Integrable sigma models at RG fixed points: quantisation as affine Gaudin models

Jörg Teschner

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University of Hamburg, Department of Mathematics
and DESY

Troubles with integrable nonlinear sigma models

In the study of many integrable (quantum) field theories one is fortunate to benefit from Yang-Baxter algebra structures. Existence of a Lax pair $\big[\partial_x + \mathcal{L}_x(\lambda),\, \partial_t + \mathcal{L}_t(\lambda)\big] = 0$,

$$\Rightarrow \quad T(\lambda) = \mathrm{Tr}(M(\lambda)), \quad M(\lambda) := \overleftarrow{\mathcal{P}} \exp\left(\int_0^{2\pi} dx\; \mathcal{L}_x(x,\lambda)\right), \quad \text{is conserved.}$$

In many cases one finds that the Poisson-structure of the fields implies

$$\big\{\, M(\lambda) \overset{\otimes}{,} M(\mu) \,\big\} \;=\; \big[\, M(\lambda) \otimes M(\mu)\,,\, r(\lambda/\mu) \big]\,. \qquad\qquad \text{(clYBE).}$$

Quantisation of such Poisson-structures is well-understood, leading to quantum groups etc.. One may benefit from existence of a large set of advanced mathematical tools.

There is, however, an important class of field theories, the nonlinear sigma models, where life is not that easy. Appearance of δ'-distributions in the Poisson-brackets of $\mathcal{L}_x$ makes it hard, if not impossible, to get Poisson-brackets of the form (clYBE).

$\rightsquigarrow$ **non-ultralocality problem**

This might indicate that completely new methods are necessary for this class.

A case study: The Klimčík model

Gauged formulation:

Configurations: Fields $(g_1, g_2) \in G \times G$ modulo $(g_1, g_2) \mapsto (g_1 h, g_2 h)$.

$$\frac{\mathcal{A}}{4K} = \int \mathrm{d}t\, \mathrm{d}x \, \Big\langle \big(g_1^{-1}\,\partial_+ g_1 - g_2^{-1}\,\partial_+ g_2\big), \big(1 - \mathrm{i}\varepsilon_1 \hat{R}_{g_1} - \mathrm{i}\varepsilon_2 \hat{R}_{g_2}\big)^{-1} \big(g_1^{-1}\partial_- g_1 - g_2^{-1}\partial_- g_2\big) \Big\rangle,$$

where $\hat{R}(\mathsf{h}) = 0$, $\hat{R}(\mathsf{e}_\pm) = \mp \mathrm{i}\, \mathsf{e}_\pm$, $\mathtt{h} \in \mathfrak{h}$, $\mathsf{e}_\pm \in \mathfrak{n}_\pm$, and $K, \varepsilon_1, \varepsilon_2$: main **parameters**.

(ZCC) $$\big[\partial_x + \mathcal{L}_x(z),\, \partial_t + \mathcal{L}_t(z)\big] = 0\,, \qquad \begin{aligned} \mathcal{L}_t(z) &= \mathcal{L}_+(z) + \mathcal{L}_-(z)\,, \\ \mathcal{L}_x(z) &= \mathcal{L}_+(z) - \mathcal{L}_-(z)\,, \end{aligned}$$ where

$$\begin{aligned}
\mathcal{L}_\pm(z) &= \tfrac{1}{4}\left(\varepsilon_2^2 - \varepsilon_1^2 \pm \mathrm{i}\varepsilon_2\, \hat{R}_{g_2} \mp \mathrm{i}\varepsilon_1\, \hat{R}_{g_1} + \xi\, z^{\pm 1} \right) \mathcal{I}_\pm + \tfrac{1}{2}\left(g_1^{-1}\,\partial_\pm g_1 + g_2^{-1}\partial_\pm g_2 \right)\,, \\
\mathcal{I}_\pm &= 2\left(1 \pm \mathrm{i}\varepsilon_1\, \hat{R}_{g_1} \pm \mathrm{i}\varepsilon_2\, \hat{R}_{g_2}\right)^{-1} \left(g_1^{-1}\partial_\pm g_1 - g_2^{-1}\partial_\pm g_2 \right)\,, \\
\xi^2 &= \left(1 - (\varepsilon_1 + \varepsilon_2)^2 \right)\left(1 - (\varepsilon_1 - \varepsilon_2)^2 \right)\,.
\end{aligned}$$

