



Depth reduction of MPLS

Will focus on weight 4.

Recap $Li_{n_1, \dots, n_r}(z_1, \dots, z_r)$ MPL

depth r , weight $n = \sum n_i$

$$MT(F) \xrightarrow{\omega} \text{Vec } \mathbb{Q}, \quad MT(F) \simeq \text{Rep}(\text{Aut}^{\otimes \omega})$$

$\mathbb{Q}_m \times V_F$

$$H(F) = \mathcal{O}(V_F)$$

$$\mathcal{E}(F) = H(F)_{>0} / H(F)_{>0}^2$$

via matrix coefficients: define $Li_n^{\text{ge}}(z) \in \mathcal{H}_n(F)$

$$Li_{n_1, \dots, n_r}^{\text{ge}}(z_1, \dots, z_r) \in \mathcal{H}_{n_1 + \dots + n_r}(F)$$

Projecting to $\mathcal{E}$ gives $Li^{\mathcal{E}}$: polylogs modulo products.

Goncharov conj $\mathcal{E}$ spanned as vector space by $Li^{\mathcal{E}}$.

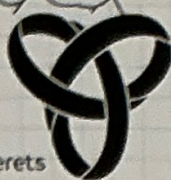
Coproduct/cobracket via iterated integrals

$$I^{\mathcal{E}}(a; x_1, \dots, x_n; b) \leftrightarrow \int \frac{dt_1}{t_1 - x_1} \dots \frac{dt_n}{t_n - x_n}$$

$a \leq t_1 \leq \dots \leq t_n \leq b$

$$Li_{n_1, \dots, n_r}(z_1, \dots, z_r) = (-1)^r I\left(0; \underbrace{\frac{1}{z_1}, \dots, \frac{1}{z_1}}_{n_1}, \underbrace{0, \dots, 0}_{n_2}, \dots, \underbrace{\frac{1}{z_r}, \dots, \frac{1}{z_r}}_{n_r}, 0, \dots, 0, 1\right)$$

$$\delta I^{\mathcal{E}}(x_0; x_1, \dots, x_n; x_{n+1}) = \sum_{i < j} I^{\mathcal{E}}(x_0; \dots, x_i, x_j, \dots, x_{n+1}) \otimes I^{\mathcal{E}}(x_i; x_{i+1}, x_{i+2}, \dots, x_{j-1}, x_j)$$



Recall $\text{Li}_2^e(x) + \text{Li}_2^e(1-x) = 0$

$$\text{Li}_2^e(x) + \text{Li}_2^e\left(\frac{1}{x}\right) = 0$$

so $\overline{\delta}$ annihilates $\widetilde{\text{Li}}_{3,1}(x,y) + \widetilde{\text{Li}}_{3,1}(1-x,y)$
 $\quad \quad \quad // \quad \quad \quad // \quad \quad \quad // \quad \frac{1}{x}$

Then (Zagier, Gangl) Depth conj holds in both above cases.

More interesting: 5-term dilog. relation

$$\sum_{i=1}^5 (-1)^i \widetilde{\text{Li}}_{3,1}\left(\underbrace{r(x_1, \dots, \hat{x}_i, \dots, x_5)}_{\text{cross-ratio}}, y\right) \stackrel{?}{=} \text{depth } 1$$

Then (Gangl) True, with sum of ≈ 122 terms
(depending on exact presentation)

Will not discuss weight 6 because time is up!