Clément Dupont minicourse III

Recap F field

▷ Tannakian category $MT(F)$ of mixed Tate motives, (Levine)
(conditional on Beilinson–Soulé)

▷ examples of objects in $MT(F)$

$$\mathbb{Q}(-n) \quad , \quad \mathbb{K}_a \ \ a\in F\setminus\{0,1\} \quad , \quad \mathcal{L}_n(z) \quad n\geq 1,\ z\in F\setminus\{0,1\}$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$\left((2\pi i)^n\right) \qquad \begin{pmatrix} 1 & \log a \\ 0 & 2\pi i \end{pmatrix} \qquad \text{e.g.} \begin{pmatrix} 1 & -\log(1-z) & Li_2(z) \\ 0 & 2\pi i & 2\pi i \log z \\ 0 & 0 & (2\pi i)^2 \end{pmatrix}$$

▷ Beilinson's formula

$$\mathrm{Ext}^i_{MT(F)}(\mathbb{Q}(-n), \mathbb{Q}(0)) \simeq K_{2n-i}(F)^{(n)}_{\mathbb{Q}}$$

Want to "construct" extensions $0 \to \mathbb{Q}(0) \to ? \to \mathbb{Q}(-n) \to 0$.

Hard for $n > 1$!

Will use Tannakian formalism.

Def A <u>Tannakian category</u> (neutral $/\mathbb{Q}$) $\mathcal{T}$ is a

$\mathbb{Q}$-linear abelian rigid sym. monoidal category s.t. $\exists$

"fiber functor" $\omega : \mathcal{T} \to \mathrm{Vect}_{\mathbb{Q}}$ exact $\otimes$-functor, faithful

$\quad$ ↳ f. dim. $\mathbb{Q}$-vsp

Construction of an affine group scheme $G = \mathrm{Aut}^{\otimes}(\omega)$

R $\mathbb{Q}$-alg. $\mapsto G(R) =$ group of autom. of $\omega \underset{\mathbb{Q}}{\otimes} R \xrightarrow{\sim} \omega \underset{\mathbb{Q}}{\otimes} R$

Thm (Tannakian reconstruction)

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\sim"] \arrow[dr, "\omega"'] & \mathrm{Rep}_{\mathbb{Q}}(G) \arrow[d, "\text{forget}"] \\ & \mathrm{Vect}_{\mathbb{Q}} \end{tikzcd}$$

Exercise $\mathcal{T}$ = graded $\mathbb{Q}$-vsp

Check that $G = \mathbb{G}_m$.

Back to $MT(F)$.

Every $M \in MT(F)$ has a weight filtration

$$0 \subseteq \cdots \subseteq W_{2(n-1)} M \subseteq W_{2n} M \subseteq \cdots \subseteq M$$

with $$\mathrm{Gr}^W_{2n} M \simeq \bigoplus \mathbb{Q}(-n)$$

Fiber functor: $$MT \xrightarrow{\omega} \mathrm{Vect}_{\mathbb{Q}}$$
$$M \longmapsto \bigoplus_n \mathrm{Hom}_{MT(F)}(\mathbb{Q}(-n), \mathrm{Gr}^W_{2n} M)$$

$\rightsquigarrow \exists$ group scheme G_F "motivic Galois group of mixed Tate motives"

s.t. $MT(F) \simeq \mathrm{Rep}_{\mathbb{Q}} G_F$.

Since ω factors through graded vector spaces, Tannakian formalism gives

$$1 \to U_F \to G_F \to \mathbb{G}_m \to 1$$

and $G_F \simeq \mathbb{G}_m \ltimes U_F$ with U_F pro-unipotent.

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From group to Hopf algebra.

$$\mathcal{H}(F) := \mathcal{O}(U_F) \quad \text{connected graded Hopf algebra}$$

$$MT(F) \simeq \mathrm{grComod}(\mathcal{H}(F))$$

From Hopf algebra to Lie coalgebra.

$$\mathcal{C}(F) = \mathcal{H}(F)_{>0} / \mathcal{H}(F)^2_{>0}$$

Lie coalgebra $\delta : \mathcal{C}(F) \to \Lambda^2 \mathcal{C}(F)$

$$MT(F) \simeq \mathrm{grComod}(\mathcal{C}(F))$$

<u>ex</u> what graded comodule corresponds to $\mathbb{Q}(-n)$?
– $\mathbb{Q}$ in degree n, trivial coaction.

