Orthogonal ring patterns, discrete surfaces and integrable systems

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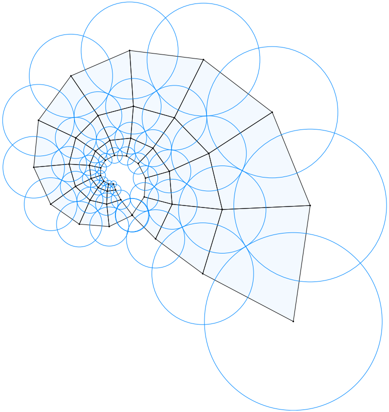
Technische Universität Berlin

Integrability in Condensed Matter Physics and Quantum Field Theory, Les Diablerets, Feb 3-12, 2023

based on joint works with
T. Hoffmann, T. Rörig, N. Smeenk, N. Schmitt, S. Heller

SFB TRR 109 Discretization in Geometry and Dynamics

Orthogonal circle patterns



- Circle patterns (with the combinatorics of the square grid) [Schramm '97]
- Convergence to conformal maps
- Circle patterns as discrete complex analysis
- Discrete Riemann mapping theorem [Thurston]
- Integrable equations

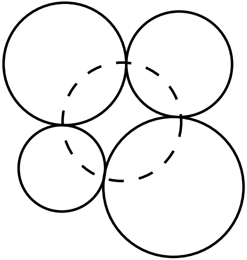
Orthogonal circle patterns in a plane and discrete Hirota equation

unknowns:

radii r

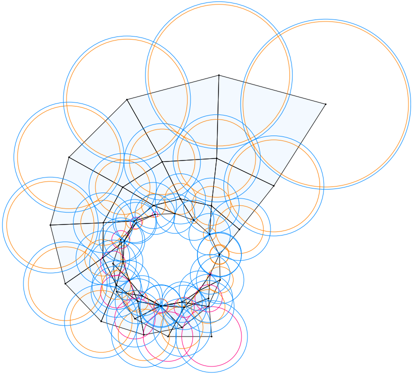
closure condition:

(integrable) Hirota equation



$$\frac{(r - ir_1)(r - ir_2)(r - ir_3)(r - ir_4)}{(r + ir_1)(r + ir_2)(r + ir_3)(r + ir_4)} = 1$$

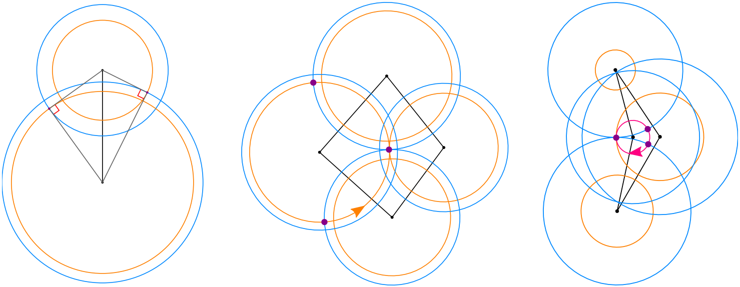
Orthogonal ring patterns



Outline

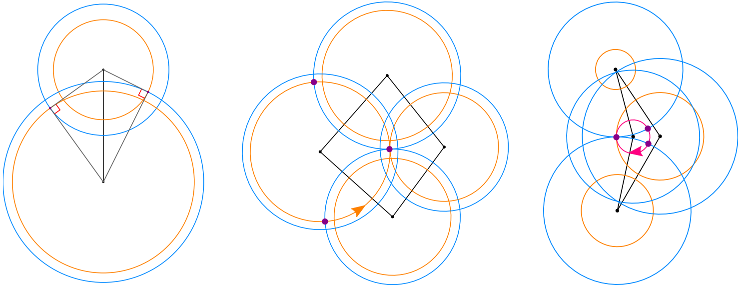
- A generalization of orthogonal circle patterns (away from conformal limit)
- Orthogonal ring patterns in a sphere and hyperbolic space
- Relation to discrete minimal and cmc surfaces
- Integrable discretizations of $\Delta u \pm \sinh u = 0$

Orthogonally intersecting rings



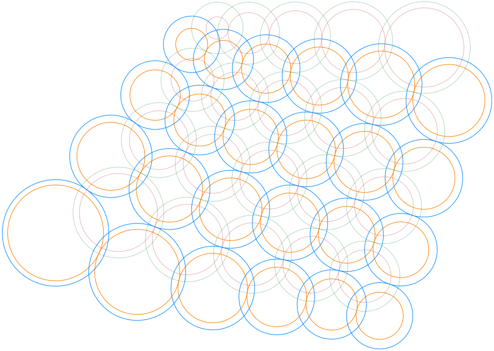
- A ring is a pair of concentric circles, inner and outer circle radii: r and R
- Combinatorics $\mathbb{Z}^2$, inner and outer circles $c_{m,n}$ and $C_{m,n}$
- Orthogonality of rings at neighboring vertices: the outer circle C_i intersects the inner circle c_j orthogonally
- $c_{m,n}, c_{m+1,n+1}$ and $C_{m+1,n}, C_{m,n+1}$ pass through one point

Orthogonally intersecting rings



- $R_i^2 + r_j^2 = R_j^2 + r_i^2 \Rightarrow R_i^2 - r_i^2 = R_j^2 - r_j^2$
- ρ-*radii*: $R_i = \cosh(\rho_i), r_i = \sinh(\rho_i)$
- Orthogonal rings have the same area

Two families of touching rings



The rings of an orthogonal ring pattern partition into two diagonal families of touching rings.

