



Clément Dupont minicourse II

Case of number fields: Zagier's conjecture.

F number field: $K_2(F)_{\mathbb{Q}} = 0$.

$$\Rightarrow K_3(F)_{\mathbb{Q}} = K_3(F)_{\mathbb{Q}}^{\text{indec.}}$$

Borel regulator: $\forall n \geq 1, K_{2n-1}(\mathbb{C}) \xrightarrow{P_n} \mathbb{C}/(2\pi i)^n \mathbb{R}$

F number field. $K_{2n}(F)_{\mathbb{Q}} = 0 \quad \forall n$

$$K_{2n-1}(F) \xrightarrow{P_n^F} \left(\bigoplus_{\sigma: F \hookrightarrow \mathbb{C}} \mathbb{C}/(2\pi i)^n \mathbb{R} \right)^+ \quad \leftarrow \begin{array}{l} \text{cx} \\ \text{conj-} \\ \text{invariants} \end{array}$$

P_n^F is injective mod torsion, and its image is a lattice.

$\rightarrow$ rank $K_{2n-1}(F)$ expressed in terms of parity of n ,
real / cx embeddings.

Moreover, volume of lattice $\sim \zeta_F(n)$.

$n=2$

$$K_3(F)_{\mathbb{Q}} \xrightarrow{P_2^F} \left(\bigoplus_{\sigma} \mathbb{C}/(2\pi i)^2 \mathbb{R} \right)^+$$

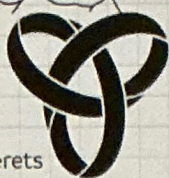
$$\downarrow \simeq$$

$$\ker(S: \mathbb{Q}[F^\times]/R(F) \rightarrow \wedge^2 F_{\mathbb{Q}}^\times)$$

$$\nearrow P_2^F$$

$\leadsto$ "formula" for $\zeta_F(2)$.

= determinant of evaluations of P_2^F



Cohomology of algebraic varieties.

X/F algebraic variety, F field, $F \hookrightarrow \mathbb{C}$ (X affine smooth for ease)

Betti cohomology $H_B(X) := H_{\text{sing}}^*(X(\mathbb{C}); \mathbb{Q})$ $\mathbb{Q}$ -vsp

de Rham cohomology $H_{\text{dR}}(X) := H^*(\Omega_{X/F}^*)$ F -vsp

comparison iso: $H_{\text{dR}}(X) \otimes_F \mathbb{C} \xrightarrow{\sim} H_B(X) \otimes_{\mathbb{Q}} \mathbb{C}$. "integration"

Period matrix: matrix of comparison iso
wrt some F -basis of H_{dR} & $\mathbb{Q}$ -basis of H_B .

More generally relative cohomology $H^*(X, Y)$.

Ex $\mathbb{Q}(-1) := H^1(A^1 \setminus \{0\})$.

Period matrix: $(2\pi i)$.

Ex $\mathcal{K}_a$ (Kummer extension) $:= H^1(A^1 \setminus \{0\}, \{1, a\})$.
 $a \in F \setminus \{0, 1\}$

Period matrix:

$$\begin{pmatrix} 1 & \log(a) \\ 0 & 2\pi i \end{pmatrix}$$

$$0 \rightarrow \mathbb{Q}(0) \rightarrow \mathcal{K}_a \rightarrow \mathbb{Q}(-1) \rightarrow 0$$

Motives (Grothendieck)

— universal coh. thry for alg. varieties $\text{Mot}(F)$
w/ realization functors $\swarrow \omega_{\text{dR}} \searrow \omega_B$
 $\text{Vect}_F \quad \text{Vect}_{\mathbb{Q}}$



- Morphisms in $\text{Mot}(F)$ should be related to algebraic cycles.
- $\text{Mot}(F)$ should have the formal properties of Vect_F (a "Tannakian category")

We primarily care about mixed Tate motives $\text{MT}(F) \subset \text{Mot}(F)$
 full subcat. spanned by iterated extensions of $\mathbb{Q}(-n)$'s.

Beilinson's formula $\text{Ext}_{\text{MT}(F)}^i(\mathbb{Q}(-n), \mathbb{Q}(0)) \simeq K_{2n-i}(F)_{\mathbb{Q}}^{(n)}$ ^{Adams eigenspace}

Ex $\text{Ext}_{\text{MT}(F)}^1(\mathbb{Q}(-1), \mathbb{Q}(0)) \simeq F_{\mathbb{Q}}^{\times}$
 $[K_a] \longleftrightarrow [a]$

Above landscape is conjectural in general, conditional on Beilinson-Soulé vanishing conjecture. Known when F number field.

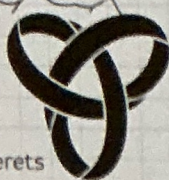
Polylogarithms as periods of mixed Tate motives.

$$\text{Li}_n(z) = \int_{[0,1]^n} \frac{z dx_1 \dots dx_n}{1 - zx_1 \dots x_n}$$

$$Z_n = \{zx_1 \dots x_n = 1\}$$

$$C_n = \bigcup_i \{x_i \in \{0,1\}\}$$

$$\mathcal{L}_n(z) := H^n(\tilde{A}^n - Z_n, C_n \setminus C_n \cap Z_n)$$



Full period matrix of $\mathcal{L}_n(z)$:

$$\begin{pmatrix} \mathbb{1} & Li_1(z) & Li_2(z) & \dots & Li_n(z) \\ 0 & 2\pi i & 2\pi i \log(z) & \dots & 2\pi i \frac{\log^{n-1}(z)}{(n-1)!} \\ & & (2\pi i)^2 & & \\ & & & \ddots & \\ & & & & (2\pi i)^{n-1} \log(z) \\ & & & & & (2\pi i)^n \end{pmatrix} =: \Lambda_n(z)$$

$$0 \rightarrow \mathbb{Q}(0) \rightarrow \mathcal{L}_n(z) \rightarrow \text{Sym}^{n-1}(\mathcal{K}_z) \rightarrow 0$$

Beilinson-Deligne: consider

$$\overline{\Lambda_n(z)}^{-1} \Lambda_n(z)$$

single-valued, because monodromy multiplies $\Lambda_n(z)$ on left by rational (in particular real) matrix

$$\log \left(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & (-1)^n \end{pmatrix} \overline{\Lambda_n(z)}^{-1} \Lambda_n(z) \right) = \begin{pmatrix} 0 & P_1(z) & \dots & P_n(z) \\ 0 & & \ddots & ? \\ & & & \ddots \\ & & 0 & \end{pmatrix}$$