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Def A Shimura datum is a pair (G, D)

- G semisimple alg. gp / \mathbb{Q}
- D locally symmetric Hermitian domain

s.t. Setting $G = G(\mathbb{R})$, $D \cong G/K$ K max. cpt
and $G \rightarrow \text{Aut}(D)$.

$\Gamma \subset G(\mathbb{Q})$ arithmetic, $X = \Gamma \backslash D$ is a Shimura variety
~~XXXX~~

It is indeed a variety (Baily-Borel) defined over a no field
(Shimura, Deligne)

Ex (A_g) $G = \text{Sp}_{2g}$ $D = \mathbb{H}_g$ Siegel's upper half space
 $= \{ X \in \text{Mat}_{g \times g}(\mathbb{C}) \mid X = X^t, \text{Im } X > 0 \}$

$$\Gamma = \text{Sp}_{2g}(\mathbb{Z})$$

A_g is the moduli space of principally polarized abelian varieties.

Ex $A_g[m]$ G, D as above, Γ principal level m subgroup.
 $\text{Sp}_{2g}(\mathbb{Z})[m] = \ker(\text{Sp}_{2g}(\mathbb{Z}) \rightarrow \text{Sp}_{2g}(\mathbb{Z}/m))$

moduli of principally polarized
abelian varieties with level m structure

(= a trivialization of the m -torsion subgroup)

Ex $A_{p,q}[E]$ $E = \mathbb{Q}(\sqrt{-d})$ imag. quadratic field

$$G = \text{SU}_{p,q}(E)$$

$$\text{w.r.t. form } J = \begin{pmatrix} & \text{Id}_q \\ -\text{Id}_p & \end{pmatrix}$$

$$G = G(\mathbb{R}) = \{ X \in \text{SL}_{p+q}(\mathbb{C}) \mid X \text{ preserves } J \}$$

$$D = \left\{ \begin{pmatrix} W \\ Z \end{pmatrix} \in \text{Mat}_q(\mathbb{C}) \times \text{Mat}_{p-q, q}(\mathbb{C}) \mid \text{im. part pos det} \right\}$$

$$\Gamma = \text{SU}_{p, q}(\mathcal{O}_E)$$

D is contractible, and Γ acts on D virtually freely.
 $\Rightarrow H^*(X; \mathbb{Q}) \cong H^*(\Gamma; \mathbb{Q})$.

X being algebraic gives its cohomology extra structure.

Deligne: rational coh. has a MHS. In particular a weight filtration.

$\forall i$, \forall variety X :

$$0 \subseteq W_0 H^i(X; \mathbb{Q}) \subseteq W_1 H^i(X; \mathbb{Q}) \subseteq \dots \subseteq W_2 H^i(X; \mathbb{Q}) = H^i(X; \mathbb{Q})$$

$$X \text{ smooth} \Rightarrow \text{Gr}_j^W H^i(X; \mathbb{Q}) \cong_{\mathbb{P}D} \text{Gr}_{2d-j}^W H_c^{2d-i}(X; \mathbb{Q})^\vee \quad d = \dim_{\mathbb{C}} X$$

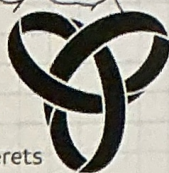
$$\text{in particular } \text{Gr}_{2d}^W H^i(X; \mathbb{Q}) \cong \text{Gr}_0^W H_c^{2d-i}(X; \mathbb{Q})^\vee$$

\uparrow
 "top weight"

What is tropicalization?

If $X = \Gamma \backslash D$ Shimura variety then we hope that X^{top}

- is a topological space
- should realize the top-wt coh of X
- should be stratified / filtered in a meaningful way,
- has a cellular structure depending on some extra data Σ
- points have combinatorial meaning



cone of pos det matrices

Example $A_2^{\text{trop}} = \text{PD}_2 / \text{GL}_2(\mathbb{Z})$

All above desiderata are satisfied by Ash-Mumford-Rapoport-Tai theory of toroidal compactifications.

Assume G of type $A, C, D(2)$ $\left(\begin{array}{l} A \text{ unitary grp} \\ C \text{ sympl. group} \end{array} \right)$

$$\Rightarrow G = \{ X \in \text{SL}(V) \mid X \text{ preserves } \mathcal{J} \}$$

for a vector space V & Hermitian form \mathcal{J}

Def $\mathcal{W}(G, D, \Gamma)$ category s.t.

objects: $W \subseteq V$ isotropic

morphisms: $W_1 \xrightarrow{\gamma} W_2$ where $\gamma \in \Gamma$ is s.t. $\gamma W_1 \subseteq W_2$

Def The tropicalization of $X = T \setminus D$ is the geom. realization of a functor

$$\Sigma: \mathcal{W}(G, D, \Gamma) \longrightarrow \text{RPF}$$

category of
rational polyhedral
fans

which satisfies some compatibility conditions.

(this re-packages the data that goes into choosing a toroidal compactification of X .)

Thm (BBCMMW, ABBCV)

- X^{trop} indep. of Σ up to homeo
- $W_0 H_c^*(X; \mathbb{Q}) \cong H_c^*(X^{\text{trop}}; \mathbb{Q})$
- there is a filtration of X^{trop}
from dim. filtration of \mathcal{W}

$$\hookrightarrow E_{p,q}' = \bigoplus_{\substack{W \in \mathcal{W}/\sim \\ \dim W = p}} H_{p+q}^{\text{BM}} \left(\frac{c(W)}{\Gamma_W}; \mathbb{Q} \right)$$

converging to $\text{Gr}_{2d}^W H^{2d-p-q}(X; \mathbb{Q})$