Equation for ρ_i

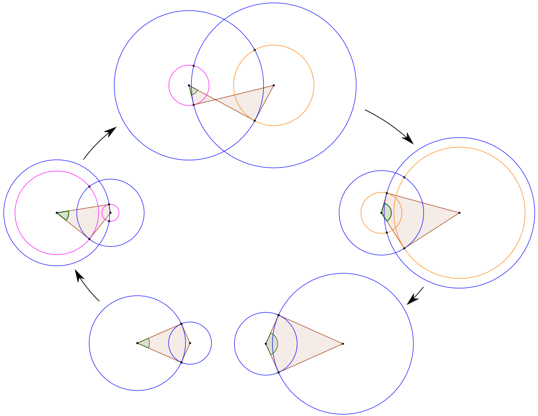
Theorem [B., Hoffmann, Rörig '19]
Orthogonal ring patterns correspond to solutions of

$$2\pi = \sum_{j:(ij)\in E} 2\arctan(e^{\rho_i - \rho_j}).$$

for ρ-radii.

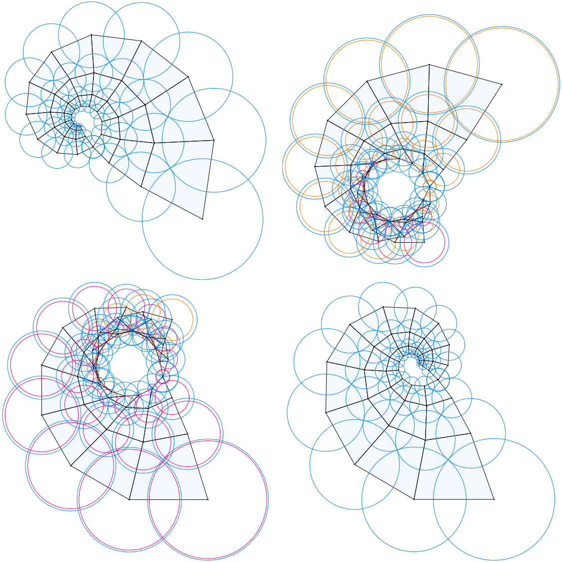
- Equation for logarithmic radii $\rho = \log R$ of orthogonal circle patterns
- One parameter family of orthogonal ring patterns $\mathcal{R}^\delta$ with radii $r_i^\delta = \sinh(\rho_i + \delta), R_i^\delta = \cosh(\rho_i + \delta)$
- Equivalent to integrable Hirota equation

Relation to orthogonal circle patterns



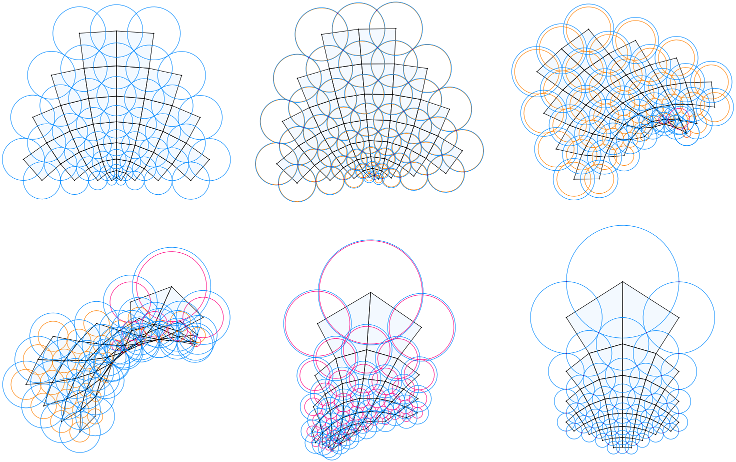
δ-family of a ring pattern.
For $\delta \to +\infty$ one obtains an orthogonal circle pattern $\mathcal{C}$ with logarithmic radii ρ_i,
$\delta \to -\infty$ gives the dual circle pattern $\mathcal{C}^*$ with log radii $-\rho_i$.

