



Erik Panzer (w/ Francis Brown, Simone Hu)

Def P_n - space of $n \times n$ pos. def. sym. matrices $\subset \mathbb{R}^{n \times n}$

Remark $GL_n \mathbb{R} / O(n) \xrightarrow{\cong} P_n$

\Rightarrow every $X \in P_n$ has unique square root $X^{\frac{1}{2}}$ st. $(X^{\frac{1}{2}})^2 = X$, $X^{\frac{1}{2}} \in P_n$.

Gives residual $GL_n \mathbb{R}$ -action on P_n (conjugation)

Remark P_n is convex in $\mathbb{R}^{n \times n} \Rightarrow$ contractible.

Remark $\text{Stab}_{\text{Id}} = O(n)$, compact group.

\Rightarrow If $\Gamma \subset GL_n \mathbb{R}$ discrete then $\Gamma \cap O(n)$ finite

$\rightarrow GL_n \mathbb{Z}$ acts by finite stabilizers

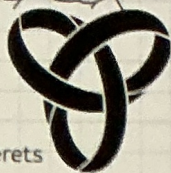
$\Rightarrow H^*(GL_n \mathbb{Z}; \mathbb{R}) = H^*(P_n/GL_n \mathbb{Z}; \mathbb{R})$.

\rightsquigarrow want $GL_n \mathbb{Z}$ -invariant diff forms on P_n
easier: $GL_n \mathbb{R}$ -invariant forms on P_n (Lie alg. coh.)
(and such are automatically closed)

(Remark: Such ω is determined by $\omega|_{\text{Id}} \in \underbrace{\wedge^k T_{\text{Id}}^* P_n}_{= \mathfrak{gl}_n / \mathfrak{so}_n}$)

Def $\beta_X^k := \text{tr}(X^{-1} dX)^k \in \Omega^k(P_n)$

(better notation β , X is only a choice of local coord, but sometimes convenient to include it in notation),



Properties

1) $GL_n \mathbb{R}$ -invariance:

$$\beta_{gXg^t}^k = \beta_X^k$$

2) $\beta^{2k} = 0 \quad \forall k, \quad \beta^{4k-1} = 0 \quad \forall k$

3) $d\beta^k = 0 \quad \forall k$

4) β^{4k+1} vanishes on \mathbb{P}_n if $n < 2k+1$.

5) $\beta^1 = d \log \det X$.

6) $\beta_{(\partial X)}^k = \beta_X^k + \beta_Y^k$ in particular the β^k are stable

So, e.g. $[1] \in H^0(GL_n \mathbb{Z}; \mathbb{R})$

$$[\beta^5] \in H^5(GL_n(\mathbb{Z}); \mathbb{R}) \quad \text{for } n \geq 3$$

Thm (Bismut-Lott) $[\beta^{4k+1}] = 0$ in $H^{4k+1}(GL_n \mathbb{Z}; \mathbb{R})$ for $n = 2k+1$

Thm (Brown) $\mathbb{R}\langle \beta^5, \beta^9, \dots, \beta^{4k-3} \rangle$ injects into $H^*(GL_n \mathbb{Z}; \mathbb{R})$ for $n \geq 2k+1$

Proof sketch: Poincaré duality

$$H^*(\mathbb{P}_n/GL_n \mathbb{Z}; \mathbb{R}) \otimes H_c^{(n+1)-*}(\mathbb{P}_n/GL_n \mathbb{Z}; \text{orientation steel})$$

= \mathbb{R} for n odd

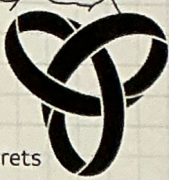
How to compute $H_c(-1)$?

do not want to write down forms w/ compact support.

Replace $\mathbb{P}_n/GL_n \mathbb{Z}$ by $LP_n/GL_n \mathbb{Z}$ $LP_n = \mathbb{P}_n/\mathbb{R}^\times$

construct a "variety" $LA_g^{\text{top, bordified}} \supset \partial LA_g^{\text{top, B}}$ ($g=n$)
 st $LP_n/GL_n \mathbb{Z} = LA_n^{\text{top, B}} \setminus \partial LA_n^{\text{top, B}}$





$$\Rightarrow H_c^*(P_n/G_{2n}\mathbb{Z}) \cong H^*(LA_n^{\text{trop}, B}, \partial LA_n^{\text{trop}, B})$$

Point is: forms extend to $LA_n^{\text{trop}, B}$

and we have blown up where the forms have poles.

• β^k extend smoothly to $LA_g^{\text{trop}, B}$

• A single blowup can be parametrized as

$$X = \begin{pmatrix} z_A & z_B \\ z_B^t & c \end{pmatrix} \quad \text{exc. divisor } z=0$$

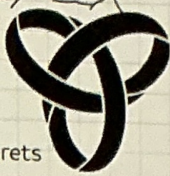
$$\beta_x^k = \beta_A^k + \beta_c^k + \mathcal{O}(z) \quad \text{describes how } \beta^k \text{ restricts to boundary.}$$

• β^{4k+1} on $\partial LA_{2k+1}^{\text{trop}, B}$ is zero.

$$\Rightarrow \text{class in } H^{4k+1}(LA_{2k+1}^{\text{trop}, B}, \partial LA_{2k+1}^{\text{trop}, B})$$

Thm $\mathbb{R}\langle \beta^5, \beta^9, \dots, \beta^{4k-3}, \beta^{4k+1} \rangle \cdot \beta \longleftrightarrow H_c^*(P_{2k+1}/G_{2k+1}\mathbb{Z}; \mathbb{R})$

Moreover, volume form on $LP_{2k+1}/G_{2k+1}\mathbb{Z}$ is proportional to $\beta^5 \wedge \beta^9 \wedge \dots \wedge \beta^{4k+1}$.



In the even rank case, $P_{2n}/GL_{2n}\mathbb{Z}$ not orientable.

$$\rightsquigarrow H_c^0(P_{2n}/GL_{2n}\mathbb{Z}; \mathbb{O})$$

Pfaffian form
$$\text{Pf } M = \frac{1}{2^n n!} \sum \text{sgn}(\sigma) M_{\sigma(1)\sigma(2)} M_{\sigma(3)\sigma(4)} \dots$$

($2n \times 2n$ matrix)

$$\phi_x^{2n} = \frac{\text{Pf}(dx x^{-1} dx)}{\sqrt{|\det x|}} \in \Omega^{2n}(P_{2n})$$

volume form in even case is $\beta^5 \wedge \beta^9 \wedge \dots \wedge \beta^{4n-3} \wedge \phi_0^{2n}$