Reduced representation: $g := g_1 g_2^{-1}$, $\mathcal{L}_\mu^{(\mathrm{inv})}(z) = g_2\, \mathcal{L}_\mu(z)\, g_2^{-1} - (\partial_\mu g_2)\, g_2^{-1}$,

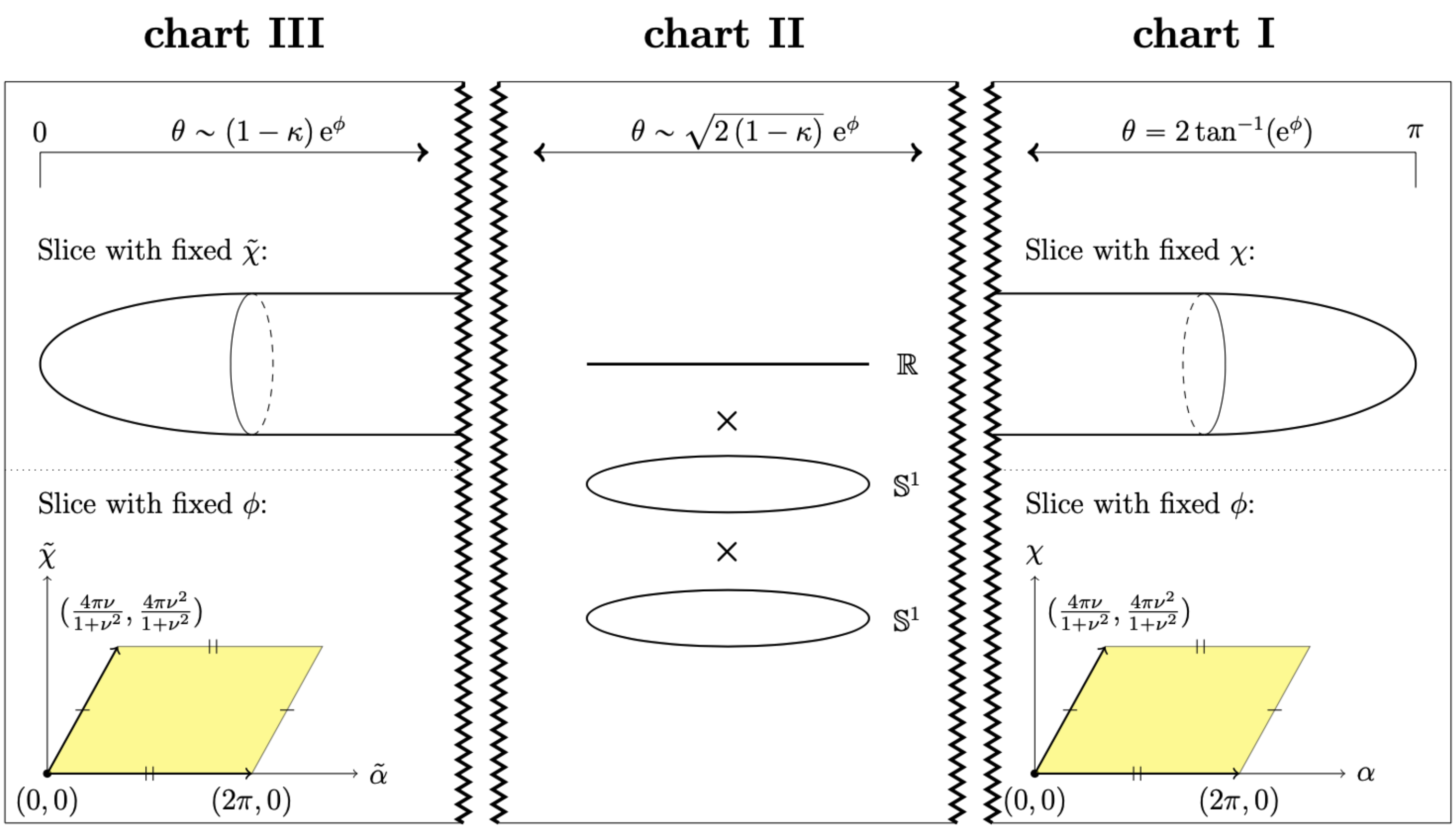
$$\mathcal{A} = 4K \int \mathrm{d}t\mathrm{d}x \, \Big\langle g^{-1}\,\partial_+ g\,,\, \big(1 - \mathrm{i}\varepsilon_1 \hat{R}_g - \mathrm{i}\varepsilon_2 \hat{R}\big)^{-1} \big(g^{-1}\partial_- g\big) \Big\rangle\,.$$

A case study: The $G = SU(2)$ Klimčík model – conformal limit

RG flow: $$\varepsilon_1 = \frac{1}{\sqrt{(1+\kappa^{-1}\nu^2)(1+\kappa\nu^2)}}\,, \qquad \varepsilon_2 = \frac{\nu^2}{\sqrt{(1+\kappa^{-1}\nu^2)(1+\kappa\nu^2)}}\,,$$

with running coupling κ (Fateev).

