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Multiple polylogarithms: $Li_{n_1,\dots,n_d}(z_1,\dots,z_d) = \sum_{0<m_1<\dots<m_d} \frac{z_1^{m_1}\cdots z_d^{m_d}}{m_1^{n_1}\cdots m_d^{n_d}}$

Clément: Li_n "enhanced" to $Li_n^{\mathcal{H}} \in \mathcal{H}_n(F)$

Similarly $Li_{n_1,\dots n_d} \rightsquigarrow Li^{\mathcal{H}}_{n_1,\dots n_d} \in \mathcal{H}_n(F)$ $\quad n = n_1+\dots+n_d$

n weight

d depth

Iterated integrals

$I(a_0,\dots,a_n;a_{n+1}) \in \mathcal{H}_n(F) \qquad a_i \in F,\ a_i \neq a_j \quad i\neq j$

$$I(0;a_1\dots a_n;1) = \int_{0\le t_1\le\dots\le t_n\le 1} \frac{dt_1}{t_1-a_1}\wedge\dots\wedge\frac{dt_n}{t_n-a_n}$$

$\gamma:[0,1]\to\mathbb{C}$
$\gamma(0)=a_0$
$\gamma(1)=a_{n+1}$
$I_\gamma(a_0,\dots,a_{n+1})\dots$

Then $Li_{n_1,\dots n_d}(x_1,\dots,x_d) = (-1)^d I(0,\frac{1}{x_1\cdots x_d},\underbrace{0,0,\dots,0}_{n_1-1},\frac{1}{x_2\cdots x_d},\underbrace{0,\dots,0}_{n_2-1},\dots,\frac{1}{x_d},\underbrace{0,\dots,0}_{n_d-1},1)$

(series converges for $|x_i|<1$)

$$\mathcal{H}_n(F) \overset{conj}{=} \mathbb{Q}\left[Li_{n_1\dots n_d}(x_1,\dots,x_d) : \begin{matrix} n_i>0 \\ n_1+\dots+n_d=n \\ x_1,\dots,x_d\in F\end{matrix}\right] \Big/ \text{(all functional equations)}$$

Functional equations:

- $\log(xy) = \log(x)+\log(y)$ $\quad (\log(x) = -Li_1(1-x))$
- 5-term rel for Li_2
- depth reduction. ex: $Li_{1,1}(x,y) = Li_2\left(\frac{x-1}{y^{-1}-1}\right) - Li_2\left(\frac{y}{y-1}\right) + Li_2(xy)$

Shuffle products.

$$I(0;a_1,\dots,a_k;1)\,I(0;a_{k+1},\dots,a_{k+\ell};1) = \sum_{\sigma\in\Sigma_{k,\ell}} I(0;a_{\sigma(1)},\dots,a_{\sigma(k+\ell)};1)$$

$$\Sigma_{k,\ell} = \left\{ \sigma\in\Sigma_{k+\ell} \,\middle|\, \begin{array}{l} \sigma^{-1}(1)<\dots<\sigma^{-1}(k) \\ \sigma^{-1}(k+1)<\dots<\sigma^{-1}(k+\ell) \end{array} \right\}$$

quasi-shuffles

$$\mathrm{Li}_{n_1\dots n_k}(x_1,\dots,x_k)\,\mathrm{Li}_{n_{k+1}\dots n_{k+\ell}}(x_{k+1},\dots,x_{k+\ell})$$
$$= \sum_{\sigma\in\Sigma_{k,\ell}} \mathrm{Li}_{n_{\sigma(1)}\dots n_{\sigma(k+\ell)}}(x_{\sigma(1)},\dots,x_{\sigma(k+\ell)}) + \left(\begin{array}{c}\text{lower}\\ \text{depth}\\ \text{terms}\end{array}\right)$$

Ex $I(0;a;1)I(0;b;1) = I(0;a,b;1) + I(0;b,a;1)$

$\mathrm{Li}_1(x)\mathrm{Li}_1(y) = \mathrm{Li}_{1,1}(x,y) + \mathrm{Li}_{1,1}(y,x) + \mathrm{Li}_2(xy)$

