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Algebraic structures of Steinberg modules.

<u>Goals</u> 1) mult. $St_n(R) \otimes St_m(R) \to St_{n+m}(R)$

2) generalizations of CFP conj.

Conventions: R PID $F = \mathrm{Frac}(R)$ k coefficients

Review of Jenny's talk.

V free R-module

$T(V)$ – realization of poset of proper nonzero summands of V

$T_n(R) = T(R^n)$.

<u>Prop</u> $T_n(R) \cong T_n(F)$.

$W \longmapsto W \otimes F$

$(U \cap R^n) \longleftarrow U$

<u>Thm</u> (Solomon-Tits) $T_n(R) \simeq \bigvee S^{n-2}$

<u>Def</u> $St_n R := \tilde{H}_{n-2}(T_n(R))$

For $\beta = \{b_1, \ldots, b_n\}$ basis, let $A(\beta)$ = full subcx of $T_n(R)$ spanned by proper nonempty subsets of β.

There is a mandatory picture of a triangle here. $(n=3)$

$A(\beta) \cong S^{n-2}$.

<u>Def</u> $[\beta]$ = image of fund. class of $A(\beta)$ in $St_n R$.

(β needs to be ordered up to sign)

<u>Thm</u> (Ash-Rudolph) R euclidean $\Rightarrow$ $St_n R$ generated by $[\beta]$

Multiplication for Steinberg.

For R euclidean, let $\mu: \mathrm{St}_n R \otimes \mathrm{St}_m R \longrightarrow \mathrm{St}_{n+m} R$

defined by $\mu([v_1,\dots,v_n] \otimes [w_1,\dots,w_m]) = [v_1,\dots,v_n,w_1,\dots,w_m]$

Why well defined? For $R=\mathbb{Z}$ use relations (Jenny's talk)

Better approach: GKRW, define a map on space level and does not require R euclidean.

uses $R^n \oplus R^m \cong R^{n+m}$

<u>Prop</u> $\bigoplus_n \mathrm{St}_n R$ associative ring.

Not commutative: $\underset{\mathrm{St}_1}{\underset{\cap}{[e_1]}} \otimes \underset{\mathrm{St}_2}{\underset{\cap}{[e_1, e_1+e_2]}} \longmapsto [e_1, e_2, e_2+e_3]$ $(\star)$

$[e_1, e_1+e_2] \otimes [e_1] \longmapsto [e_1, e_1+e_2, e_3]$

VB: groupoid of f.g. free R-modules. $(\simeq \bigsqcup GL_n R)$

VB-module: functor $VB \longrightarrow k\text{-mod}$ ($\simeq$ family of $GL_n R$-representations)

<u>Ex</u> $\underline{k}$ constant VB-module

<u>Ex</u> $\mathrm{St} = \{\mathrm{St}_n\}$

Day convolution of VB-modules.

M, N VB-modules: $(M \otimes_{\mathrm{Day}} N)(V) = \bigoplus_{A \oplus B = V} M(A) \otimes_k N(B)$

i.e. $(M \otimes_{\mathrm{Day}} N)(R^n) \bigoplus_{a+b=n} \mathrm{Ind}_{GL_a \times GL_b}^{GL_n} M(R^a) \otimes_k M(R^b)$

<u>Def</u> A VB-ring is a monoid object in $(\mathrm{Mod}_{VB}, \otimes_{\mathrm{Day}})$

$\underline{k}$ and St are both VB-rings.

Both are commutative!

<u>Exercise</u> work out example (☆)

CFP conjecture $k=\mathbb{Q}$

$H_i(SL_n\mathbb{Z}; St_n)=0$ for $n\geq i+2$

If A is a VB-ring, then $H_0(SL;A)=\bigoplus_n H_0(SL_nR;A(n))$ is a graded ring, and $H_i(SL;A)=\bigoplus_n H_i(SL_nR;A(n))$ is an $H_0(SL;A)$-module.

<u>Ex</u> $H_0(SL(\mathbb{Z});St)=\begin{cases}\mathbb{Q} & n\leq 1\\ 0 & n\geq 2\end{cases}$ Lee–Szczarba

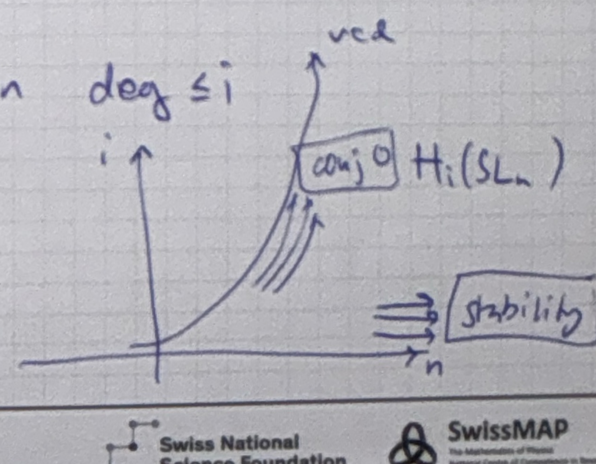
as a ring, exterior alg. on one generator, $\Lambda(x)$ $|x|=1$

<u>Rmk</u> M a $\Lambda(x)$-module generated in degree $\leq d$

$\Rightarrow M_n=0$ for $n\geq d+2$

CFP reformulated:

$H_i(SL(\mathbb{Z});St)$ is generated in deg $\leq i$ as an $H_0(SL(\mathbb{Z});St)$-module.



Homological stability

$$H_0(SL;k) = k[x] \qquad |x|=1$$

Thm (Borel, Li-Sun) $\quad H_i(SL_n(\mathbb{Z});\mathbb{Q}) \to H_i(SL_{n+1}(\mathbb{Z});\mathbb{Q})$

surj $\quad i \le n+1$

iso $\quad i \le n$

Reformulation $H_i(SL;\mathbb{Q})$ gen'd in degree $\le i+1$
as $H_0(SL;\mathbb{Q})$-module, and presented in deg $\le i+2$

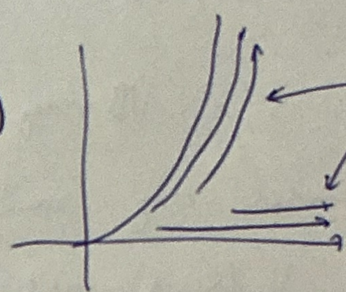
Congruence subgroups:

Def $\Gamma_n(L) = \ker(SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}/L\mathbb{Z}))$

Thm (Lee-Szczarba) $H_0(\Gamma_n(3); St) \cong St_n(\mathbb{F}_3) \cong \mathbb{Z}^{3^{\binom{n}{2}}}$

$n \geq 3 \qquad H_1(\Gamma_n(L)) \cong \mathfrak{sl}_n(\mathbb{Z}/L) \cong (\mathbb{Z}/L)^{n^2-1}$

Thm (Putman) $\quad H_i(\Gamma;k)$
f.g module over $H_0(\Gamma;\underline{k})$



"uniform description"
= "rep. stability"

Conj $\quad H_i(\Gamma;St)$ f.g over $H_0(\Gamma;St)$