Target space near **conformal limit** $\kappa \to 1^-$:



More general: Classical Affine Gaudin Models[1] (AGMs)

All field theory parameters encoded in function $\varphi(z)$ of **spectral parameter** z

$$\varphi(z) = \frac{K}{\xi^2} \frac{\prod_{i=1}^{M}(z-\zeta_i^+)(z-\zeta_i^-)}{\prod_{r=1}^{M+1}(z-z_r^+)(z-z_r^-)}$$

$$\left(\begin{array}{l}\text{holomorphic on surface } C_{0,2M+2} \\ C_{0,2N} := \mathbb{CP}^1 \setminus \{z_1^\pm, \ldots, z_N^\pm\}.\end{array}\right)$$

Kinematics: Poles $z_r^\pm \mapsto \mathfrak{g}^{\mathbb{C}}$-Kac-Moody currents $J_r^\pm(x)$ with levels $\ell_r^\pm = \operatorname*{Res}_{z=z_r^\pm} \varphi(z)$,

$$\left\{J_r^\rho(x) \stackrel{\otimes}{,} J_s^\sigma(y)\right\} = \delta^{\rho\sigma}\delta_{rs}\Big(\big[\big(1\otimes J_r^\rho(x)\big), \mathsf{C}_2\big]\,\delta(x-y) - \ell_r^\rho\,\mathsf{C}_2\,\partial_x\delta(x-y)\Big).$$

Constraint generating G gauge symmetry:

$$\mathcal{C}(x) = \sum_{r=1}^{M+1} J_r^+(x) + \sum_{r=1}^{M+1} J_r^-(x) \approx 0.$$

Dynamics generated by light-cone Hamiltonians $\mathcal{P}_\pm \approx \pm\sum_{i=1}^{M} \operatorname*{Res}_{z=\zeta_i^\pm} \mathcal{Q}(z)\,\mathrm{d}z$, where

$$\mathcal{Q}(z) = \frac{-1}{2\varphi(z)}\int \mathrm{d}x\,\big\langle\Gamma(z,x),\Gamma(z,x)\big\rangle, \qquad \Gamma(z,x) = \sum_{\rho=\pm}\sum_{r=1}^{M+1}\frac{J_r^\rho(x)}{z-z_r^\rho}\,.$$

[1] Following Feigin-Frenkel; Vicedo; Delduc-Lacroix-Magro-Vicedo; Lacroix, and specialising to a class of such models.

Affine Gaudin models 2

Classical integrability: Zero curvature condition:

$$\big[\,\partial_+ + \mathcal{L}_+(z), \partial_- + \mathcal{L}_-(z)\big] = 0\,, \qquad \text{(ZCC)}$$

$$\mathcal{L}_\pm(z) = \mathcal{B}_\pm + \sum_{i=1}^{M} \frac{\mathcal{K}_i^\pm}{z - \zeta_i^\pm}\,, \qquad \mathcal{K}_i^\pm = \frac{\Gamma(\zeta_i^\pm)}{\varphi'(\zeta_i^\pm)}\,,$$

where z: spectral parameter.

Recovering the Klimčík model: Choosing $M = 1$, and

$$z_1^\pm = +\frac{1 + \varepsilon_1^2 - \varepsilon_2^2 \mp 2\varepsilon_1}{\xi}\,, \qquad \ell_1^\pm = \mp\frac{K}{\varepsilon_1}\,,$$

$$z_2^\pm = -\frac{1 + \varepsilon_2^2 - \varepsilon_1^2 \mp 2\varepsilon_2}{\xi}\,, \qquad \ell_2^\pm = \mp\frac{K}{\varepsilon_2}\,.$$

reproduces the Klimčík model.

Perturbatively quantised AGMs in the UV

Conjecture[2]**:** a) One-loop **RG-flow** $\Leftrightarrow$ flow of function $\varphi(z)$

$$\frac{d}{d\tau}\varphi(z) = \hbar\, h^\vee \frac{d}{dz}\big(f(z)\varphi(z)\big) + O(\hbar^2), \quad \text{with certain } f = f[\varphi].$$

b) Limit $\kappa \to 1^- \Rightarrow z_i^+ \to 0$, $z_i^- \to \infty$, **RG fixed point**.

