combinatorially on $Li(\cdots)$, $I(\cdots)$

Relations in weight n:

$f \in \mathcal{L}_n(F(t))$ $\quad \delta f = 0$

$\Rightarrow f(0) - f(1) = 0$ in $\mathcal{L}_n(F)$ $\quad n \geq 2$

$T_d = \operatorname{Spec} \overline{\mathbb{Q}}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$ $\qquad \mathcal{H}_n(F)$, $\quad F = \overline{\overline{\mathbb{Q}}(T_d)}$.

We defined $Li_{\underline{n}, A}$ $\quad A \in \mathbb{Z}^{d \times d}$, $|\det A| = N > 0$ $\quad \underline{n} = (n_1, \dots, n_k)$, $n_i > 0$, $k \leq d$.

It is pushforward of $Li_{\underline{n}}$ along $T^d \xrightarrow{A} T^d$.

$GL_d(\mathbb{Q})$ acts via $g\, Li_{\underline{n},A} = Li_{\underline{n}, gA}$ $\quad$ (g integer matrix)
and rescaling, if $g = \lambda \cdot Id$, $\lambda \in \mathbb{Q}^\times$
(by λ^{n-d})

$$\mathbb{L}_n(T^d) = \frac{\mathbb{Q}\text{-span of } \{ Li_{\underline{n},A},\ \Sigma n_i = n,\ k \leq d \}}{(\text{smaller depth}) + \log(x) \cdot (\ \Sigma n_i = n-1\ )}$$

Thm (Charlton-R-Rudenko) $\mathbb{L}_n(T^d) \xrightarrow{\sim} St^{\mathcal{H}}(V) \otimes Sym^{n-d}(V)$
where $St^{\mathcal{H}}(V) = St(V) \otimes St(V)$ $\qquad [V = \mathbb{Q}^d]$

defined by

$$\mathrm{Li}_{n_1\cdots n_d}(x_1,\dots,x_d) \longmapsto L[e_1,\dots,e_d] \otimes \prod_{j=1}^{d} \frac{e_j^{n_j-1}}{(n_j-1)!}$$

$$L[e_1,\dots,e_d] = [v_d,\, v_d+v_{d-1},\, \dots,\, v_d+\dots+v_1] \otimes [v_d,\dots,v_1]$$

$\alpha+\beta=\gamma$

$z=xy$

$$\mathrm{Li}_{1,1}(y,x) + \frac{\beta}{\gamma}\mathrm{Li}_2(xy) = \underbrace{\sum_{\substack{X^\alpha=x\\ Y^\delta=y}} \mathrm{Li}_{1,1}\left(\frac{X}{Y},Y\right)}_{A\cdot \mathrm{Li}_{1,1}(x,y)} - \sum_{\substack{Y^\beta=y\\ Z^\gamma=z}} \mathrm{Li}_{1,1}\left(\frac{Z}{Y},Y\right) + \sum_{\substack{X^\alpha=x\\ Z^\gamma=z}} \mathrm{Li}_{1,1}\left(\frac{Z}{X},X\right)$$

$$A=\begin{pmatrix} \frac{1}{\alpha} & 0 \\ -\frac{1}{\beta} & \frac{1}{\beta}\end{pmatrix}$$

similarly for other two terms.

<u>Thm</u> (CRR) In weight n, depth d, all multiple polylogs can be expressed in terms of $\mathrm{Li}_{1,1,\dots,1,n-d+1}$.

e.g. $\mathrm{Li}_{2,2,2}(x,y,z) = \sum_i c_i\, \mathrm{Li}_{1,1,4}(\underline{r}_i(x,y,z))$ $\underline{r}_i$ tuple of rational functions.

Δ coproduct on MPL

$\rightsquigarrow St^{\mathcal{H}}$ VB-Hopf algebra

$(St^{\mathcal{H}}, \Delta, m)$ explicit formula for Δ which will not be written out for lack of time.

$$St^{\mathcal{H}}(V) \simeq \mathrm{Ker}\big(B_d\, St(V) \to B_{d-1}\, St(V)\big)$$

$$B_m\, St\, V = \bigoplus_{\substack{V_1 \oplus \cdots \oplus V_m = V \\ V_i \neq 0}} St(V_1) \otimes \cdots \otimes St(V_m)$$

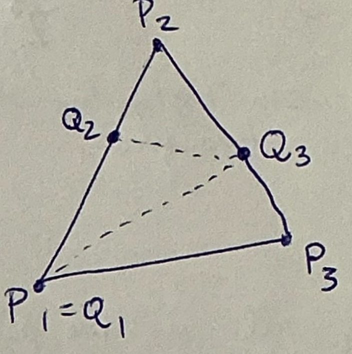
<u>conj</u> $\mathrm{Ker}\big(B_d^{\mathbb{Z}}\, St\, V \to B_{d-1}^{\mathbb{Z}}\, St\, V\big) \cong GL_d(\mathbb{Z}) \circ L[e_1, \dots, e_d]$

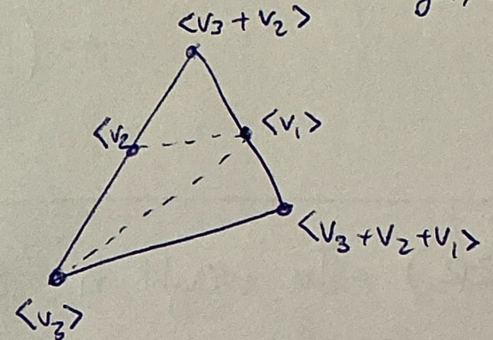
<u>Special pairs</u> ("Coxeter pairs" in CRR)

$P = (P_1, \dots, P_d) \in (\mathbb{P}V)^d$ dim $V = d$

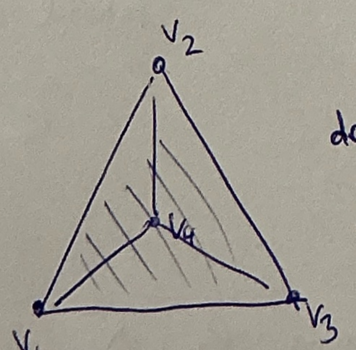
– simplex

P, Q simplices are a <u>special pair</u> if Q_i lies in the span of $\underbrace{(P_{j-1}, P_j)}$ for all i.





compare formula for $L[\cdots]$



decompose triangle as union of 3 triangles

$\longrightarrow [v_1, v_2, v_3] = [v_1, v_2, v_4] + [v_1, v_4, v_3] + [v_4, v_2, v_3]$

$\longrightarrow$ can prove relations by drawing pictures.