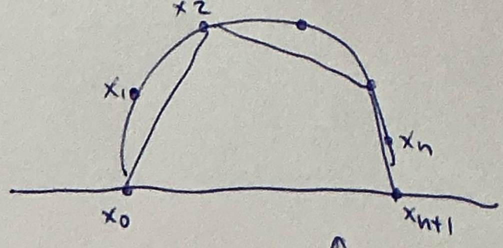
(proof: decompose product of simplices into simplices)

Distribution relation

$$\sum_{\substack{y_1^N = x_1 \\ \vdots \\ y_d^N = x_d}} \mathrm{Li}_{n_1,\dots,n_d}(y_1,\dots,y_d) = N^{d-n}\,\mathrm{Li}_{n_1,\dots,n_d}(x_1,\dots,x_d)$$

(true also for $N<0$ suitably interpreted but more difficult)

Coproduct $$\Delta I(x_0,\dots,x_{n+1}) = \sum_{0=i_0<i_1<\dots<i_{k+1}=n+1} I(x_{i_0},\dots,x_{i_{k+1}}) \otimes \prod_{j=1}^{k} I(x_{i_j},x_{i_j+1},\dots)$$



sum over such polygons

left tensor factor : polygon

product on right $\otimes$ factor : product of all little slivers outside polygon.

Δ' reduced coproduct (ignore terms $k=0$ and $k=n$)

$$\Delta' Li_2(x) = Li_1(x) \otimes \log(x)$$

$$\Delta' Li_{1,1}(x,y) = Li_1(xy) \otimes Li_1(y) - Li_1(xy) \otimes Li_1(x^{-1}) + Li_1(y) \otimes Li_1(x)$$

$x \rightsquigarrow v_1 \quad y \rightsquigarrow v_2$

$[v_1 + v_2, v_2] - [v_1 + v_2, -v_1] + [v_2, v_1] = 0$ in $St(V)$

[Rmk $\quad Li_1(x) = Li_1(x^{-1}) + \log x$]

but $[v_1, \dots, v_n] = [\mathbb{Q}v_1, \dots, \mathbb{Q}v_n]$

$T_d = \operatorname{Spec} \overline{\mathbb{Q}}[x_1^{\pm 1}, \dots, x_d^{\pm 1}] \qquad F = \overline{\overline{\mathbb{Q}}(T_d)}$

$A \in \mathbb{Z}^{d \times d}, \quad \det A \neq 0, \quad \underline{n} = (n_1, \dots, n_d) \quad x = (x_1, \dots, x_d)$

$N = |\det A| \qquad x^N = (x_1^N, \dots, x_d^N) \qquad x^A = \begin{pmatrix} x_1^{a_{11}} & & x_d^{a_{d1}} \\ & \ddots & \\ x_1^{a_{1d}} & & x_d^{a_{dd}} \end{pmatrix}$

$$Li_{n_1 \dots n_k; A}(x_1, \dots, x_d) = N^{n-d-1} \sum_{y^N = x} Li_{n_1 \dots n_k}(y^A)$$

$(k \leq d)$

$$L_n(T^d) = (\mathbb{Q}\text{-span of } Li_{\underline{n},A}) \,/\, (\text{lower depth terms})$$

$n_1 + \dots + n_k = n, \quad k \leq d$

$k < d$ OR terms $\log(x_i) Li_{\underline{n},A}$

Thm (Charlton-R-Rudenko)

$\exists$ map $L_n(T^d) \xrightarrow{\sim} St(\mathbb{Q}^d) \otimes St(\mathbb{Q}^d) \otimes Sym^{n-d}(\mathbb{Q}^d)$

Takes $Li_{n_1,\ldots,n_d}$ to $[e_d,\ldots,e_d+\cdots+e_1] \otimes [e_d,\ldots,e_1]$

$$\otimes \prod \frac{e_i^{n_i-1}}{(n_i-1)!}$$