$\rightsquigarrow$ **conformal/chiral limit**: Defining $\mathcal{L}_\pm^{(\mathrm{L})}\big(z^{(\mathrm{L})}\big) = \lim_{\xi\to 0}\, \mathcal{L}_\pm\big(\frac{z^{(\mathrm{L})}}{\xi}\big)$ we get

$$\mathcal{L}_+^{(\mathrm{L})}\big(z^{(\mathrm{L})}\big) \approx \mathcal{B}_+^{(\mathrm{L})} + \sum_{i=1}^{M} \frac{\mathcal{K}_i^{(\mathrm{L})}}{z^{(\mathrm{L})} - \zeta_i^{(\mathrm{L})}}\,, \qquad \mathcal{L}_-^{(\mathrm{L})}\big(z^{(\mathrm{L})}\big) \sim 0\,,$$

up to gauge transformations. (ZCC) implies $\partial_- \mathcal{L}_+^{(\mathrm{L})} = 0$, **chiral**!

There is a second limit for the other chirality $\Rightarrow$ **Decoupling!**

Let's focus on the chiral theory defined by $\mathcal{L}_+^{(\mathrm{L})}\big(z^{(\mathrm{L})}\big)$, and drop (L), $+$ in notations.

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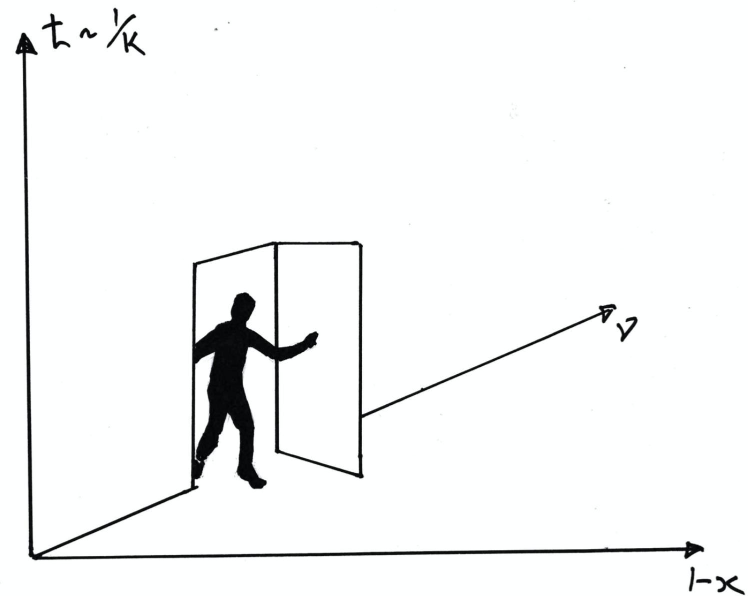
[2] Part a) Delduc-Lacroix-Sfetsos-Siampos; b) Fateev for $M = 1$ (Klimčík model); numerical checks for low M.

Our strategy: Entering quantum world through the side entrance

Usual approaches (canonical quantisation) start from the classical limit $\hbar \sim 1/K \sim \alpha' \to 0$. It may help to start from **conformal limits** instead.

Conformal limit: Chiral (VOA) structure offers useful starting point for quantisation of such models (algebra of local observables) and quantisation of integrable structure.



Strategy: Understand integrable structure **at** conformal limit & deform **away**.

However:

Problems with non-ultralocality do not go away in the conformal limits!

Quantising chiral AGMs

Quantize Kac-Moody-currents $J_r(x) = \sum_a J_{r,a}(x) T^a$, with T^a: basis for $\mathfrak{g}$:

$$\big[\, \mathsf{J}_{r,a}(x)\,,\, \mathsf{J}_{s,b}(y)\,\big] \;=\; -2\pi\, \delta_{rs}\, (f^c_{ab}\, \mathsf{J}_{r,c}(x)\, \delta(x-y) + \mathrm{i}\, k_r\, \eta_{ab}\, \partial_x \delta(x-y))\,,$$

