Goncharov's conjecture and higher Chow group

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Conventions

1. All fields have characteristic zero.
2. Everywhere we work over $\mathbb{Q}$. So any abelian group is supposed to be tensored by $\mathbb{Q}$. For example, when we write $F^\times$ this actually means $F^\times \otimes_{\mathbb{Z}} \mathbb{Q}$. All exterior powers and tensor products are over $\mathbb{Q}$.

Second K-theory

Denote by $F^\times$ the multiplicative group of the field F.
For an abelian group A, denote by $\Lambda^2 A$ the quotient of $A \otimes A$ by the group generated by the elements of the form $a \otimes b + b \otimes a$.

Theorem (Matsumoto)

$$K_2(F) = \Lambda^2 F^\times / \langle a \wedge (1-a), a \neq 0, 1 \rangle .$$

Bloch group

Let F be a field. Denote by $\mathbb{Q}[F\backslash\{0,1\}]$ a free vector space with basis $[a], a \in F\backslash\{0,1\}$.
The group $B_2(F)$ is the quotient of $\mathbb{Q}[F\backslash\{0,1\}]$ by the subgroup generated by the following elements:

$$\sum_{i=1}^{5}(-1)^i[c.r.(x_1,\dots,\widehat{x_i},\dots,x_5)].$$

In this formula $x_1,\dots,x_5$ are five different points on $\mathbb{P}^1(F)$.

$$c.r.(a,b,c,d)=\frac{(a-c)(b-d)}{(a-d)(b-c)}.$$

Polylogarithmic complex in weight 2

Consider the following complex placed in degrees 1 and 2:

$$\Gamma(F,2): \qquad B_2(F) \to \Lambda^2 F^\times.$$

$$d([a]) = a \wedge (1-a).$$

Theorem (A.Suslin)

We have $K_3^{(2)} \cong H^1(\Gamma(F,2))$.

Milnor K-theory

By definition,

$$K_n^M(F) = \Lambda^n F^\times / \langle a \wedge (1-a) \wedge a_3 \cdots \wedge a_n \rangle .$$

Theorem (A. Suslin, Yu. Nesterenko, B. Totaro)

For any field and $m \geq 1$ we have

$$K_m^{(m)}(F) \cong K_n^M(F).$$

Truncated polylogarithmic complexes

Let F be a field and $m \geq 2$.
Denote by $\mathbb{Q}[F^\times]$ free vector space with generators $[a], a \in F^\times$.
Denote by $\Gamma_{\geq m-1}(F, m)$ the following complex placed in degrees $m-2, m-1, m$:

$$0 \to \mathbb{Q}[F^\times] \otimes \Lambda^{m-3} F^\times \to B_2(F) \otimes \Lambda^{m-2} F^\times \to \Lambda^m F^\times \to 0.$$

$$d([a] \otimes b) = \{a\}_2 \otimes a \wedge b.$$

$$d(\{a\}_2 \otimes b) = a \wedge (1-a) \wedge b.$$

By definition, $H^m(\Gamma_{\geq m-1}(F, m)) = K_m^M(F)$.

Main result

Recall that $\Gamma_{\geq m-1}(F, m)$ has the following form

$$0 \to \mathbb{Q}[F^\times] \otimes \Lambda^{m-3} F^\times \to B_2(F) \otimes \Lambda^{m-2} F^\times \to \Lambda^m F^\times \to 0$$

Theorem

Let F be a field of characteristic zero and $m \geq 2$. We have the canonical isomorphism

$$K^{(m)}_{m+1}(F) \cong H^{m-1}(\Gamma_{\geq m-1}(F, m))$$

.

Our approach: present K-theory as Bloch's higher Chow groups.

Reduction to algebraically closed field

- K-theory (rationally) satisfied Galois descent:

$$K_{m+1}^{(m)}(\overline{F})^{\mathrm{Gal}(\overline{F}/F)} = K_{m+1}^{(m)}(F).$$

- The group

$$H^{m-1}(\Gamma_{\geq m-1}(F,m)).$$

  satisfies Galois descent by result of D. Rudenko.
- We can assume that F is algebraically closed.
- Rudenko's result uses Suslin's theorem. For algebraically closed field the proof is independent.

Higher Chow groups

- We will consider algebraic cyles in $\mathbb{A}^n$.
  An algebraic cycle of dimension r is a formal linear combination of irreducible subvarities of dimension r with rational coefficients.
- *A face* is a subvariety of $\mathbb{A}^n$ given by some numbers of equations of the form $x_i = 0, 1$.
- Cycle $Z \subset \mathbb{A}^n$ is called *admissible* if it intersects each face in expected dimension.
- Denote by $z_r(n)$ the subgroup of dimension r admissible cycles on $\mathbb{A}^n$.

