R. Hain Voevodsky, Levine, Deligne-Goncharov:

F no. field $\Rightarrow \exists$ tannakian cat. $MT(F)$
of mixed Tate motives $/F$

more generally: $MT(\mathcal{O}_{F,S})$ mixed Tate motives over $\mathcal{O}_{F,S}$.

Simple objects $\mathbb{Q}(n)$ $n \in \mathbb{Z}$.

$$\mathrm{Ext}^1_{MT(\mathcal{O}_{F,S})}(\mathbb{Q}, \mathbb{Q}(n)) = K_{2n-1}(\mathcal{O}_{F,S}) \quad n>0$$

tannakian fund. grp.

$$1 \to \mathcal{K}_{F,S} \to \pi_1(MT(\mathcal{O}_{F,S})) \to \mathbb{G}_m \to 1$$

$\mathcal{K}_{F,S}$ pro-unipotent

$\mathrm{Lie}(\mathcal{K}_{F,S}) = \mathfrak{k}_{F,S}$ is free ($\mathrm{Ext}^i = 0$ $i>1$)

ex $\mathcal{O}_{F,S} = \mathbb{Z} \Rightarrow \mathfrak{k} = \mathrm{Lie}(\sigma_3, \sigma_5, \sigma_7, \ldots)^\wedge$

Depth filtration

Let $\mathcal{O}_N = \mathbb{Z}[\mu_N, \frac{1}{N}] \subset \mathbb{Q}(\mu_N)$ $\mathfrak{k}_N = \mathfrak{k}_{\mathcal{O}_N}$.

$N>2$. $K_{\mathrm{even}}(-)_{\mathbb{Q}} = 0$

$K_{\mathrm{odd}}(-)_{\mathbb{Q}}$ has rank $r_2 = \frac{\varphi(N)}{2}$

filtration $\mathfrak{k}_N = D^0 \supseteq D^1 \supseteq D^2 \supseteq \ldots$ $[D^j, D^k] \subseteq D^{j+k}$

$$Gr_D\, \mathfrak{k}_N = \bigoplus_j D^j / D^{j+1}$$

Let $U_N = \mathbb{G}_m \setminus \mu_N$ ex $U_1 = \mathbb{P}^1 \setminus \{0, 1, \infty\}$

<u>Digression</u> Unipotent completion. (Malcev compl)

Γ discrete group.

for simplicity: $\dim H_1(\Gamma;\mathbb{Q}) < \infty$.

$\mathrm{URep}\,\Gamma = \{\text{cat. of unipotent reps of } \Gamma/\mathbb{Q}\}$

$\Gamma^{un}_{\mathbb{Q}} := \pi_1(\mathrm{URep}\,\Gamma)$, pro-unipotent group.

other perspective: for $\mathbb{Q} \hookrightarrow K$, $\Gamma^{un}(K) =$ gp-like elements of

$K\Gamma^\wedge = \varprojlim_n K\Gamma/I^n$

(see appendix of Quillen "Rat'l homotopy theory")

universal map $\Gamma \to \Gamma^{un}(\mathbb{Q})$.

<u>Ex</u> $\Gamma = \mathrm{free}\langle s_1, \ldots, s_n\rangle$

$\Gamma^{un} = \exp(\mathbb{L}\langle S_1, \ldots, S_n\rangle^\wedge) \subseteq \mathbb{Q}\langle\langle S_1, \ldots, S_n\rangle\rangle$

$\Gamma \to \Gamma^{un}(\mathbb{Q})$

$s_i \mapsto \exp(S_i)$

Deligne–Goncharov

$$\mathcal{O}(\pi_1(U_N, \vec{v})) \text{ is in } \mathrm{Ind}\text{-}\mathrm{MT}(\mathcal{O}_N)$$

$$\wr$$

$$\mathfrak{f}_N := \mathrm{Lie}\, \pi_1^{un}(U_N, \vec{v}) \text{ is in } \mathrm{Pro}\text{-}\mathrm{MT}(\mathcal{O}_N)$$

$\left[\vec{v} = \frac{\partial}{\partial x} \text{ at } 1 \text{ tangential basept}\right]$

so both have an action of $\underline{\mathfrak{k}}_N$.

$U_N \hookrightarrow \mathbb{G}_m$, $\pi_1^{un}(\mathbb{G}_m, \vec{v}) = \mathbb{Q}e_0$ $e_0 =$ "$\log$"

gives $\mathfrak{f}_N \to \mathbb{L}(e_0) = \mathbb{Q}e_0$ with kernel $\mathfrak{a}$.

