Depth reductions of MPL's

Originally: wt 6, but focus on wt 4 initially, continuing Dupont.

§1: Recap: Multiple polylogs:

$$Li_{n_1,\dots,n_r}(z_1,\dots,z_r) = \sum_{1 \le k_1 < \dots < k_r} \frac{z_1^{k_1} \cdots z_r^{k_r}}{k_1^{n_1} \cdots k_r^{n_r}}$$

with depth $= r$, weight $= n_1 + \dots + n_r$.

Have $MT(F)$, w/ fibre functor $\omega : MT(F) \to \mathrm{Vect}_{\mathbb{Q}}$

Reconstruction: $MT(F) \simeq \mathrm{Rep}_{\mathbb{Q}}(\underbrace{\underline{\mathrm{Aut}}^{\otimes} \omega})$

"motivic Galois gp"

$\simeq \mathbb{G}_m \ltimes U_F$

$\rightsquigarrow$ graded (for $\mathbb{G}_m$); unipotent via wt filt (for U_F)

Hopf algebra: $\mathcal{H}(F) := \mathcal{O}(U_F)$

regular functions

$\mathbb{N}$-graded, connected, ...

Lie coalgebra $\mathcal{C}(F) := \mathcal{H}(F)_{>0} / \mathcal{H}(F)_{>0} \cdot \mathcal{H}(F)_{>0}$

Via matrix coefficients, construct

$$\log^H a \in \mathcal{H}_1(F), \quad Li_n^H(z) \in \mathcal{H}_n(F).$$

⌜ Eg: Extension

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{Q}(0) & \to & K_a & \to & \mathbb{Q}(-1) & \to & 0 \\
 & & \wr\| & & \wr\| & & \wr\| & & \\
 & & H^0(pt) & & H^1(\mathbb{A}^1\setminus\{0\}, \{1,a\}) & & H^1(\mathbb{A}^1\setminus\{0\}) & & \\
 & & \frac{dz}{z} & & & & & &
\end{array}$$

period matrix $\begin{pmatrix} 1 & \log a \\ 0 & 2\pi i \end{pmatrix}$ $\overset{\frown}{0,a}$

matrix coefficients $\begin{pmatrix} 1 & \log^H a \\ 0 & 1 \end{pmatrix}$, where

$\log^H a \in \mathcal{H}_1(F)$ is function

$$(g \in U_F) \longmapsto \langle \phi, g \cdot v \rangle$$

$$\mathbb{Q}(0) \xrightarrow{\phi = id} gr_0^W K_a$$
$$\mathbb{Q}(1) \xrightarrow[v = id]{} gr_2^W K_a$$ ⌟

Also get $Li^H_{n_1,\dots,n_r}(z_1,\dots,z_r) \in \mathcal{H}_{n_1+\dots+n_r}(F)$

Projecting to $\mathcal{C}(F)$, $\log^{\mathcal{C}}$, $Li^{\mathcal{C}}_{n_1,\dots,n_r}$.

Expect (Goncharov) $\mathcal{C}(F) = \mathcal{L}(F)$ generated by polylogs.

§2 Coproduct formula: "Iterated integrals"

$$I^{\mathcal{C}}(a; x_1, \dots, x_n; b) \longleftrightarrow \int_{a<t_1<\dots<t_n<b} \frac{dt_1}{t_1 - x_1} \cdots \frac{dt_n}{t_n - x_n}$$

$$\mathrm{Li}_{n_1, \dots, n_r}(z_1, \dots, z_r)$$

$$(*) \quad = (-1)^r I(0; \underbrace{\frac{1}{z_1 \cdots z_r}, 0 \cdots 0}_{n_1}, \underbrace{\frac{1}{z_2 \cdots z_r}, 0 \cdots 0}_{n_2} \cdots \underbrace{\frac{1}{z_r}, 0 \cdots 0}_{n_r}; 1)$$

Then Goncharov coproduct formula gives

$$\delta I^{\mathcal{C}}(x_0; x_1, \dots, x_n; x_{n+1})$$

$$= \sum_{i<j} I^{\mathcal{C}}(x_0; x_1, \dots, x_i \,\vdots\, x_j \cdots x_n; x_{n+1}) \wedge I^{\mathcal{C}}(x_i; x_{i+1}, \dots, x_{j-1}; x_j)$$