Doyle spirals deformation



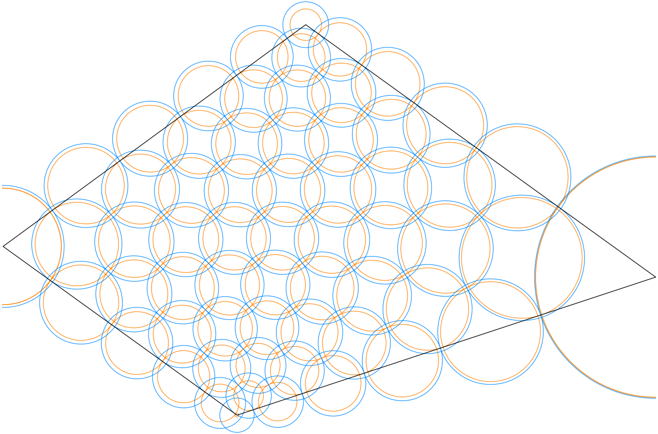
$\rho_{m,n} = mx - ny,\ x + iy \in \mathbb{C}$, Schramm ['97]

z^a ring patterns



Ring patterns between z^2 and $\log z$ circle patterns; $\rho_{m,n}$ solves a discrete Painlevé equation, B. ['99], Agafonov, B. ['00], B., Its ['16]

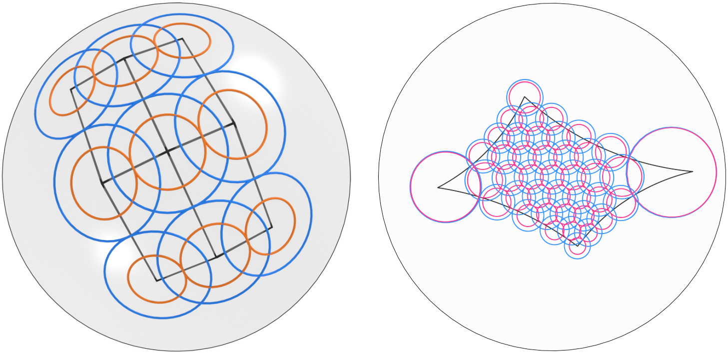
Variational principle



[B., Hoffmann, Rörig ’19]

- Neumann boundary conditions: angles Φ_i on the boundary

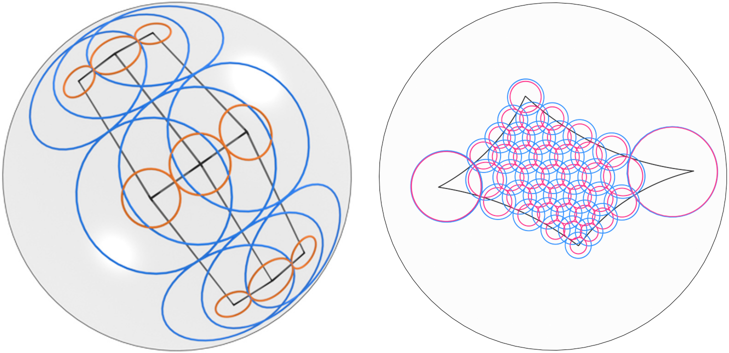
Orthogonal ring patterns in a sphere and hyperbolic plane



[B., Hoffmann, Smeenk '23]

- not related by a stereographic projection
- equations essentially different from the euclidean case

Orthogonal ring patterns in a sphere and hyperbolic plane



[B., Hoffmann, Smeenk '23]

- Relation to integrable systems
- Relation to discrete (s-isothermic) cmc surfaces

Spherical and hyperbolic orthogonal ring patterns

Concentric rings with spherical (hyperbolic) radii R_i, r_i.
Spherical (hyperbolic) Pythagoras' Theorem

$$\begin{aligned} \text{spherical:} \quad & \cos(R_j)\cos(r_k) = \cos(r_j)\cos(R_k), \\ \text{hyperbolic:} \quad & \cosh(R_j)\cosh(r_k) = \cosh(r_j)\cosh(R_k) \end{aligned}$$

implies for all rings, $q < 1$:

$$\begin{aligned} \text{spherical:} \quad & q\cos(r) = \cos(R), \\ \text{hyperbolic:} \quad & \cosh(r) = q\cosh(R). \end{aligned}$$

- Parametrization in elliptic functions with modulus $q \leq 1$.

Spherical orthogonal ring patterns. Parametrization in elliptic functions

Concentric rings with spherical radii R_j, r_j.

$$q\cos(r) = \cos(R),$$
$$\cos r = -\operatorname{sn}(\rho, q),\ \sin r = \operatorname{cn}(\rho, q),\ \sin R = \operatorname{dn}(\rho, q).$$

- one to one correspondence $(R, r) \Leftrightarrow \rho \in [-2K, 2K]$

Equations

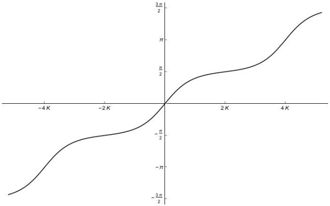
Theorem

$\rho \in (-2K, 0]$ are radii of orthogonal spherical ring pattern with $R < \frac{\pi}{2}$ if and only if they satisfy

$$\sum_{k:(jk)\in E} g(\rho_j - \rho_k) - g(\rho_j + \rho_k) = 2\pi,$$

where the sum is over the neighboring rings.

$$g(x) = \arctan \frac{(1+q)\,\mathrm{sn}\,\frac{x}{2}}{\mathrm{cn}\,\frac{x}{2}\,\mathrm{dn}\,\frac{x}{2}}.$$



