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Def A Shimura datum is a pair $(\mathbb{G}, D)$

- $\mathbb{G}$ semisimple alg. gp $/\mathbb{Q}$
- D locally symmetric Hermitian domain

s.t. Setting $G = \mathbb{G}(\mathbb{R})$, $D \cong G/K$ K max. cpt and $G \twoheadrightarrow \mathrm{Aut}(D)$.

$\Gamma \subset \mathbb{G}(\mathbb{Q})$ arithmetic, $X = \Gamma \backslash D$ is a Shimura variety

It is indeed a variety (Baily-Borel) defined over a nf field (Shimura, Deligne)

Ex $\mathcal{A}_g$ $\quad \mathbb{G} = Sp_{2g}$ $\quad D = \mathcal{H}_g$ Siegel's upper half space

$$= \{ X \in \mathrm{Mat}_{g\times g}(\mathbb{C}) \mid X = X^t,\ \mathrm{Im}\, X > 0 \}$$

$\Gamma = Sp_{2g}(\mathbb{Z})$

$\mathcal{A}_g$ is the moduli space of principally polarized abelian varieties.

Ex $\mathcal{A}_g[m]$ $\quad \mathbb{G}, D$ as above, Γ principal level m subgroup.

$$Sp_{2g}(\mathbb{Z})[m] = \mathrm{Ker}(Sp_{2g}(\mathbb{Z}) \to Sp_{2g}(\mathbb{Z}/m))$$

moduli of principally polarized abelian varieties with level m structure (= a trivialization of the m-torsion subgroup)

Ex $\mathcal{A}_{p,q}[E]$ $\quad E = \mathbb{Q}(\sqrt{-d})$ imag. quadratic field

$\mathbb{G} = SU_{p,q}(E)$

w.r.t. form $J = \begin{pmatrix} & & \mathrm{Id}_q \\ & S & \\ -\mathrm{Id}_q & & \end{pmatrix}$

$$G = \mathbb{G}(\mathbb{R}) = \{ X \in SL_{p+q}(\mathbb{C}) \mid X \text{ preserves } J \}$$

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$$D = \left\{ \begin{pmatrix} W \\ Z \end{pmatrix} \in \mathrm{Mat}_q(\mathbb{C}) \times \mathrm{Mat}_{p-q,q}(\mathbb{C}) \;\middle|\; \text{im. part pos def} \right\}$$

$$\Gamma = SU_{p,q}(\mathcal{O}_E)$$

D is contractible, and Γ acts on D virtually freely.

$\Rightarrow H^*(X;\mathbb{Q}) \cong H^*(\Gamma;\mathbb{Q})$.

X being algebraic gives its cohomology extra structure.

Deligne: rational coh. has a MHS. In particular a weight filtration.

$\forall i$, $\forall$ variety X:

$$0 \subseteq W_0 H^i(X;\mathbb{Q}) \subseteq W_1 H^i(X;\mathbb{Q}) \subseteq \cdots \subseteq W_{2i} H^i(X;\mathbb{Q}) = H^i(X;\mathbb{Q})$$

X smooth $\Rightarrow$ $\mathrm{Gr}^W_j H^i(X;\mathbb{Q}) \underset{PD}{\cong} \mathrm{Gr}^W_{2d-j} H^{2d-i}_c(X;\mathbb{Q})^\vee$ $\quad d = \dim_{\mathbb{C}} X$

in particular $\underset{\text{"top weight"}}{\mathrm{Gr}^W_{2d}} H^i(X;\mathbb{Q}) \cong \mathrm{Gr}^W_0 H^{2d-i}_c(X;\mathbb{Q})^\vee$

What is tropicalization?

If $X = \Gamma \backslash D$ Shimura variety then we hope that X^{trop}

- is a topological space
- should realize the top-wt coh of X
- should be stratified / filtered in a meaningful way,
- has a cellular structure depending on some extra data Σ
- points have combinatorial meaning

cone of pos def matrices

Example $A_2^{trop} = PD_2 / GL_2(\mathbb{Z})$

All above desiderata are satisfied by Ash-Mumford-Rapoport-Tai theory of toroidal compactifications.

Assume G of type $A, C, D(2)$ $\left(\begin{array}{ll} A & \text{unitary grp} \\ C & \text{sympl. group} \end{array} \right)$

$\Rightarrow \mathbb{G} = \{ X \in SL(V) \mid X \text{ preserves } J \}$

for a vector space V & Hermitian form J

Def $\mathcal{W}(\mathbb{G}, D, \Gamma)$ category s.t.

objects : $W \subseteq V$ isotropic

morphisms : $W_1 \xrightarrow{\gamma} W_2$ where $\gamma \in \Gamma$ is s.t. $\gamma W_1 \subseteq W_2$

Def The tropicalization of $X = \Gamma \backslash D$ is the geom. realization of a functor

$$\Sigma : \mathcal{W}(\mathbb{G}, D, \Gamma) \longrightarrow RPF$$

category of rational polyhedral fans

which satisfies some compatibility conditions.

(this re-packages the data that goes into choosing a toroidal compactification of X.)

Thm (BBCMMW, ABBCV)

- X^{trop} indep. of Σ up to homeo
- $W_0 H^*_c(X;\mathbb{Q}) \cong H^*_c(X^{trop};\mathbb{Q})$
- there is a filtration of X^{trop} from dim. filtration of $\mathcal{W}$

$$\rightsquigarrow E^1_{p,q} = \bigoplus_{\substack{W \in \mathcal{W}/\Gamma \\ \dim W = p}} H^{BM}_{p+q}\left(C(W)/\Gamma_W ; \mathbb{Q}\right)$$

converging to $Gr^W_{2d} H^{2d-p-q}(X;\mathbb{Q})$