Erik Panzer (w Francis Brown, Simone Hu)

Def P_n - space of $n\times n$ pos. def. sym. matrices $\subset \mathbb{R}^{n\times n}$

Rmk $GL_n\mathbb{R}/O(n) \xrightarrow{\cong} P_n$

$\Rightarrow$ every $X\in P_n$ has unique square root $X^{\frac12}$ s.t. $(X^{\frac12})^2 = X$, $X^{\frac12}\in P_n$.

Gives residual $GL_n\mathbb{R}$-action on P_n (conjugation)

Rmk P_n is convex in $\mathbb{R}^{n\times n}$ $\Rightarrow$ contractible.

Rmk $\mathrm{Stab}_{Id} = O(n)$, compact group.

$\Rightarrow$ If $\Gamma\subset GL_n\mathbb{R}$ discrete then $\Gamma\cap O(n)$ finite

$\Rightarrow$ $GL_n\mathbb{Z}$ acts by finite stabilizers

$\Rightarrow$ $H^\bullet(GL_n\mathbb{Z};\mathbb{R}) = H^\bullet(P_n/GL_n\mathbb{Z};\mathbb{R})$.

$\rightsquigarrow$ want $GL_n\mathbb{Z}$-invariant diff forms on P_n

easier: $GL_n\mathbb{R}$-invariant forms on P_n (Lie alg. coh.)

(and such are automatically closed)

(Rmk: Such ω is determined by $\omega|_{id}\in\Lambda^\bullet \underbrace{T^*_{id}P_n}_{=\mathfrak{gl}_n/\mathfrak{so}_n}$)

Def $\beta^k_X := \mathrm{tr}(X^{-1}dX)^k \in \Omega^k(P_n)$

(better notation β^k, X is only a choice of local coord, but sometimes convenient to include it in notation).

Properties 1) $GL_n\mathbb{R}$- invariance:

$$\beta^k_{gXg^t} = \beta^k_X$$

2) $\beta^{2k} = 0 \ \forall k, \quad \beta^{4k-1} = 0 \quad \forall k$

3) $d\beta^k = 0 \quad \forall k$

4) β^{4k+1} vanishes on P_n if $n < 2k+1$.

5) $\beta^1 = d \log \det X$.

6) $\beta^k_{\left(\begin{smallmatrix} X & 0 \\ 0 & Y \end{smallmatrix}\right)} = \beta^k_X + \beta^k_Y$ in particular the β^k are <u>stable</u>

So, e.g. $[1] \in H^0(GL_n\mathbb{Z}; \mathbb{R})$

$[\beta^5] \in H^5(GL_n(\mathbb{Z}); \mathbb{R})$ for $n \geq 3$

<u>Thm</u> (Bismut-Lott) $[\beta^{4k+1}] = 0$ in $H^{4k+1}(GL_n\mathbb{Z}; \mathbb{R})$ for $n = 2k+1$

<u>Thm</u> (Brown) $\mathbb{R}\langle \beta^5, \beta^9, \ldots, \beta^{4k-3} \rangle$ injects into $H^*(GL_n\mathbb{Z}; \mathbb{R})$ for $n \geq 2k+1$

<u>Proof sketch:</u> Poincaré duality

$$H^\bullet(P_n/GL_n\mathbb{Z}; \mathbb{R}) \otimes H_c^{\binom{n+1}{2}-\bullet}(P_n/GL_n\mathbb{Z}; \text{orientation sheaf})$$

$= \mathbb{R}$ for n odd

How to compute $H_c(-)$?
do not want to write down forms w/ compact support.

Replace $P_n/GL_n\mathbb{Z}$ by $LP_n/GL_n\mathbb{Z}$ $\quad LP_n = P_n/\mathbb{R}^\times$

construct a "variety" $LA_g^{\text{trop, } \underbrace{\text{bordified}}_{=B}} \supset \partial LA_g^{\text{trop}, B}$ $\quad (g = n)$

st $LP_n/GL_n\mathbb{Z} = LA_n^{\text{trop}, B} \setminus \partial LA_n^{\text{trop}, B}$

$$\Longrightarrow H_c^\bullet(P_n/GL_n\mathbb{Z}) \cong H^\bullet(LA_n^{trop,B}, \partial LA_n^{trop,B})$$

Point is: forms extend to $LA_n^{trop,B}$
and we have blown up where the forms have poles.

- β^k extend smoothly to $LA_g^{trop,B}$
- A single blowup can be parametrized as

$$X = \begin{pmatrix} zA & zB \\ zB^t & C \end{pmatrix} \quad \text{exc. divisor } z=0$$

$$\beta_X^k = \beta_A^k + \beta_C^k + O(z)$$ describes how β^k restricts to boundary.

- β^{4k+1} on $\partial LA_{2k+1}^{trop,B}$ is zero.

$\Rightarrow$ class in $H^{4k+1}(LA_{2k+1}^{trop,B}, \partial LA_{2k+1}^{trop,B})$

<u>Thm</u> $$\mathbb{R}\langle \beta^5, \beta^9, \dots, \beta^{4k-3}\rangle \cdot \beta^{4k+1} \hookrightarrow H_c^*(P_{2k+1}/GL_{2k+1}\mathbb{Z}; \mathbb{R})$$

Moreover, volume form on $LP_{2k+1}/GL_{2k+1}\mathbb{Z}$ is proportional to $\beta^5 \wedge \beta^9 \wedge \dots \wedge \beta^{4k+1}$.

In the even rank case, $P_{2n}/GL_{2n}\mathbb{Z}$ not orientable.

$\rightsquigarrow H_c^{\bullet}(P_{2n}/GL_{2n}(\mathbb{Z}) ; \mathcal{O})$

<u>Pfaffian form</u> $\quad \mathrm{Pf}\, M = \frac{1}{2^n n!} \sum \mathrm{sgn}(\sigma) M_{\sigma(1)\sigma(2)} M_{\sigma(3)\sigma(4)} \cdots$

($2n \times 2n$ matrix)

$$\phi_X^{2n} = \frac{\mathrm{Pf}(dX\, X^{-1}\, dX)}{\sqrt{\det X}} \in \Omega^{2n}(P_{2n})$$

volume form in even case is $\beta^5 \wedge \beta^9 \wedge \cdots \wedge \beta^{4n-3} \wedge \phi^{2n}$.