Variational description

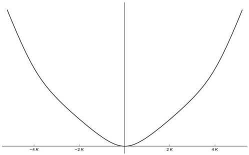
Theorem

Orthogonal ring pattern with $R < \frac{\pi}{2}$ are critical points of the functional

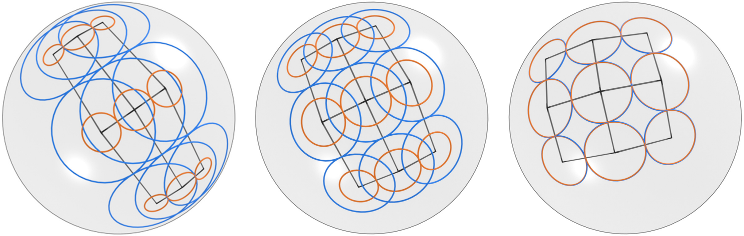
$$S_{sph}(\rho) := \sum_{(jk)} \big(F(\rho_j - \rho_k) - F(\rho_j + \rho_k)\big) - 2\pi \sum_j \Phi_j \rho_j,$$

where the first sum is over all pairs of neighboring rings.

$$F(x) = \int_0^x \arctan \frac{(1+q)\,\mathrm{sn}\,\frac{u}{2}}{\mathrm{cn}\,\frac{u}{2}\,\mathrm{dn}\,\frac{u}{2}} du.$$



Orthogonal circle pattern $q \to 1, \quad \rho \to \log\tan\frac{R}{2}$.



$q = 0.95$ $q = 0.98$ $q = 0.9999$

Theorem (Away from conformal limit)

For any rigid orthogonal circle pattern and small ϵ there exists an orthogonal ring pattern with $q = 1 - \epsilon$.

Hyperbolic orthogonal ring patterns. Parametrization in elliptic functions

Concentric rings with hyperbolic radii R_j, r_j.

$$\cosh(r) = q\cosh(R),$$
$$\cosh r = -\frac{1}{q\,\mathrm{sn}(\rho,q)},\ \tanh R = \mathrm{dn}(\rho,q),\ \sinh r = \frac{\mathrm{cn}(\rho,q)}{\mathrm{sn}(\rho,q)}.$$

- one to one correspondence $(R,r) \Leftrightarrow \rho \in [-2K, 0]$

Equations. Variational description

Theorem

$\rho \in (-2K, 0]$ are radii of orthogonal hyperbolic ring pattern if and only if they satisfy

$$\sum_{k:(jk)\in E} g(\rho_j - \rho_k) + g(\rho_j + \rho_k) = -2\pi,$$

where the sum is over the neighboring rings. They are critical points of the functional

$$S_{hyp}(\rho) := \sum_{(jk)} \big(F(\rho_j - \rho_k) + F(\rho_j + \rho_k)\big) + 2\pi \sum_j \rho_j.$$

Convexity

Theorem

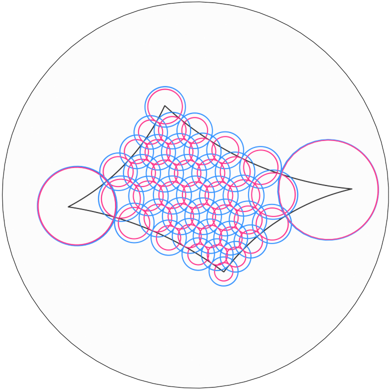
The functional $S_{hyp}(\rho)$ is convex:

$$D^2 S_{hyp} = \frac{1}{2}\sum_{(jk)}(\mathrm{dn}(\rho_j - \rho_k) + q\,\mathrm{cn}(\rho_j - \rho_k))(d\rho_j - d\rho_k)^2 + (\mathrm{dn}(\rho_j + \rho_k) + q\,\mathrm{cn}(\rho_j + \rho_k))(d\rho_j + d\rho_k)^2.$$

Boundary valued problems

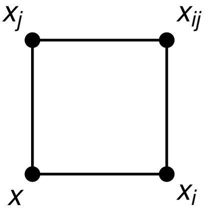
Theorem

For any choice of the boundary radii or angles (Dirichlet and Neumann boundary conditions) there exists a unique orthogonal hyperbolic ring pattern.



Integrability as Consistency

- 2D Equation
- 3D Consistency



$$Q(x, x_i, x_j, x_{ij}) = 0$$

Integrability as Consistency

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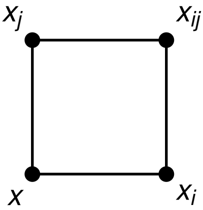

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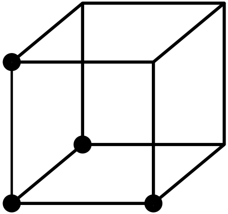

Integrability as Consistency

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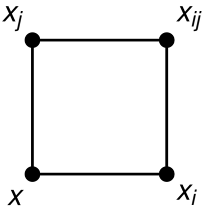

