SRS
SwissMAP Research
Station in Les Diablerets

Fun with Tits building (Complexes)

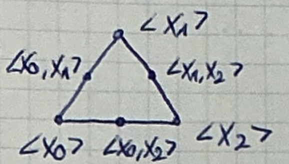
$T(\mathbb{R}^n) = T(S^{n-1}) = |\text{poset of linear } 0 \subsetneq V \subsetneq \mathbb{R}^n|$

$T^{\text{aff}}(\mathbb{R}^n) = T(E^n) = |\text{poset of affine-linear } \emptyset \subsetneq M \subsetneq \mathbb{R}^n|$

<u>Theorem:</u> (Solomon-Tits) $T(S^{n-1}) \simeq \bigvee S^{n-2}$, $T(E^n) \simeq \bigvee S^{n-1}$

$St(E^n) \cong \tilde{H}_{n-1}(T(E^n))$

If $x_0, \dots, x_n \in E^n$, $(\partial\Delta^n) \longrightarrow T(E^n)$



$$\begin{array}{c} (\partial\Delta^n) \xrightarrow{\text{apt}(x_0,\dots,x_n)} T(E^n) \\ \wr\!\shortparallel \\ S^{n-1} \end{array}$$

generates but relations: $\langle x_0, \dots, x_n\rangle \subsetneq E^n$, then apt $= 0$

$\sum_i (-1)^i \text{apt}(x_0, \dots, \hat{x}_i, \dots, x_n)$

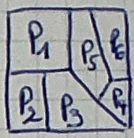
Every convex polytope in E^n gives an "apt": $(\partial \mathring{P}) \longrightarrow T(E^n)$

$\wr\!\shortparallel$
S^{n-1}

k-simplex for $F_0 \subseteq F_1 \subseteq \dots \subseteq F_k$ faces of P

$\rightsquigarrow \langle F_0\rangle \subseteq \dots \subseteq \langle F_k\rangle$

<u>Relations:</u> If P weakly subdivides into $P_1, \dots, P_m$



$\text{apt}(P) = \sum_i \text{apt}(P_i)$

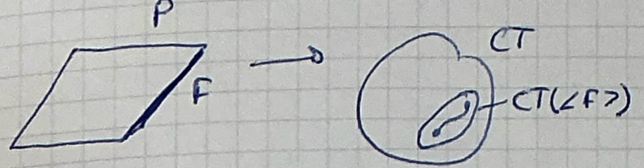
<u>Definition:</u> Polytope group $Pt(E^n) = \mathbb{Z}^{\oplus \text{conv poly}} / [P] = \sum_i [P_i]$

<u>Claim:</u> $Pt(E^n) \underset{\text{apt}}{\cong} St(E^n)$

$Pt(E^n) \longrightarrow St(E^n)$ well-defined?

First, $\Sigma\,\text{apt}: \dfrac{P}{\partial P} \longrightarrow \dfrac{CT(E^n)}{T(E^n)} = \dfrac{|\emptyset \subsetneq M \subseteq E^n|}{|\emptyset \subsetneq M \subsetneq E^n|}$

$P \longrightarrow CT(E^n)$ is "apt-like" if $\forall$ faces $F \subseteq P$, $a(F) \subseteq |\emptyset \subsetneq M \subseteq \langle F\rangle|$



The space of apt-like maps is weakly contractible

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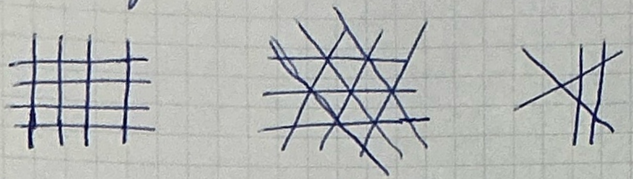
$$\begin{tikzcd} P/\partial P \arrow[r, "apt"] \arrow[d] & CT/T \\ \bigoplus_i P_i/\partial P_i \arrow[ur, "apt"'] & \end{tikzcd}$$

$\mathcal{H}$ = set of hyperplanes in E^n

$\mathcal{L}$ = set of intersections

Ex:

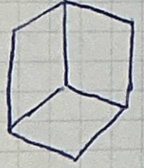


$P_t = \mathbb{Z}^2$

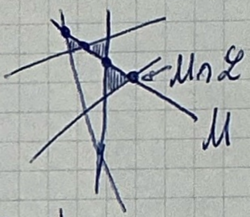
$T^{\mathcal{L}}(E^n) = |\emptyset \subsetneq U \subsetneq E^n, U \in \mathcal{L}|$, $Pt^{\mathcal{L}}(E_n)$ = group of polytopes P such that $\langle F \rangle \in \mathcal{L}$

Theorem: (KLMMS) If $\mathcal{L}$ contains a 0-dimensional subspace $T^{\mathcal{L}}(E^n) \simeq \vee S^{n-1}$
$Pt^{\mathcal{L}}(E^n) \xrightarrow[apt]{\cong} H_{n-1}(T^{\mathcal{L}}(E^n))$

Proof: By induction on dim and # hyperplanes (transfinite)



$Pt = 0$, $St = 0$



Pt, St both increase by
$Pt^{\mathcal{L} \cap U} \cong St^{\mathcal{L} \cap U}(U)$

Why? Scissors congruence! $K_0(E^n, G)$ = group of polytopes up to G s.c =
$= (Pt(E^n) \otimes sgn)_G = (St(E)^n \otimes sym)_G$

Higher S.C. $K(E^n, G)$ Spectrum, $K_*(E^n, G) \underset{(KLMMS)}{=}$ indecomposable H_* (group of scissors automorphisms)

$K(E^n, G) \simeq (\Sigma^{-TE^n} ST(E^n))_{hG}$ (M, BGMMZ)

$K_0(E^1) = \mathbb{R}$ (length), $K_1(E^1) = 0$, $K_2(E^1) = \Lambda^3_{\mathbb{Q}} \mathbb{R}$, $K_3(E^1) = 0$, $K_4(E^1) =$
$= \Lambda^5_{\mathbb{Q}} \mathbb{R}$ (M 2022)

Problem: really really hard above E^1

$$K_m(E^2_{\mathbb{Q}}) = \begin{cases} \Lambda^m\left(\bigoplus_{\substack{p \equiv 1 \bmod 4 \\ p \text{ primes}}} \mathbb{Q}\right) & m \text{ even} \\ 0 & m \text{ odd} \end{cases}$$

$SO(\mathbb{Q}) \cong C_4 \oplus \bigoplus_{p \equiv 1(4)} \mathbb{Z}$, also give a geometric interpretation to $H_*(SL_n(\mathbb{Z}), St(\mathbb{Q}^n))$

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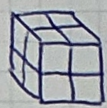
$$apt : \partial \mathring{\Delta}^{n-1} \to T(\mathbb{R}^n), \quad S(apt) : \frac{\mathring{\Delta}^{n-1}}{\partial \mathring{\Delta}^{n-1}} \to ST(\mathbb{R}^n)$$

$$\Sigma S(apt) = \frac{C\mathring{\Delta}^{n-1}}{\partial} \to \Sigma ST(\mathbb{R}^n)$$

$$\frac{C\mathring{\Delta}^{n-1}}{\partial} = \frac{I^n}{\partial I^n}$$



E_1 - coalgebra