Then $\delta \mathrm{Li}^{\mathcal{C}}_{n_1 \cdots n_r}(z_1, \dots, z_r)$ via $(*)$

Exercise: Check $\delta \mathrm{Li}_n(z) = \mathrm{Li}_{n-1}(z) \wedge \log z$

Recall: $I^{\mathcal{C}}(0; 1) = 0$, $I^{\mathcal{H}}(a; x_1 \dots x_n; a) = 0 \quad n \geq 1$

$$I^{\mathcal{C}}(a; \underbrace{x \cdots x}_{n}; b) = \frac{1}{n!} I^{\mathcal{C}}(a; x; b)^n$$

$$I^{\mathcal{H}}(a;b;c) = \log^{\mathcal{H}}(b-c) - \log^{\mathcal{H}}(b-a)$$
$$\text{w/ } \log^{\mathcal{H}}(0) := 0.$$
[regularisation.]

Note: $\delta \mathrm{Li}_n^{\mathcal{C}}(z) \in C_{n-1}(F) \wedge C_1(F)$

Exercise: $\delta \mathrm{Li}_{3,1}^{\mathcal{C}}(x,y) = \text{wt } 3 \wedge \text{wt } 1$
$$+ \underbrace{\mathrm{Li}_2^{\mathcal{C}}(xy) \wedge \mathrm{Li}_2^{\mathcal{C}}(y)}_{\neq 0 \;\in\; C_2(F) \wedge C_2(F)}$$

Cor: $\mathrm{Li}_{3,1}^{\mathcal{C}}(x,y) \neq \sum \lambda_i \mathrm{Li}_4^{\mathcal{C}}(z_i)$

Notation: $\bar{\delta} := \delta \mod C_1(F)$

Goncharov Depth Conjecture (special case)

$\bar{\delta} f = 0 \iff f$ has depth 1

i.e. $f = \sum \lambda_i \mathrm{Li}_n(z_i)$

§ $\underline{\text{wt } 2 \text{ \& wt } 3}$

Check: $\bar{\delta}\, Li^{C}_{1,1}(x,y) = 0$

$\bar{\delta}\, Li^{C}_{1,1,1}(x,y,z) = 0$

⌜no space for $C_2 \wedge C_2$⌟

These are indeed depth 1

$$Li_{1,1}(x,y) = Li_2\left(\frac{y(x-1)}{1-y}\right) - Li_2\left(\frac{-y}{1-y}\right) - Li_2(xy)$$

⌜Exercise: check this on power series level.⌟

$$Li_{1,1,1}(x,y,z) = \sum 8 \times Li_3$$

args: $\frac{1-xyz}{xy(1-z)}$

⌜Known since at least Lewin (1980's), popularisation via Zagier (wt 2), Goncharov / Zhao⌟

§ The weight 4 story

Fact: All wt 4 MPL's expressible via $Li_{3,1}$ & Li_4 ⌜general reduction to depth $n-2$ known⌟

When does $\bar{\delta} \sum \lambda_i \mathrm{Li}^{\mathcal{C}}_{3,1}(x_i, y_i) = 0$?

$$\bar{\delta}\, \mathrm{Li}^{\mathcal{C}}_{3,1}\left(\frac{x}{y}, y\right) = \mathrm{Li}^{\mathcal{C}}_2(x) \wedge \mathrm{Li}^{\mathcal{C}}_2(y)$$

$$\underset{\widetilde{\mathrm{Li}}_{3,1}(x,y)}{\shortparallel}$$ for simplicity.

Get $\bar{\delta} = 0$ by substituting $x = \mathrm{Li}_2$ (motivic) equations?

Recall: $\begin{cases} \mathrm{Li}^{\mathcal{C}}_2(z) + \mathrm{Li}^{\mathcal{C}}_2(1-z) = 0 \\ \mathrm{Li}^{\mathcal{C}}_2(z) + \mathrm{Li}^{\mathcal{C}}_2(1/z) = 0 \end{cases}$

So expect

$$\begin{cases} \widetilde{\mathrm{Li}}_{3,1}(z,y) + \widetilde{\mathrm{Li}}_{3,1}(1-z,y) \overset{?}{=} \text{depth } 1 \\ \widetilde{\mathrm{Li}}_{3,1}(z,y) + \widetilde{\mathrm{Li}}_{3,1}(1/z,y) \overset{?}{=} \text{depth } 1 \end{cases}$$

Thm (Zagier, Gangl) Both are true.