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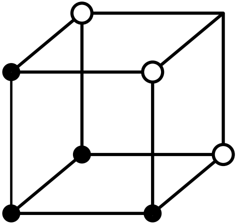

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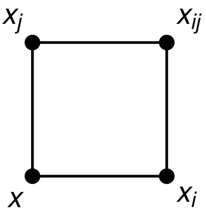

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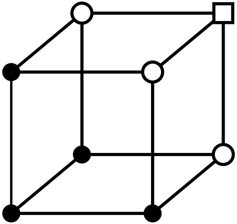

Classification of 2D integrable equations

Theorem [Adler,B.,Suris '09]
Up to Möbius transformations $(PSL_2(\mathbb{C}))^8$, any 3D-consistent system with multi-affine Q's (and with nondegenerate biquadratics) is one of the following list ($\alpha = \alpha^{(i)}$, $\beta = \alpha^{(j)}$, $\mathrm{sn}(\alpha) = \mathrm{sn}(\alpha; k)$):

$$\alpha(x - x_j)(x_i - x_{ij}) - \beta(x - x_i)(x_j - x_{ij}) = \delta\alpha\beta(\beta - \alpha) \qquad (Q1)$$

$$\begin{aligned} &\alpha(x - x_j)(x_i - x_{ij}) - \beta(x - x_i)(x_j - x_{ij}) \\ &\quad + \alpha\beta(\alpha - \beta)(x + x_i + x_j + x_{ij}) \\ &\quad = \alpha\beta(\alpha - \beta)(\alpha^2 - \alpha\beta + \beta^2) \end{aligned} \qquad (Q2)$$

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$$\begin{aligned} &\Big(\alpha - \frac{1}{\beta}\Big)(xx_i + x_jx_{ij}) - \Big(\beta - \frac{1}{\alpha}\Big)(xx_j + x_ix_{ij}) \\ &-\Big(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\Big)(xx_{ij} + x_ix_j) = \frac{\delta}{4}\Big(\alpha - \frac{1}{\alpha}\Big)\Big(\beta - \frac{1}{\beta}\Big)\Big(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\Big), \end{aligned} \qquad (Q3)$$

$$\begin{aligned} &\mathrm{sn}(\alpha)\,\mathrm{sn}(\beta)\,\mathrm{sn}(\alpha-\beta)(k^2xx_ix_jx_{ij} + 1) + \mathrm{sn}(\alpha)(xx_i + x_jx_{ij}) \\ &- \mathrm{sn}(\beta)(xx_j + x_ix_{ij}) - \mathrm{sn}(\alpha-\beta)(xx_{ij} + x_ix_j) = 0. \end{aligned} \qquad (Q4)$$

Q4 integrable equation

$$\begin{aligned}\mathrm{sn}(\alpha)\,\mathrm{sn}(\beta)\,\mathrm{sn}(\alpha-\beta)(k^2 x x_i x_j x_{ij}+1)+\mathrm{sn}(\alpha)(x x_i + x_j x_{ij})\\ -\mathrm{sn}(\beta)(x x_j + x_i x_{ij}) - \mathrm{sn}(\alpha-\beta)(x x_{ij} + x_i x_j) = 0.\end{aligned}$$

- Master 2D integrable equation. All others as appropriate limits
- Parametrization in elliptic functions
- Geometric interpretation - long standing problem

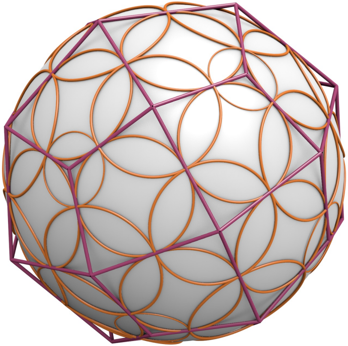
Spherical and hyperbolic orthogonal ring pattern equation as a (Laplace equation for) Q4 equation

- Equation for the ρ-radii of the spherical and hyperbolic orthogonal ring patterns is a special case of the Q4 integrable equation on the square grid.

$$\frac{\operatorname{sn}\frac{1}{2}(\rho+\rho_1+iK')}{\operatorname{sn}\frac{1}{2}(\rho-\rho_1+iK')}\frac{\operatorname{sn}\frac{1}{2}(\rho+\rho_2+iK')}{\operatorname{sn}\frac{1}{2}(\rho-\rho_2+iK')}\times$$
$$\frac{\operatorname{sn}\frac{1}{2}(\rho+\rho_3+iK')}{\operatorname{sn}\frac{1}{2}(\rho-\rho_3+iK')}\frac{\operatorname{sn}\frac{1}{2}(\rho+\rho_4+iK')}{\operatorname{sn}\frac{1}{2}(\rho-\rho_4+iK')}=1.$$

Koebe polyhedra and orthogonal circle patterns

Minimal surface case



- Orthogonal circle pattern$\leftrightarrow$ Koebe polyheder with its dual
- Circumscribed polyhedron with touching edges (and quadrilateral faces)

[Koebe, Andreev, Thurston,...]