Higher Chow group

Let $i \in \{1, \dots, n\}$ and $\epsilon \in \{0, 1\}$. Denote by

$$\partial_{i,\epsilon} \colon z_r(n) \to z_{r-1}(n-1)$$

the intersection with the face given by the equation $x_i = \epsilon$.
Consider the following cochain complex:

$$\mathcal{CH}(F, m) \colon \qquad \dots \to z_r(r+m) \to z_{r-1}(r+m-1) \to \dots$$

$z_r(n)$ sits in degree $m - r$.
Differential is

$$d = \sum_{i=1}^{n} (-1)^i (\partial_{i,0} - \partial_{i,1})$$

Bloch's result

Theorem (Bloch)

We have the canonical isomorphism

$$K_n^{(m)}(F) \cong H^{2m-n}(\mathcal{CH}(F, m)).$$

Case $m = 2$

Starting from here we assume that $m = 2$ and F is algebraically closed field. We have:

$$\mathcal{CH}(F,2): \quad \mathcal{CH}(F,2)_0 \to \mathcal{CH}(F,2)_1 \to \mathcal{CH}(F,2)_2 \to 0$$

- $\mathcal{CH}(F,2)_0$ - surfaces in $\mathbb{A}^4$
- $\mathcal{CH}(F,2)_1$ - curves in $\mathbb{A}^3$
- $\mathcal{CH}(F,2)_2$ - points in $\mathbb{A}^2$.

Main result reformulation

We will define some subcomplex $M(F,2) \subset \mathcal{CH}(F,2)$ which is acyclic in degrees 1 and 2.

Theorem (Main result, explicit version)

Let F be algebraically closed. We have the following isomorphism of complexes:

$$\begin{array}{ccc} B_2(F) & \xrightarrow{\quad d \quad} & \Lambda^2 F^\times \\ \downarrow\cong & & \downarrow\cong \\ \mathcal{CH}(F,2)_1/(M(F,2)_1 + \mathrm{Im}\, d) & \xrightarrow{d} & \mathcal{CH}(F,2)_2/M(F,2)_2 \end{array}$$

Parametric cycles

Let Y be a variety and $f_1, \ldots, f_n$ are rational functions on Y. Consider rational map

$$\psi \colon Y \to \mathbb{A}^n$$

$$\psi(y) \mapsto \left(\frac{f_1(y)}{f_1(y)-1}, \ldots, \frac{f_n(y)}{f_n(y)-1}\right).$$

Let $Z \subset \mathbb{A}^n$ be the image of ψ. Let $\widetilde{\psi} \colon Y \to Z$ be the natural map. We set

$$[Y, f_1, \ldots, f_n] = \begin{cases} 0 & \text{if } \dim Z < \dim X \\ (\deg \psi)[\overline{Z}] & \text{if } \dim Z = \dim X \end{cases}$$

The complex $\mathcal{M}(F, 2)$

Define subcomplex $\mathcal{M}(F, 2) \subset \mathcal{CH}(F, 2)$ placed in degrees 1 and 2 as follows. It is generated by the following elements:

1. In degree 2
$$(a, b) + (b, a).$$
$$(a_1 a_2, b) - (a_1, b) - (a_2, b).$$
2. In degree 1
$$[X, f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)}] - sgn(\sigma)[X, f_1, f_2, f_3].$$
$$[X, f_1 f_2, g_2, g_3] - [X, f_1, g_2, g_3] - [X, f_2, g_2, g_3]$$
3. In degree 0 we set
$$M(F, 2)_0 = d^{-1}(M(F, 0)_1).$$

Main result, explicit version

Define map

$$\mathcal{T}_2 \colon B_2(F) \to \mathcal{CH}(F,2)_1/M(F,2)_1$$

by the formula

$$[a] \mapsto [\mathbb{P}^1, t, (1-t), 1-a/t].$$

Theorem (Main result, explicit version)

1. *The map $\mathcal{T}_2$ is an isomorphism*
2. *The group $d(\mathcal{CH}(F,2)_0)$ is contained in $M(F,2)_1$.*

Corollary

We have

$$B_2(F) \cong \mathcal{CH}(F,2)_1/(M(F,2)_1 + \mathrm{Im}(d)).$$

The group $\mathcal{C}H(F,2)_1/M(F,2)_1$ explicitely

Generators: Cycles of the form $[X, f_1, f_2, f_3]$ which are generic with respect to faces

Relations:

1. $[X, f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)}] = sgn(\sigma)[X, f_1, f_2, f_3]$.
2. $[X, g_1g_2, f_2, f_3] = [X, g_1, f_2, f_3] + [X, g_2, f_2, f_3]$
3. If $\varphi\colon X \to Y$ is a non-constant map, then $[Y, f_1, f_2, f_3] = (\deg\varphi)^{-1}[X, \varphi^*(f_1), \varphi^*(f_2), \varphi^*(f_3)]$

The group $N_{1,3}$

Drop the condition of generic position!

Define the group $N_{1,3}$ as the the group with following generators and relations:

Generators:

(X, f_1, f_2, f_3), where $f_1, f_2, f_3 \in F(X)^\times$.

Relations:

1. $(X, f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)}) = sgn(\sigma)(X, f_1, f_2, f_3)$.
2. $(X, g_1 g_2, f_2, f_3) = (X, g_1, f_2, f_3) + (X, g_2, f_2, f_3)$
3. If $\varphi \colon X \to Y$ is a non-constant map, then $(Y, f_1, f_2, f_3) = (\deg \varphi)^{-1}(X, \varphi^*(f_1), \varphi^*(f_2), \varphi^*(f_3))$

Theorem

The natural map $\mathcal{CH}(F, 2)_1 / M(F, 2)_1 \to N_{1,3}$ is isomorphism.