Quantised constraint $\mathsf{J}_{\mathrm{diag}}(x) := \sum_{r=1}^{N} \mathsf{J}_r(x) \rightsquigarrow$ Algebra of local observables:

coset VOA $$\frac{\hat{\mathfrak{g}}_{k_1} \oplus \ldots \oplus \hat{\mathfrak{g}}_{k_{M+1}}}{\hat{\mathfrak{g}}_{k_1+\ldots+k_{M+1}}}\,, \qquad k_r = -\frac{2\pi \ell_r + O(\hbar)}{\hbar}\,.$$

Conjectured form of **local charges**[3]:

$$\boxed{\mathsf{Q}_{\gamma,p} = \oint_\gamma \mathcal{P}(z)^{-p/h^\vee}\, \mathsf{V}_p(z)\, dz} \qquad \mathsf{V}_p(z) = \int \mathsf{S}_{p+1}(z,x)\, dx\,,$$

$$\mathsf{S}_{p+1}(z,x) = \tau_p^{a_1 \ldots a_{p+1}} \;:\!\Gamma_{a_1}(z,x) \cdots \Gamma_{a_{p+1}}(z,x)\!: + \;\ldots\;,$$

$$\mathcal{P}(z) = \prod_{r=1}^{N-1} (z-z_r)^{k_r}, \qquad \Gamma_a(z,x) = \sum_{r=1}^{N} \frac{\mathsf{J}_{r,a}(x)}{z-z_r}, \qquad \gamma \text{ Pochhammer contour,}$$

with S_{p+1} satisfying VOA conditions ensuring gauge invariance and $[\mathsf{Q}_{\gamma,p}, \mathsf{Q}_{\gamma',q}] = 0$.

[3] Lacroix-Vicedo-Young

Relation AGM to Klimčík model

Klimčík model has been studied extensively[4] using the **gauge-fixed** formulation.

Central part of our work is a detailed **comparison** of **gauged** (AGM) and **gauge-fixed** formulations of Klimčík model on classical and quantum level, including:

- Classical level:

Gauge fixing of AGM Lax matrix $\rightsquigarrow$ Chiral limit of Klimčík Lax matrix

$$\mathcal{L}_{\mathrm{UV}}(x,z) = -\big(\chi z\, V_+\, \mathsf{e}_+ + (1+\chi z) V_-\, \mathsf{e}_-\big) - \frac{\chi}{2}\, z\, V_0\, \mathsf{h}\,, \qquad \chi := \frac{4\nu^2}{1+\nu^2},$$

$$V_\pm = \big(\mathrm{i}\sqrt{1+\nu^2}\ \partial_+\phi_3 + \partial_+\phi_2 \pm \nu\ \partial_+\phi_1\big)\, e^{\pm 2\,\phi_3}, \qquad \mathsf{e}_+ = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right),$$

$$V_0 = -2\left(\partial_+\phi_3 - \mathrm{i}\sqrt{1+\nu^2}\,\partial_+\phi_2\right), \qquad \mathsf{h} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \qquad \mathsf{e}_- = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right).$$

Proof: Intricate computation of Dirac brackets.

- Quantum level: **Comparison of quartic charge** $\mathrm{Q}_{\gamma,4}$.

[4] Bazhanov-Lukyanov; Bazhanov-Kotousov-Lukyanov

Application I – ODE/IQFT-conjectures for sigma models

Recall: Relativistic AGM $\rightsquigarrow$ chiral AGMs in UV-limit. Thanks to Feigin and Frenkel:

UV limit of AGM: model for which there is ODE/IQFT-conjecture:

There is a one-to-one correspondence between the eigenstates of the quantum affine Gaudin model associated to $(\widehat{\mathfrak{sl}}_2, \varphi)$, and differential operators

$$\mathcal{D}_v = -\partial_z^2 + v(z) + \chi\,\mathcal{P}(z),$$

where $\mathcal{P}(z) = \prod_{r=1}^N (z - z_r)^{k_r}$, and the function $v(z)$ has second order poles at z_r, $r = 1, \ldots, N$, and singularities $x_1, \ldots, x_K$ with trivial monodromy.

(Such singularities are called apparent singularities. Triviality of monodromy of ∇_z around $x_1, \ldots, x_K$, implies algebraic equations for $x_1, \ldots, x_K$ having a discrete set of solutions.)

The corresponding eigenvalues of local IM $\mathsf{Q}_{\gamma,p}$ are given in terms of $v(z)$ as[5]

$$I_{\gamma,p} = \oint_\gamma \mathcal{P}(z)^{-p/h^\vee} v_p(z)\mathrm{d}z\,, \quad v_p(z)\text{: certain differential polynomial in } v(z).$$

[5] Bazhanov-Lukyanov; generalisation: Lacroix-Vicedo-Young.

