Clément Dupont minicourse II

Case of number fields: Zagier's conjecture.

F number field: $K_2(F)_{\mathbb{Q}} = 0$.

$$\Rightarrow K_3(F)_{\mathbb{Q}} = K_3(F)_{\mathbb{Q}}^{\text{indec.}}$$

Borel regulator: $\forall n \geq 1$, $K_{2n-1}(\mathbb{C}) \xrightarrow{\rho_n} \mathbb{C}/(2\pi i)^n\mathbb{R}$

F number field. $K_{2n}(F)_{\mathbb{Q}} = 0 \quad \forall n$

$$K_{2n-1}(F) \xrightarrow{\rho_n^F} \Big(\bigoplus_{\sigma: F \hookrightarrow \mathbb{C}} \mathbb{C}/(2\pi i)^n\mathbb{R}\Big)^+$$

cx conj. invariants

ρ_n^F is injective mod torsion, and its image is a lattice.

$\rightarrow$ rank $K_{2n-1}(F)$ expressed in terms of parity of n, # real / cx embeddings.

Moreover, covolume of lattice $\sim \zeta_F(n)$.

$\boxed{n=2}$

$$\begin{tikzcd}
K_3(F)_{\mathbb{Q}} \arrow[r, "\rho_2^F"] & \Big(\bigoplus_\sigma \mathbb{C}/(2\pi i)^2\mathbb{R}\Big)^+ \\
\ker(\delta: \mathbb{Q}[F^\times]/R(F) \to \Lambda^2 F^\times_{\mathbb{Q}}) \arrow[u, "\simeq"'] \arrow[ur, "P_2^F"'] &
\end{tikzcd}$$

$\rightsquigarrow$ "formula" for $\zeta_F(2)$.

= determinant of evaluations of P_2^F

Cohomology of algebraic varieties.

X/F algebraic variety, F field, $F \hookrightarrow \mathbb{C}$ (X affine smooth for ease)

Betti cohomology $\quad H_B(X) := H^*_{sing}(X(\mathbb{C}); \mathbb{Q})$ $\quad \mathbb{Q}$-vsp

de Rham cohomology $\quad H_{dR}(X) := H^*(\Omega^\bullet_{X/F})$ $\quad F$-vsp

comparison iso: $\quad H_{dR}(X) \otimes_F \mathbb{C} \xrightarrow{\sim} H_B(X) \otimes_{\mathbb{Q}} \mathbb{C}$. $\quad$ "integration"

Period matrix: matrix of comparison iso wrt some F-basis of H_{dR} & $\mathbb{Q}$-basis of H_B.

More generally relative cohomology $\quad H^\bullet(X, Y)$.

<u>Ex</u> $\quad \mathbb{Q}(-1) := H^1(\mathbb{A}^1 \setminus \{0\})$.

Period matrix: $(2\pi i)$.

<u>Ex</u> $\quad \mathcal{K}_a$ (Kummer extension) $:= H^1(\mathbb{A}^1 \setminus \{0\}, \{1, a\})$.

$a \in F \setminus \{0,1\}$

Period matrix:

$$\begin{pmatrix} 1 & \log(a) \\ 0 & 2\pi i \end{pmatrix}$$

$$0 \to \mathbb{Q}(0) \to \mathcal{K}_a \to \mathbb{Q}(-1) \to 0$$

Motives (Grothendieck)

— universal coh. thy for alg. varieties $\quad \mathrm{Mot}(F)$

w/ realization functors

$$\begin{tikzcd} & \mathrm{Mot}(F) \arrow[dl, "\omega_{dR}"'] \arrow[dr, "\omega_B"] & \\ \mathrm{Vect}_F & & \mathrm{Vect}_{\mathbb{Q}} \end{tikzcd}$$

— Morphisms in Mot(F) should be related to algebraic cycles.

— Mot(F) should have the formal properties of Vect_F (a "Tannakian category")

We primarily care about <u>mixed Tate motives</u> $MT(F) \subset Mot(F)$ full subcat. spanned by iterated extensions of $\mathbb{Q}(-n)$'s.

Beilinson's formula $\mathrm{Ext}^i_{MT(F)}(\mathbb{Q}(-n), \mathbb{Q}(0)) \simeq K_{2n-i}(F)^{(n)}_{\mathbb{Q}}$ Adams eigenspace

<u>Ex</u> $\mathrm{Ext}^1_{MT(F)}(\mathbb{Q}(-1), \mathbb{Q}(0)) \simeq F^\times_{\mathbb{Q}}$

$[K_a] \longleftrightarrow [a]$

Above landscape is conjectural in general, conditional on Beilinson-Soulé vanishing conjecture. Known when F number field.

Polylogarithms as periods of mixed Tate motives.

$$\mathrm{Li}_n(z) = \int_{[0,1]^n} \frac{z\, dx_1 \cdots dx_n}{1 - z x_1 \cdots x_n}$$

$Z_n = \{ z x_1 \cdots x_n = 1 \}$

$C_n = \bigcup_i \{ x_i \in \{0,1\} \}$

$$\mathcal{L}_n(z) := H^n(\mathbb{A}^n - Z_n, C_n \setminus C_n \cap Z_n)$$

Full period matrix of $\mathcal{L}_n(z)$:

$$\left(\begin{array}{c|cccc} 1 & \mathrm{Li}_1(z) & \mathrm{Li}_2(z) & \cdots & \mathrm{Li}_n(z) \\ \hline & 2\pi i & 2\pi i \log(z) & \cdots & 2\pi i \frac{\log^{n-1}(z)}{(n-1)!} \\ & & (2\pi i)^2 & & \vdots \\ 0 & & & & (2\pi i)^{n-1}\log(z) \\ & & & & (2\pi i)^n \end{array}\right) =: \Lambda_n(z)$$

$$0 \to \mathbb{Q}(0) \to \mathcal{L}_n(z) \to \mathrm{Sym}^{n-1}(\mathcal{K}_z)(1) \to 0$$

Beilinson-Deligne: consider

$$\underbrace{\overline{\Lambda_n(z)}^{-1} \Lambda_n(z)}$$

single-valued, because monodromy multiplies $\Lambda_n(z)$ on left by rational (in particular real) matrix

$$\log\left(\begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & \ddots \; (-1)^n \end{pmatrix} \overline{\Lambda_n(z)}^{-1} \Lambda_n(z)\right) = \left(\begin{array}{c|ccc} 0 & P_1(z) & \cdots & P_n(z) \\ \hline & \ddots & & ? \\ 0 & & 0 & \ddots \end{array}\right)$$