$N_{1,3}$ as a colimit

We can present $N_{1,3}$ as the following colimit:

$$N_{1,3} = \operatorname{colim}_X \Lambda^3 F(X)^\times$$

The colimit is taken over category of curves and non constant morphisms. If $\varphi\colon X \to Y$ is a map than the map $\Lambda^3 F(Y)^\times \to \Lambda^3 F(X)^\times$ is defined by the formula $f_1 \wedge f_2 \wedge f_3 \mapsto (\deg \varphi)^{-1} \varphi^*(f_1) \wedge \varphi^*(f_2) \wedge \varphi^*(f_3)$.
The map is given by the formula

$$(X, f_1, f_2, f_3) \mapsto (X, f_1 \wedge f_2 \wedge f_3).$$

Idea of the proof - moving lemma

Proposition

Let X be a curve. The group $\Lambda^3 F(X)^\times$ is geneerated by elements of the form $f_1 \wedge f_2 \wedge f_3$, such that f_i has disjoint divisors.

Example

$$(t-a) \wedge (t-b) = \\ 1/2\left(\frac{t-a}{t-c} \wedge (t-b) + (t-a) \wedge \frac{(t-b)}{t-c} + \frac{t-a}{t-b} \wedge (t-c)\right)$$

This shows that $N_{1,3}$ is generated by elements of the form $(X, f_1 \wedge f_2 \wedge f_3)$ so that (f_i) are dosjoint. But in this case the cycle $[X, f_1, f_2, f_3]$ is admissible. So $\widetilde{\mathcal{C}H}(F, 2)_1 \to N_{1,3}$ is surjective.

What remains to prove

Recall that

$$N_{1,3} = \mathrm{colim}_X \Lambda^3 F(X)^\times.$$

Define a map $\widetilde{\mathcal{T}}_2 \colon B_2(F) \to N_{1,3}$ by the formula

$$[a] \mapsto (\mathbb{P}^1, t \wedge (1-t) \wedge (1-a/t)).$$

It remains to show that $\widetilde{\mathcal{T}}_2$ is isomorphism.
Its surjectivity follows easily from Galois descent for K_3^M.

Injectivity - idea of the proof

Recall that

$$N_{1,3} = \operatorname{colim}_X \Lambda^3 F(X)^\times.$$

And I defined map $\widetilde{\mathcal{T}}_2 \colon B_2(F) \to N_{1,3}$ by the formula

$$[a] \mapsto (\mathbb{P}^1, t \wedge (1-t) \wedge (1-a/t)).$$

To prove that $\widetilde{\mathcal{T}}_2$ is injective we need to construct a left inverse map. This means that for any curve X over F one need to construct a map $\mathcal{H}_X \colon \Lambda^3 F(X)^\times \to B_2(F)$, such that:

1. For any nonconstant map $\varphi \colon X \to Y$ we have $\mathcal{H}_Y(a) = (\deg \varphi^{-1})\mathcal{H}_X(\varphi^*(a))$
2. $\mathcal{H}_{\mathbb{P}^1}(t \wedge (1-t) \wedge (1-a/t)) = [a]$.

Reciprocity map

Theorem

Let F be algebraically closed. To any curve X over F one can assign a map $\mathcal{H}_X\colon \Lambda^3 F(X)^\times \to B_2(F)$ such that:

1. *For any nonconstant map $\varphi\colon X \to Y$ we have* $\mathcal{H}_Y(a) = (\deg \varphi^{-1})\mathcal{H}_X(\varphi^*(a))$
2. $\mathcal{H}_{\mathbb{P}^1}(t \wedge (1-t) \wedge (1-a/t)) = [a]$.

Moreover the family of the maps $\mathcal{H}_X$ uniquely determined by this properties.

Example:

$$\mathcal{H}_{\mathbb{P}^1}((t-a) \wedge (t-b) \wedge (t-c)) = [c.r.(a,b,c,\infty)].$$

Complex $\Lambda(F, m)$

Similarly to $N_{1,3}$, define $N_{d,n}$ by the following formula:

$$N_{d,n} = \mathrm{colim}_{\dim X = d} \Lambda^n F(X)^\times.$$

Consider the following complex:

$$\to N_{d+1,n+1} \to N_{d,n} \to N_{d-1,n-1} \to$$

Differential is given by the formula

$$d((X, a)) = \sum_{D \subset X} (D, \partial_D(a)).$$

Denote this complex by $\Lambda(F, m)$, where $N_{d,n}$ sits in degree $m - d$ and $n - d = m$.

Results

Theorem

For $j \in \{m-1, m\}$ *we have* $H^j(\Lambda(F,m)) \cong H^j(\Gamma(F,m)) \cong K^{(m)}_{2m-j}(F)$.

Conjecture

For any j *we have* $H^j(\Lambda(F,m)) \cong K^{(m)}_{2m-j}(F)$.

Thank you for your attention!