Application II – Non-local IM in AGM 1

Non-ultralocality strikes back:

– So far no general proof of commutativity of quantum non-local IM in AGMs.

A conjecture of Bazhanov-Kotousov-Lukyanov (2018) suggests a solution: Let

$$\mathsf{X}_0 = \frac{1}{q-q^{-1}} \int_0^{2\pi} \mathrm{d}x \, \mathsf{V}_+(x) \,, \qquad \mathsf{X}_1 = \frac{1}{q-q^{-1}} \int_0^{2\pi} \mathrm{d}x \, \mathsf{V}_-(x) \,,$$

$$\mathsf{V}_\pm = : \frac{\mathrm{i}\sqrt{k_3+2}\,\partial\varphi_3 + \sqrt{k_2+2}\,\partial\varphi_2 \pm \sqrt{k_1+2}\,\partial\varphi_1}{\sqrt{k_2(k_3+2)}} \; e^{\pm\frac{2\varphi_3}{\sqrt{k_3+2}}} :, \quad k_3 = k_1 + k_2.$$

Fact: Polynomials formed out of X_0, X_1 can be renormalised such that

$$\mathsf{M}(\lambda) := \; e^{-\frac{\pi\,\mathsf{h}}{\sqrt{k_3+2}}\,p^{(3)}} \; \overleftarrow{\mathcal{P}} \exp_{\mathrm{ren}} \left(\lambda \int_0^{2\pi} \mathrm{d}x \, \left(\mathsf{V}_+ \, q^{\frac{\mathsf{h}}{2}} \, \mathsf{e}_+ + \mathsf{V}_- \, q^{-\frac{\mathsf{h}}{2}} \, \mathsf{e}_- \right) \right)$$

can be defined as a formal power series in $\mathsf{X}_0, \mathsf{X}_1, \lambda$ satisfying Yang-Baxter equation[6]

$$R(\lambda/\mu) \big(\mathsf{M}(\lambda) \otimes \mathrm{id} \big) \big(\mathrm{id} \otimes \mathsf{M}(\mu) \big) \; = \; \big(\mathrm{id} \otimes \mathsf{M}(\mu) \big) \big(\mathsf{M}(\lambda) \otimes \mathrm{id} \big) \, R(\lambda/\mu) \qquad \text{(YBE)}.$$

[6] There exist a renormalisation such that X_0, X_1 satisfy Serre relations of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_2)$. Claim thereby follows from results of Bazhanov-Lukyanov-Zamolodchikov, Bazhanov-Khoroshkin-Hibberd relating $\mathsf{M}(\lambda)$ to the universal R-matrix of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_2)$.

Application II – Non-local IM in AGM 2

Conjecture: (Bazhanov-Kotousov-Lukyanov) $\mathsf{M}(\lambda)$ admits a classical limit $k_i \to \infty$

$$M(\lambda) = e^{-\pi P_3 \mathsf{h}} \overleftarrow{\mathcal{P}} \exp\left(- \int_0^{2\pi} \mathrm{d}x \, \mathcal{L}_{\mathrm{UV}}\big(x, z(\lambda)\big) \right),$$

where $z(\lambda)$ is a formal power series in λ, and $\mathcal{L}_{\mathrm{UV}}(x, z)$ is the classical Lax matrix of the Klimcik model in the chiral limit.

The limit is **subtle**[7] (renormalisation; spectral parameter $z(\lambda)$ scheme dependent).

However, validity of this conjecture would establish that

$\mathsf{M}(\lambda)$: quantum counterpart of the classical monodromy matrix, satisfying YBE.

Unclear if the classical monodromy matrix satisfies an equally simple Poisson-algebra.

$\rightsquigarrow$ **Suggestion:** Construction of non-local IM in q-AGMs can follow YBE-paradigm.

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[7] Kotousov-Lacroix-T., work in progress

Summary: Hidden structures in chiral AGMs

- AGM-formulation ($\rightsquigarrow$ spin chain with **affine Lie algebra symmetry**) enables
  - quantisation of algebra of local observables (VOA, W-algebra),
  - construction of local conserved quantities.

  Complex geometry of spectral parameter surface (affine Hitchin system) helps understanding

  - RG-flows,
  - construction of local IM,
  - spectra (affine opers).

- Relations to **quantum affine algebras** help understanding non-local IM.

One may hope that these structures admit

deformations away from the conformal limits.