Idea: Can by hand find explicit expressions, involving things like

$$\mathrm{Li}_4\left(\frac{x(1-y)}{y(1-x)}\right) + \ldots$$

More interesting Li_2 (five-term)

Thm (Gangl)

$$\sum_{i=0}^{5} (-1)^i \, \widetilde{Li}_{3,1}\big(r(z_1,\dots,\hat{z_i},\dots,z_5),\, y\big)$$

$$= \sum_{j=1}^{122(+\varepsilon)} \lambda_j \, Li_4^{c}\big(\underbrace{f_j(z_1,\dots,z_5,y)}_{\text{products of } \frac{4\ r\text{'s}}{4\ r\text{'s}}}\big)$$

$$r(abcd) = \frac{a-c}{a-d} \Big/ \frac{b-c}{b-d}$$

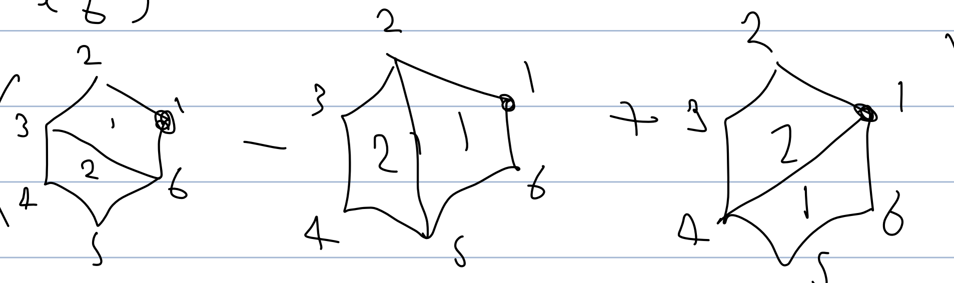
cross-ratio.

[Idea for conceptual proof:

Goncharov-Rudenko / Rudenko-Matveiakin give nice "cluster" / "quadrangular" polylogarithms w/ nice properties

[singularities at $x_i = x_j$ only, good structure, symmetries, nice functional eqn]

$QLi_4(x_1,\dots,x_6)$

$= \widetilde{Li}_{3,1}\Big($  $\Big)$

$\mathbb{I}$

$$\widetilde{Li}_{3,1}([1236],[3456])$$
$$-\widetilde{Li}_{3,1}([1256],[3452])$$
$$+Li_{3,1}([1456],[1234])$$

$$[abcd] = \frac{(a-b)(c-d)}{(b-c)(d-a)}$$

$$\longleftrightarrow \quad 1 - r(abcd)$$

Then

$$\sum (-1)^i \, Q Li_4(x_1,\dots,\hat{x_i},\dots,x_7)$$
$$= \text{depth } 1 \quad (\text{explicit})$$

By specialising / degenerating in judicious ways, obtain previous results.

- Show $\widehat{Li}_{3,1}(xy) + \widehat{Li}_{3,1}(1-x, 1-y) = \text{dp}$
- Find relation between $(xy)(x\ 1-y)$ & $(x'y')(x'\ \frac{1}{y'})$
- disentangle using some projective involution of special configuration.

- Show $\widetilde{\mathrm{Li}_{3,1}}(\mathrm{five}(z_1, \ldots, z_5), y)$ satisfies 'exotic' symmetry not just from $x \mapsto 1-x, \frac{1}{x}$

$\hookrightarrow$ this forces representation of S_6 to be trivial

$\Longrightarrow$ Goncharov's 122-term relation. $\lrcorner$

$\leadsto$ in particular SES

$$0 \to B_4(F) \to L_4(F) \xrightarrow{\delta} L_2(F) \wedge L_2(F) \to 0$$

So Relation is key step in showing quasi-iso

$$\begin{tikzcd}
B_4(F) \arrow[r] \arrow[d] & B_3(F) \otimes F^\times_{\mathbb{Q}} \arrow[r] \arrow[d] & B_2(F) \otimes \Lambda^2 F^\times_{\mathbb{Q}} \arrow[r] \arrow[d] & \Lambda^4 F^\times_{\mathbb{Q}} \arrow[d] \\
L_4(F) \arrow[r] & L_3(F) \otimes L_1(F) \oplus \Lambda^2 L_2(F) \arrow[r] & L_2(F) \otimes \Lambda^2 L_1(F) \arrow[r] & \Lambda^4 L_1(F)
\end{tikzcd}$$

Clearly injective; kernel identified w/ $\Lambda^2 L_2(F) \xrightarrow{=} \Lambda^2 B_2(F)$ qi to 0.