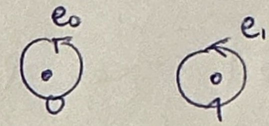
$D^0\mathfrak{f}_N = \mathfrak{f}_N$ $D^1\mathfrak{f}_N = \mathfrak{a}$ (free) $D^d\mathfrak{f}_N = L^d\mathfrak{a}$

$\rightsquigarrow$ filtration on $\mathrm{Der}\,\mathfrak{p}_N$, $\delta \in D^m \iff \delta D^j \mathfrak{p}_N \subseteq D^{j+m}\mathfrak{p}_N \;\forall j$

$\rightsquigarrow$ filtration on $\mathfrak{k}_N$ via pullback $\mathfrak{k}_N \to \mathrm{Der}\,\mathfrak{p}_N$.

$N=1$. $U_1 = \mathbb{P}^1 \setminus \{0,1,\infty\}$

$\mathfrak{p} = \mathfrak{p}_1 = \mathrm{Lie}(e_0, e_1)^\wedge$



$$0 \to \mathbb{L}(e_0^j \cdot e_1)_{j \geq 0} \to \mathbb{L}(e_0, e_1) \longrightarrow \mathbb{L}(e_0) \longrightarrow 1$$

ex $\quad e_0^2 \cdot e_1 = [e_0, [e_0, e_1]]$

e_0, e_1 both of weight -2 (look like $\mathbb{Q}(1)$)

depth \ wt	-2	-4	-6	-8	-10	-12
0	e_0 $\mathbb{Q}(1)$					
1	e_1	$e_0 e_1$	$e_0^2 e_1$	$e_0^3 e_1$ $\mathbb{Q}(4)$	$e_0^4 e_1$	$e_0^5 e_1$ $\mathbb{Q}(6)$
2						
3						

$\zeta(3)$ $\quad \zeta(5)$ $\quad$ nontrivial extensions

polylog quotient

so $\delta_{2j+1} \neq 0$, $j \geq 1$, in $\mathrm{Gr}_D^1 \mathfrak{k}$ (it takes e_0 to $e_0^{2j+1} e_1$)

<u>Question</u> Is $\mathrm{Gr}_D^{\bullet} \mathfrak{k}_N$ free?

<u>$N=1$</u> (1999) Ihara–Takao: not free – depth 2 rel'ns – coming from cusp forms.

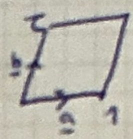
<u>Deligne</u> (2010) $N = 2,3,4,8$ $\quad \mathrm{Gr}_D^{\bullet} \mathfrak{k}_N$ free (also 6?)

Goncharov N prime > 5, $Gr_D^{\cdot}\, \underline{\mathfrak{k}}_N$ not free

Gangl ?

Period polynomials. $SL_2(\mathbb{Z}) \circlearrowright H = H_1(E;\mathbb{Q})$ E elliptic curve

$H = \mathbb{Q}\underline{a} \oplus \mathbb{Q}\underline{b}$



$m>0$

$$0 \to H^1_{cusp}(SL_2\mathbb{Z}, S^{2m}H) \to H^1(SL_2\mathbb{Z}, S^{2m}H) \longrightarrow H^1\left(\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}; S^{2m}H\right) \to 0$$

$\left(H^{2m+1,0} \oplus H^{0,2m+1}\right)$ over $\mathbb{C}$
spanned by cusp forms for $SL_2\mathbb{Z}$ of weight $2m+2$

$\cong$ $\mathbb{Q}(-m-1)$ spanned by Eisenstein series
weight $2m+2$

$\rho : SL_2\mathbb{Z} \longrightarrow S^{2m}H$ 1-cocycle
$\langle s, t\rangle$

$s = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$

$t = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$

ρ cuspidal $\Rightarrow$ $\rho(t) = 0$

so ρ determined by $\rho(s) \in S^{2m}H$

polynomial $r(\underline{a}, \underline{b})$

$$(1+s)r = (1+u+u^2)r = 0 \qquad u = st$$

Ihara–Takao relations between $[\sigma_j, \sigma_k]$ in $Gr_D^2\, \underline{\mathfrak{k}}_1$

$\updownarrow$

r_+ (even part of $r = r_+ + r_-$) for cusp form.

<u>Ex</u> $\Delta = q\prod(1-q^n)^{24}$ $\sim$ $[\sigma_3, \sigma_9] - 3[\sigma_5, \sigma_7] \equiv 0 \mod D^3$