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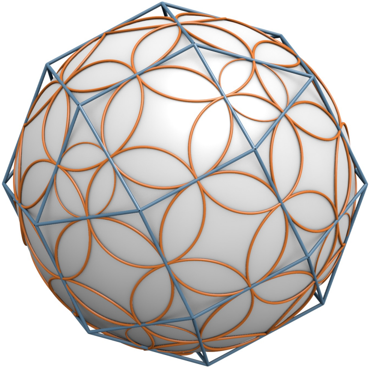
Minimal surface case



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[Koebe, Andreev, Thurston,...]

Construction method for discrete minimal surfaces

continuous minimal surface

$\Downarrow$

image of curvature lines under Gauss-map

$\Downarrow$

cell decomposition of (a branched cover of) the sphere

$\Downarrow$

Koebe polyhedron (variational principle)

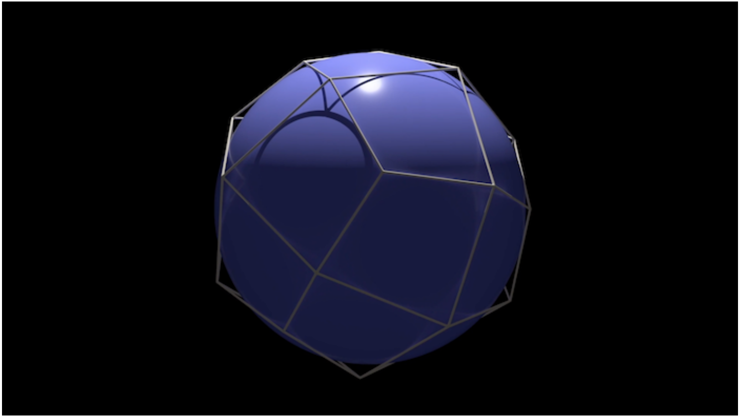
$\Downarrow$

discrete minimal surface (Christoffel dual)

- Geometry from combinatorics of curvature lines

[B., Hoffmann, Springborn, Annals ’06]

Koebe polyhedra and minimal surfaces

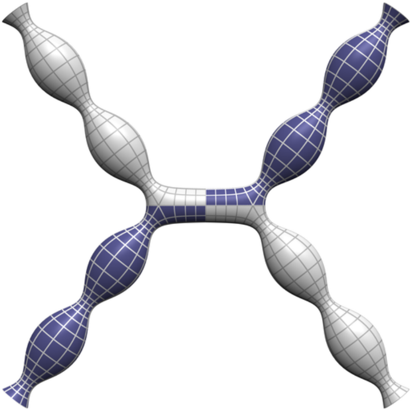


Bobenko, Newjoto, Techter, Koebe polyhedra and minimal surfaces, Movie, 2018

Smooth cmc surfaces from loop group factorization

Which surfaces ?

- cmc
  $\Delta u + e^u - |Q(z)|^2 e^{-u} = 0$,
  sinh-Gordon equation on a Riemann surface, $Q(z)$ holo quadratic differential
- N-noids $\to$ punctured spheres
- periodic reflection surfaces $\to$ fundamental polygons
- DPW-method based on Iwasawa loop group factorization
- works numerically, proof of existence problematic



[B., Heller, Schmitt ’21]

Changing mean curvature, movie

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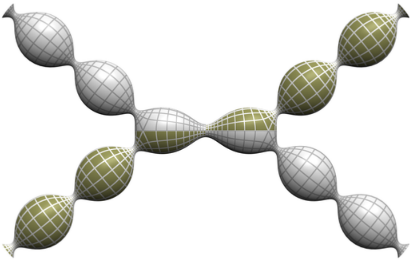

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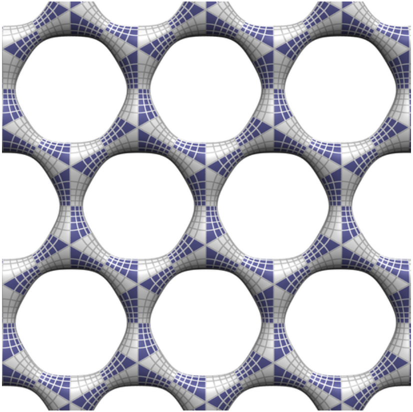

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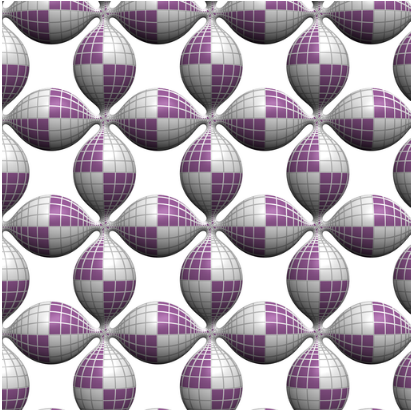

