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Clément Dupont minicourse I.

Classical polylog of weight n:

$$\mathrm{Li}_n(z) = \sum_{k\geq 1} \frac{z^k}{k^n}$$

Rmk Also exist multiple polylogarithms $\mathrm{Li}_{n_1,\dots,n_r}(z_1,\dots,z_r) = \sum_{1\leq k_1<\dots<k_r} \frac{z_1^{k_1}\cdots z_r^{k_r}}{k_1^{n_1}\cdots k_r^{n_r}}$

$$\mathrm{Li}_1(z) = -\log(1-z) = \int_0^z \frac{dt}{1-t}.$$

$n\geq 2$: $\quad d\,\mathrm{Li}_n(z) = \mathrm{Li}_{n-1}(z)\frac{dz}{z} \rightsquigarrow \mathrm{Li}_n(z) = \int_0^z \mathrm{Li}_{n-1}(t)\frac{dt}{t}$

multivalued holomorphic function on $z \in \mathbb{C}\setminus\{0,1\}$

Exercise : monodromy around 1 is $\mathrm{Li}_n(z) \rightsquigarrow \mathrm{Li}_n(z) - 2\pi i \frac{\log^{n-1}(z)}{(n-1)!}$

Single-valued variants:

Bernoulli no

$$P_n(z) = \begin{cases} \mathrm{Re}\Big(& (n \text{ odd}) \\ \mathrm{Im}\Big(& (n \text{ even}) \end{cases} \sum_{k=0}^{n-1} \frac{2^k B_k}{k!} \log^k|z|\, \mathrm{Li}_{n-k}(z) \Big)$$

$$\Big(\mathrm{Li}_n(z) - \log|z|\,\mathrm{Li}_{n-1}(z) + \dots \Big)$$

$$P_1(z) = -\log|1-z|$$

$$P_2(z) = \mathrm{Im}\big(\mathrm{Li}_2(z) + \log|z| \log(1-z)\big)$$

Bloch-Wigner dilog.

Exercise P_n is single-valued.

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Rmk If you do the exercise, not many properties of Bernoulli n°s needed (really only one)
→ many potential other single-valued variants, but this is the "right" one (will explain later)

Functional equations:

$$\left.\begin{aligned} &P_2\left(\tfrac{1}{z}\right)+P_2(z)=0, \qquad P_2(1-z)+P_2(z)=0, \\ &P_2(x)+P_2(y)+P_2\left(\frac{1-x}{1-xy}\right)+P_2(1-xy)+P_2\left(\frac{1-y}{1-xy}\right)=0 \end{aligned}\right] \quad (*)$$

[if we worked with Li_2 instead of P_2, would get similar relations w/ funny right-hand sides.
Ex $\mathrm{Li}_2(1-z)+\mathrm{Li}_2(z)=\frac{\pi^2}{6}-\log(z)\log(1-z)$

Let's prove (*). Differentiate, show derivative is zero.

$$dP_2(z)=-\log|z|\,d\arg(1-z)+\log|1-z|\,d\arg z$$

let's organize this. F is the field $\mathbb{C}(x,y)$

$$\begin{aligned} \Lambda^2 F^\times &\longrightarrow \{1\text{-forms}\} \\ f\wedge g &\longmapsto \log|f|\,d\arg g-\log|g|\,d\arg f \end{aligned}$$

$$\begin{aligned} \mathbb{Z}[F^\times] &\xrightarrow{\delta} \Lambda^2 F^\times \\ f &\longmapsto -(1-f)\wedge f \\ 1 &\longmapsto 0 \end{aligned}$$

Enough to prove $\delta\left(\underset{x_1}{[x]}+\underset{x_4}{[y]}+\underset{x_2}{\left[\frac{1-x}{1-xy}\right]}+\underset{x_5}{[1-xy]}+\underset{x_3}{\left[\frac{1-y}{1-xy}\right]}\right)=0$

Can check: $1-x_i=x_{i-1}x_{i+1}$ (indices mod 5)
→ easy to finish.

Another way to write the 5-term relation: cross-ratios.

$$r(a,b,c,d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$$

$$a_0, \dots, a_4 \in \mathbb{P}^1(\mathbb{C}) : \quad \sum_{i=0}^{4} (-1)^i P_2(r(a_0, \dots, \hat{a}_i, \dots, a_4)) = 0 .$$

$$GL_2(\mathbb{C})^4 \longrightarrow \mathbb{R}$$

$$(g_0, \dots, g_3) \longmapsto P_2(r(g_0(z), \dots, g_3(z))) \qquad z \in \mathbb{C} \setminus \{0,1\} \text{ fixed}$$

is a 3-cocycle for $GL_2(\mathbb{C})$

$$\rightsquigarrow [P_2] \in H^3(GL_2(\mathbb{C}); \mathbb{R})$$

For any field F: $H_3(GL_2(F); \mathbb{Q}) \longrightarrow \overbrace{\mathbb{Q}[F^\times]}^{\text{space of "formal evaluations of } P_2\text{, } P_2(x)\ x \in F^\times\text{"}} / R(F)$

$R(F)$ = $\mathbb{Q}$-subspace of $\mathbb{Q}[F^\times]$ spanned by the 5-term relations.

(the other relations follow rationally from 5-term)

~~Resolution complex~~ $H_\cdot(GL_2(F); \mathbb{Q})$ for F infinite:

$$\dots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} \mathbb{Q} \to 0$$

C_n: $\mathbb{Q}$-vsp. w/ basis $(x_0, \dots, x_n)$ pairwise distinct in $\mathbb{P}^1(F)$

$$\partial = \sum (-1)^i (x_0, \dots, \hat{x}_i, \dots, x_n)$$

this is not an acyclic resolution but it is a resolution of $\mathbb{Q}$ so we get a map $H_3(GL_2(F); \mathbb{Q}) \to \mathbb{Q}[F^\times]/R(F)$ from this construction.

Thm (Suslin) For any field F, $\exists$ exact sequence

$$0 \to K_3(F)_{\mathbb{Q}}^{\text{indec}} \longrightarrow \underbrace{\frac{\mathbb{Q}[F^\times]}{R(F)} \xrightarrow{\delta} \wedge^2 F^\times_{\mathbb{Q}}}_{\substack{\text{Bloch-Suslin complex} \\ \text{(two term chain cx)}}} \longrightarrow K_2(F)_{\mathbb{Q}} \to 0$$

$[V_{\mathbb{Q}} = V \underset{\mathbb{Z}}{\otimes} \mathbb{Q}]$

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remarks on K-theory.

$$F \rightsquigarrow K_\bullet(F)$$

$K_0 F = \mathbb{Z}$
$K_1 F = F^\times$
$K_2 F = \wedge^2 F^\times / (-(1-x)\wedge x)$ rationally only ?

"Goncharov's program": extend to higher K-groups.
(in particular Bloch-Suslin cx)

$$K_n(F)_{\mathbb{Q}} = (\text{Prim } H_n(GL(F); \mathbb{Q})$$