S. Charlton

Depth reduction of MPLs

Will focus on weight 4.

Recap $\mathrm{Li}_{n_1,\dots,n_r}(z_1,\dots,z_r)$ MPL

depth r, weight $n=\sum n_i$

$$MT(F) \xrightarrow{\omega} \mathrm{Vect}\,\mathbb{Q} \,, \quad MT(F) \simeq \mathrm{Rep}(\underset{\substack{\rotatebox{90}{$\simeq$} \\ \mathbb{G}_m \ltimes U_F}}{\mathrm{Aut}^{\otimes}\omega})$$

$$\mathcal{H}(F) = \mathcal{O}(U_F)$$

$$\mathcal{L}(F) = \mathcal{H}(F)_{>0} / \mathcal{H}(F)_{>0}^2$$

via matrix coefficients: define $\mathrm{Li}_n^{\mathcal{H}}(z) \in \mathcal{H}_n(F)$

$$\mathrm{Li}^{\mathcal{H}}_{n_1,\dots,n_r}(z_1,\dots,z_r) \in \mathcal{H}_{n_1+\dots+n_r}(F)$$

Projecting to $\mathcal{L}$ gives $\mathrm{Li}^{\mathcal{L}}$: polylogs modulo products.

Goncharov conj $\mathcal{L}$ spanned as vector space by $\mathrm{Li}^{\mathcal{L}}$.

Coproduct / cobracket via iterated integrals

$$I^{\mathcal{L}}(a; x_1,\dots,x_n; b) \Leftrightarrow \int_{a \le t_1 \le \dots \le t_n \le b} \frac{dt_1}{t_1 - x_1} \cdots \frac{dt_n}{t_n - x_n}$$

$$\mathrm{Li}_{n_1,\dots,n_r}(z_1,\dots,z_r) = (-1)^r I(0; \underbrace{\frac{1}{z_1\cdots z_r}, 0, 0, \dots, 0}_{n_1}, \underbrace{\frac{1}{z_2\cdots z_r}, 0, \dots}_{n_2}, \underbrace{\frac{1}{z_r}, 0, 0, \dots 0}_{n_r}; 1)$$

$$\delta I^{\mathcal{L}}(x_0; x_1,\dots,x_n; x_{n+1}) = \sum_{i<j} I^{\mathcal{L}}(x_0; \dots, x_i, x_j, \dots; x_{n+1}) \otimes I^{\mathcal{L}}(x_i; x_{i+1}, x_{i+2}, \dots, x_{j-1}; x_j)$$

Ex $\delta \mathrm{Li}^{e}_{n} z = \mathrm{Li}^{e}_{n-1}(z) \wedge \log^{e}(z)$

need to use some identities like $I^{e}(0;1) = 0$

$I^{\mathfrak{a}}(a; \underbrace{x_1, \ldots, x_n}_{\geq 1}; a) = 0$

$I^{\mathfrak{a}}(a; x, \ldots, x; b) = \dfrac{I^{\mathfrak{a}}(a;x;b)^n}{n!}$ ← goes to zero in I^{e} if $n>1$

$I^{\mathfrak{a}}(a;b;c) = \log^{\mathfrak{a}}(b-c) - \log^{\mathfrak{a}}(b-a)$

with $\log^{\mathfrak{a}}(0) = 0$.

Note $\delta \mathrm{Li}^{e}_{n}(z) = \underset{\mathcal{C}_{n-1}(F)}{\underset{\cap}{\mathrm{Li}^{e}_{n-1}(z)}} \wedge \underset{\mathcal{C}_1(F)}{\underset{\cap}{\log^{e} z}}$

Ex $\delta \mathrm{Li}^{e}_{3,1}(x,y) = \mathrm{wt}\,3 \wedge \mathrm{wt}\,1 + \mathrm{Li}^{e}_{2}(xy) \wedge \mathrm{Li}^{e}_{2}(y)$

Introduce $\bar{\delta} = \delta \bmod \mathcal{C}_1(F)$.

Conj (Goncharov depth conjecture)

$\ker \bar{\delta} =$ depth 1 MPL's.

Wt 2, wt 3.

Ex $\bar{\delta}\, \mathrm{Li}^{e}_{1,1} = 0$

$\bar{\delta}\, \mathrm{Li}^{e}_{1,1,1} = 0$

(no space for $\mathcal{C}_{\geq 2}(F) \wedge \mathcal{C}_{\geq 2}(F)$)

indeed these reduce to depth 1 eg. $\mathrm{Li}_{1,1}(x,y) = \mathrm{Li}_2\left(\frac{y(x-1)}{1-y}\right) - \mathrm{Li}_2\left(\frac{y}{y-1}\right) - \mathrm{Li}_2(xy)$

Wt 4 Fact: every wt 4 MPL via $\mathrm{Li}_{3,1}$, Li_4 (Nicuşor Dan)

$\mathrm{Li}^{e}_{3,1}\left(\frac{x}{y}, y\right) =: \widetilde{\mathrm{Li}}_{3,1}(x,y)$

$\delta \widetilde{\mathrm{Li}}_{3,1}(x,y) = \mathrm{Li}^{e}_{2}(x) \wedge \mathrm{Li}^{e}_{2}(y)$

set $\bar{\delta} = 0$ by substituting $x =$ Li_2 funct. eq.

<u>Recall</u> $\mathrm{Li}_2^e(x) + \mathrm{Li}_2^e(1-x) = 0$

$$\mathrm{Li}_2^e(x) + \mathrm{Li}_2^e\left(\tfrac{1}{x}\right) = 0$$

so $\bar{\delta}$ annihilates $\widetilde{\mathrm{Li}}_{3,1}(x,y) + \widetilde{\mathrm{Li}}_{3,1}(1-x,y)$

" " " $\frac{1}{x}$

<u>Thm</u> (Zagier, Gangl) Depth conj holds in both above cases.

More interesting: 5-term dilog. relation

$$\sum_{i=1}^{5} (-1)^i \widetilde{\mathrm{Li}}_{3,1}\Big(\underbrace{r(x_1,\dots,\hat{x}_i,\dots,x_5)}_{\text{cross-ratio}}, y\Big) \overset{?}{=} \text{depth } 1$$

<u>Thm</u> (Gangl) True, with sum of ≈ 122 terms
(depending on exact presentation)

Will not discuss weight 6 because time is up!