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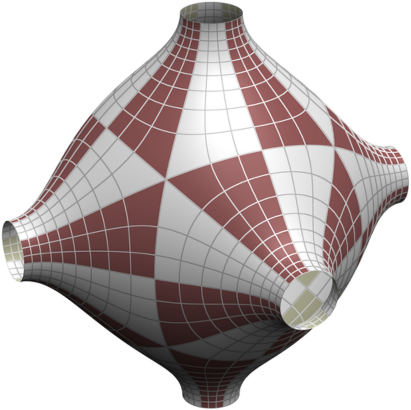
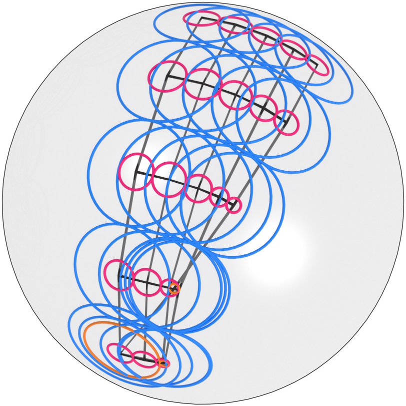


[B., Heller, Schmitt '21]

Changing mean curvature, movie

Discrete (s-isothermic) cmc surfaces

$q = 0.98291015625$, most of the rings are negatively oriented



[B., Hoffmann, Smeenk ’23]

Spherical ring pattern $\Rightarrow$ Gauss map $\Rightarrow$ discrete cmc surface

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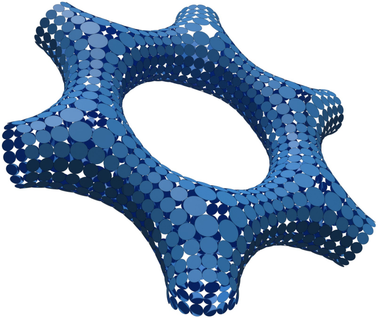

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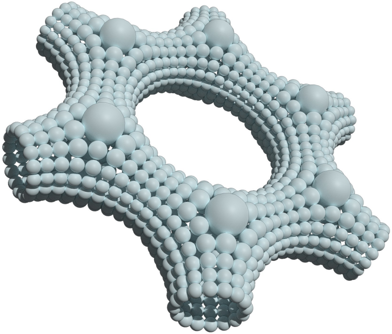

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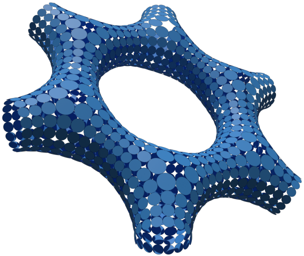
s-isothermic surfaces

- **s-isothermic**
  [B., Pinkall ’99], [B., Suris ’07], [Hoffmann]
- **minimal**
  [B., Springborn, Hoffmann ’06]
- **cmc**
  [Hoffmann ’10], [B., Hoffmann ’16], [Tellier, Hauswirth, Douthe, Baverel ’18]



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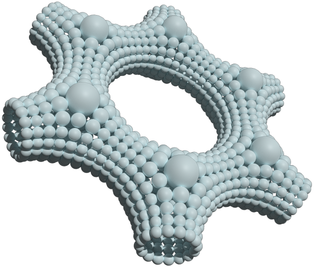
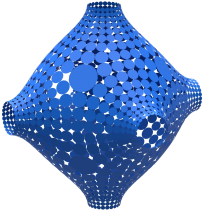
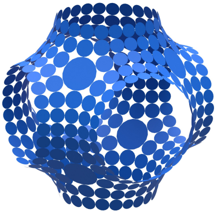


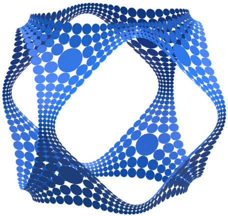
Discrete cmc surfaces



cmc



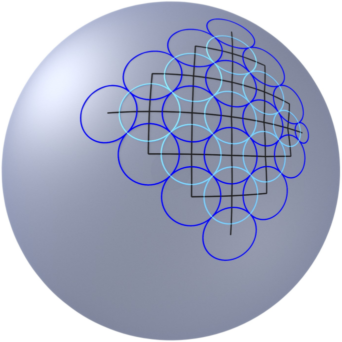
minimal



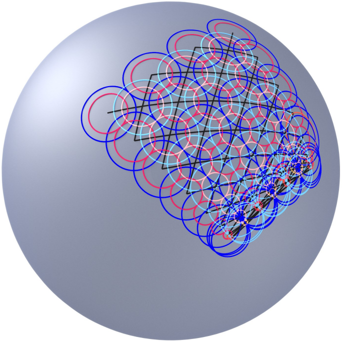
cmc

[B., Hoffmann, Smeenk '23]

Gauss maps. Orthogonal ring (circle) patterns



minimal



cmc

[B., Hoffmann, Smeenk ’23]

Convergence

C^∞ convergence known for minimal surfaces (circle patterns)
[B., Hoffmann, Springborn '06]

