Benjamin Brück

§1 Main result.

R commutative ring

$CB(R^n)$ simplicial complex w/ vertices: nonzero proper free summands of R^n

simplex $M_0, \dots, M_p \iff \exists$ basis of R^n that contains a basis of each M_i

Ex $R^3 = \langle e_1, e_2, e_3 \rangle$

A maximal simplex is $\{\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1, e_2\rangle, \langle e_1, e_3\rangle, \langle e_2, e_3\rangle\}$

We usually blur distinction between simplicial cx and its poset of simplices

Rognes : $H_k(CB(R^n))$ for $k \geq 2n-2$

NB $\dim CB(R^n) = 2^n - 3$

Conjecture R local or euclidean
$\Rightarrow CB(R^n)$ is $(2n-4)$-connected

$\tilde{H}_{2n-3}(CB(R^n)) =: St^\infty(R^n)$

[Jeremy: some people call it St^{Lie} "Steinberg-Lie"]

$PD(R^n)$ poset of partial decompositions

elements are $\{N_1, \dots, N_k\}$ s.t. each N_i proper nonzero free R-module summand of R^n

and $R^n = N_1 \oplus \cdots \oplus N_k \oplus M$

ordered by refinement:

$$\{N_1, \dots, N_k\} \leq \{N_1', \dots, N_k'\} \text{ if}$$

$$\forall i \; \exists j \text{ st. } N_i \subseteq N_j'$$

<u>Ex</u> Maximal chains in $PD(R^3)$:

$$\begin{tikzcd}[column sep=small]
 & \{\langle e_1, e_2\rangle, \langle e_3\rangle\} \arrow[dl, dash] \arrow[dr, dash] & \\
\{\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle\} \arrow[dr, dash] & & \{\langle e_1, e_2\rangle\} \arrow[dl, dash] \\
 & \{\langle e_1\rangle, \langle e_2\rangle\} \arrow[d, dash] & \\
 & \{\langle e_1\rangle\} &
\end{tikzcd}$$

<u>Thm</u> (Brück-Piterman-Welker)

If R is Hermite then there exists a $GL_n(R)$-equivariant homotopy equiv $CB(R^n) \simeq |PD(R^n)|$

A ring is <u>Hermite</u> if every stably free R-module is free.

Thm is a special case of more general statement that also works in some other situations (symplectic groups, free factor complexes)

...

Miller-Patzt-Wilson: Rognes conjecture true for fields.

Hanlon-Hersh-Shareshian : R finite field $\Rightarrow PD(R^n)$ is $(2n-3)$-spherical and describe $GL_n(R) \circlearrowright St^{\infty}(R^n) \otimes \mathbb{C}$

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§2 Definitions

Let $\mathcal{S}$ be a poset.

$x, y \in \mathcal{S}$ write $z = x \vee y$ if z is minimal among elements $\geq x$ and $\geq y$, the <u>join</u>.

An <u>antichain</u> is a subset $\mathcal{E} \subset \mathcal{S}$ s.t. all elements of $\mathcal{E}$ incomparable.

R comm ring, $n \in \mathbb{N}$ (☆)

$\mathcal{S}$ poset of nonzero proper summands of R^n

If $N, N' \in \mathcal{S}$ and $N + N' \in \mathcal{S}$ then $N + N' = N \vee N'$.

A <u>frame</u> is $\emptyset \neq \mathcal{J} \subseteq \mathcal{S}$ is an antichain satisfying certain conditions (?!)

$\mathcal{F}$ fixed set of frames in $\mathcal{S}$

<u>Def</u> $CB(\mathcal{S}, \mathcal{F})$ and $PD(\mathcal{S}, \mathcal{F})$.

For $\mathcal{J} \in \mathcal{F}$ let $\Sigma(\mathcal{J}) \subseteq \mathcal{S}$ be the subposet of joins of subsets of $\mathcal{J}$.

$\mathcal{S} \supseteq \sigma$ is <u>basis-compatible</u> if $\exists \mathcal{J} \in \mathcal{F}$ s.t. $\forall y \in \sigma, y \in \Sigma(\mathcal{J})$ (*)

$CB(\mathcal{S}, \mathcal{F})$ poset of all basis-compatible subsets under containment

σ is a partial decomposition if $(*)$ holds and $\forall x \in \mathcal{T}$ there is at most one $y \in \sigma$ st $x \leq y$.

$PD(\mathcal{S}, \mathcal{F})$ partial decompositions up to refinement (not containment)

For $\mathcal{S}$ as in $\bigstar$, $\mathcal{F}$ as in $(\bigstar\bigstar)$

$$CB(\mathbb{R}^n) = CB(\mathcal{S}, \mathcal{F})$$
$$PD(\mathbb{R}^n) = PD(\mathcal{S}, \mathcal{F})$$

More precise version of thm:

G group

$\mathcal{S}$ poset with G-action

$\mathcal{F} \neq \emptyset$ G-invariant set of frames with (E.P.) "extension property"

then $CB(\mathcal{S}, \mathcal{F}) \overset{G}{\simeq} PD(\mathcal{S}, \mathcal{F})$

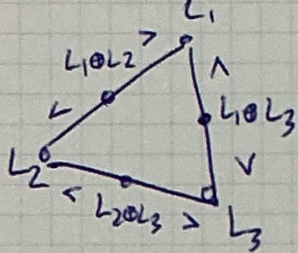
Def $\mathcal{F}$ has E.P. if $\forall \sigma, \sigma' \in PD(\mathcal{S}, \mathcal{F})$ s.t. $\sigma \leq \sigma'$, $\exists\, \mathcal{T} \in \mathcal{F}$ s.t. $(*)$ holds for both σ and σ'.

In situation of $(\bigstar\bigstar)$, R is Hermite $\Leftrightarrow$ E.P. is satisfied

$$\mathcal{F} = \{ \mathcal{J} \mid \mathcal{J} = \{N_1, \dots, N_n\},\ \operatorname{rk} N_i = 1,\ N_1 \oplus \cdots \oplus N_n = R^n \}$$

$n = 3$, $\mathcal{J} = \{L_1, L_2, L_3\}$ (★★)

$\Sigma(\mathcal{J}) =$



Other examples: $\mathcal{S}$ = isotropics in symplectic v.sp.

$\mathcal{F}$ = symplectic bases

$\mathcal{S}$ = free factors in F_n

$\mathcal{F}$ = decompositions of F_n into rank 1 free factors.