§ The story in wt 6

One can define "iterated" coboundary
(not just $\delta \circ \delta = 0$ in (E complex))

$$\delta^{[2]} = (\Delta \otimes \mathrm{id}) \circ \Delta \quad (\text{mod } pr)$$

Set $\bar{\delta}^{[2]} = \bar{\delta}$ mod $C_1(F)$

Then check:

$$\bar{\delta}^{[2]} \mathrm{Li}^{C}_{a,b}(x,y) = 0$$

$$\bar{\delta}^{[2]} \mathrm{Li}^{C}_{411}(x,y,z) = \mathrm{Li}_2(xyz) \wedge \mathrm{Li}_2(y) \wedge \mathrm{Li}_2(z).$$

Cor: $\mathrm{Li}^{C}_{411}(xyz) \neq \sum$ depth 2.

Write $\widetilde{\mathrm{Li}}_{411}(xyz) = \mathrm{Li}^{C}_{411}\left(\frac{x}{yz}, y, z\right)$

Then we expect:

$$\begin{cases} \widetilde{\mathrm{Li}}_{411}(xyz) + \widehat{\mathrm{Li}}_{411}(1-x\,yz) = \mathrm{dp}\,2 \\ \widetilde{\mathrm{Li}}_{411}(xyz) + \widehat{\mathrm{Li}}_{411}\left(\frac{1}{x}\,yz\right) = \mathrm{dp}\,2 \end{cases}$$

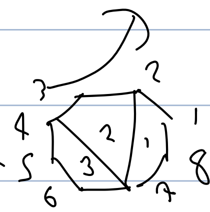
and

$$\sum (-1)^i \widetilde{Li}_{4,1,1}\left(r(w_1,\dots,\hat{w}_i,\dots,w_5), y, z\right) = dp\,2 .$$

Thm (MR)

The 5-term for $\widetilde{Li}_{4,1,1}$ holds modulo the dilog symmetries.

⌜ Rep theory / group sing calculation by degeneracy and using S_9-action on $\sum (-1)^i \, QLi_6(\underbrace{x_1,\dots,\hat{x}_i,\dots,x_9}_{8 \text{ pt function}})$

$\sum$ 15 terms like $\widetilde{Li}_{4,1,1}\Big($  $\Big)$ 8 pt function. ⌟

Thm (C)

The symmetries hold unconditionally.

⌜ — proof more intricate than wt 4,

— needs several layers of reductions

— Eg $\widetilde{Li}_{4,1,1}(1,1,z) = dp\,2$

$\iff \overset{<}{Li}_2(1) = 0$

— Then $\widetilde{Li}_{4,1,1}(1\ x\ y) = dp\ 2$.
$\rightarrow$ 2 symmetries + inverse $\rightsquigarrow$ trivialises $\widetilde{Li}_{4,1,1}(1\ x\ y)$

— Finally find synchronized symmetries

$$\widetilde{Li}_{4,1,1}(x\ y\ z) + \widetilde{Li}_{4,1,1}\left(1-x\ \frac{1}{1-y}\ 1-z\right) = dp2$$

— disentangle via more complicated combinations of identities ...

§ Final results

Charlton's depth conjecture:

$\ker \delta^{[k]}$ = exactly $dp \leq k$ MPL
"depth filtration" = "motivic coproduct filtration"

Proof sol, wt 4, wt 6 rely on some degenerations / specialisations of particular identities
$\rightsquigarrow$ complicated structure, some hint of general inductive patterns (investigate)

However, new source of general geometric identities by Kapranov-Rudenko-Sierra

$H_1(PGL_n, St_n^{\mathcal{L}}) \rightarrow$ { weight n MPL's with F-values mod products }
$=: \mathcal{L}_n(F)$