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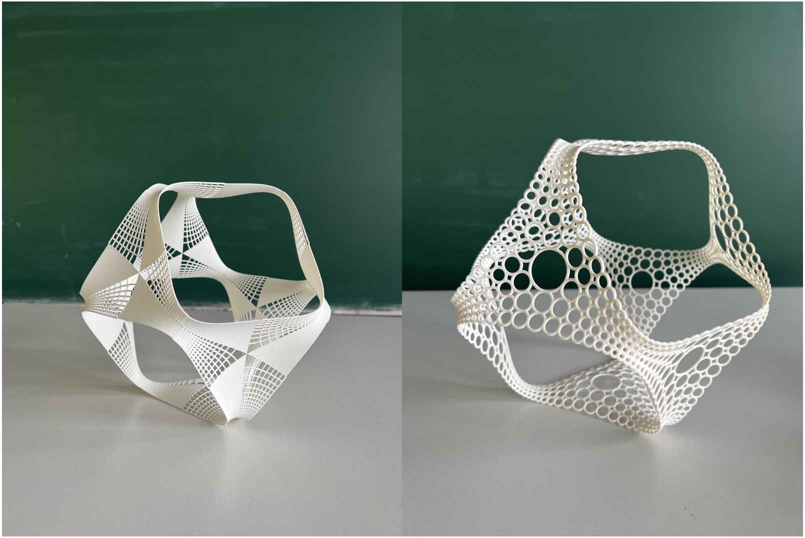


Convergence

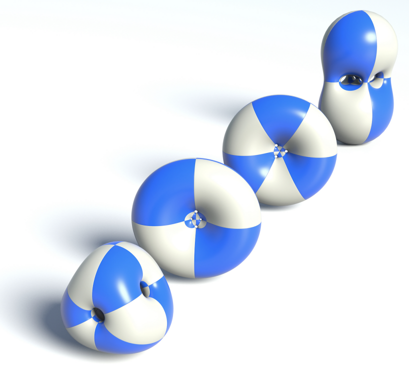
C^∞ convergence known for minimal surfaces (circle patterns)
[B., Hoffmann, Springborn '06]

Discrete minimal surfaces in space forms. Outline

minimal S^3

- isometric to cmc in R^3
- $\Delta u + \sinh u = 0$
- Lawson correspondence
- Discrete Lawson correspondence? Known for slightly different class [B., Romon '17]

Smooth minimal compact surface in S^3



[B., Heller, Schmitt '21]

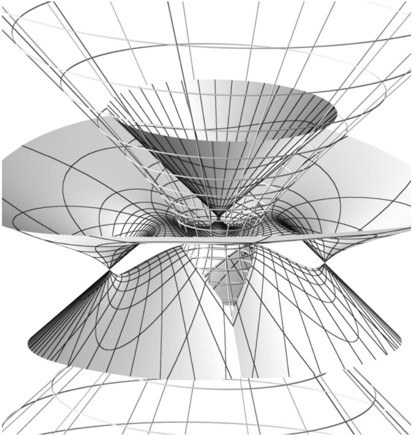
Spherical orthogonal ring patterns described by integrable $Q4$ equation with a variational principle

Discrete minimal surfaces in space forms. Outline

minimal AdS_3

- isometric to cmc in $R^{2,1}$
- $\Delta u - \sinh u = 0$
- Lawson correspondence?
- Discrete Lawson correspondence?

Smooth minimal trinoid in AdS_3



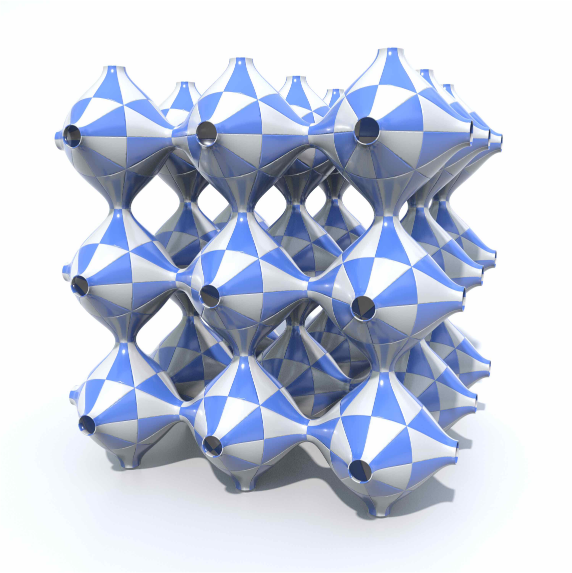
[B., Heller, Schmitt '21]

Hyperbolic orthogonal ring patterns described by integrable $Q4$ equation with convex variational principle

Papers

- A.I. Bobenko, T. Hoffmann, T. Rörig, Orthogonal ring patterns, 2019
- A.I. Bobenko, T. Hoffmann, N. Smeenk, Discrete constant mean curvature surfaces and orthogonal ring patterns. Geometry from combinatorics, 2023 (in preparation)
- A.I. Bobenko, J. Newjoto, J. Techter, Koebe polyhedra and minimal surfaces, Movie, 2018
- A.I. Bobenko, S. Heller, N. Schmitt, Minimal *n*-noids in hyperbolic and anti-de Sitter 3-space, 2019
- A.I. Bobenko, S. Heller, N. Schmitt, Constant mean curvature surfaces based on fundamental quadrilaterals, 2021

Integrable discretization



Integrable discretization