$$K_{2n-i}(F)^{(n)}_{\mathbb{Q}} \simeq H^i(\mathcal{C}(F))_n \quad \text{Lie coalgebra cohomology}$$

$$0 \to \mathcal{C}(F)_n \to (\Lambda^2 \mathcal{C}(F))_n \to (\Lambda^3 \mathcal{C}(F))_n \to \cdots \Lambda^n(\mathcal{C}(F)_1) \to 0$$

For $n=2$ get Bloch complex in disguise.

$$0 \to \underset{\substack{\| \\ K_3(F)^{\mathrm{indec},(2)}_{\mathbb{Q}}}}{K_3(F)^{(2)}_{\mathbb{Q}}} \to \mathcal{C}_2 F \to \Lambda^2 \mathcal{C}_1 F \to K_2(F)^{(2)}_{\mathbb{Q}} = K_2(F)_{\mathbb{Q}} \to 0$$

Matrix coefficients of representations of G_F give functions on G_F.

For $MT(F)$, matrix coefficients give "motivic periods", polylogarithms.

$$0 \to \mathbb{Q}(0) \to K_a \to \mathbb{Q}(-1) \to 0$$

$$\begin{pmatrix} 1 & \log a \\ 0 & 2\pi i \end{pmatrix}$$

$$gr^W_0 K_a \xrightarrow{\varphi} \mathbb{Q}(0)$$

$$gr^W_2 K_a \xleftarrow{v} \mathbb{Q}(-1)$$

$$\log^{\mathcal{H}}(a) \in \mathcal{H}_1(F)$$

s.t. $\quad g \in U_F \longmapsto \langle \varphi, g v \rangle$

action of U_F on $\omega(K_a)$ is $\begin{pmatrix} 1 & \log^{\mathcal{H}}(a) \\ 0 & 1 \end{pmatrix}$

<u>Exercise</u> $\log^{\mathcal{H}}(ab) = \log^{\mathcal{H}}(a) + \log^{\mathcal{H}}(b)$

<u>Exercise</u> $\mathcal{H}_1(F) \simeq F^{\times}_{\mathbb{Q}}$

$$\log^{\mathcal{H}}(a) \longleftrightarrow a$$

Similar construction. Start w/ polylog motive of weights $0, 2, 4, \dots, 2n$.

$$gr^W_0 \mathcal{L}_n(z) \xrightarrow{\varphi} \mathbb{Q}(0)$$

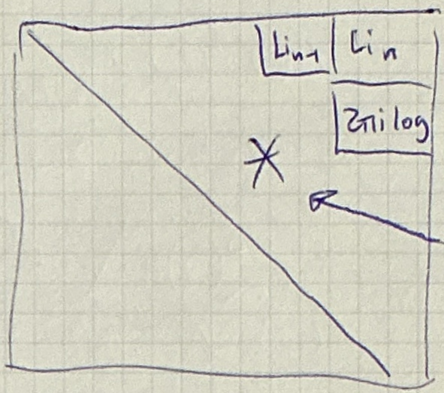
$$gr^W_{2n} \mathcal{L}_n(z) \xleftarrow{v} \mathbb{Q}(-n)$$

$\longrightarrow$ matrix coefficient

$$\mathrm{Li}^{\mathcal{H}}_n(z) \in \mathcal{H}_n(F), \quad \mathrm{Li}^{\mathcal{C}}_n(z) \in \mathcal{C}_n(F)$$

One can check:

$$\delta\, \mathrm{Li}_n^{\mathcal{L}}(x) = \mathrm{Li}_{n-1}^{\mathcal{L}}(x) \wedge \log^{\mathcal{L}}(x)$$ (Goncharov, Beilinson-Deligne)



and everything below is a nontrivial product

Suslin's thm: $\mathcal{L}_2 F$ spanned by $\mathrm{Li}_2^{\mathcal{L}}(x)$ modulo 5-term relations.

So $\mathcal{L}_2 F \longrightarrow \Lambda^2 \mathcal{L}_1 F$ is exactly Bloch cx.

⚠ at weight 4 and higher we also need multiple polylogs.

Conjecturally, multiple polylogs span $\mathcal{L}_n F$ $\forall n$
classicall polylogs definitely do not.

However, conjecture of Goncharov: classical polylogs $\mathrm{Li}_n^{\mathcal{L}}$ are enough to compute Chevalley-Eilenberg cohomology $H^\bullet(\mathcal{L})$

"depth reduction"