't Hooft loops and Integrability

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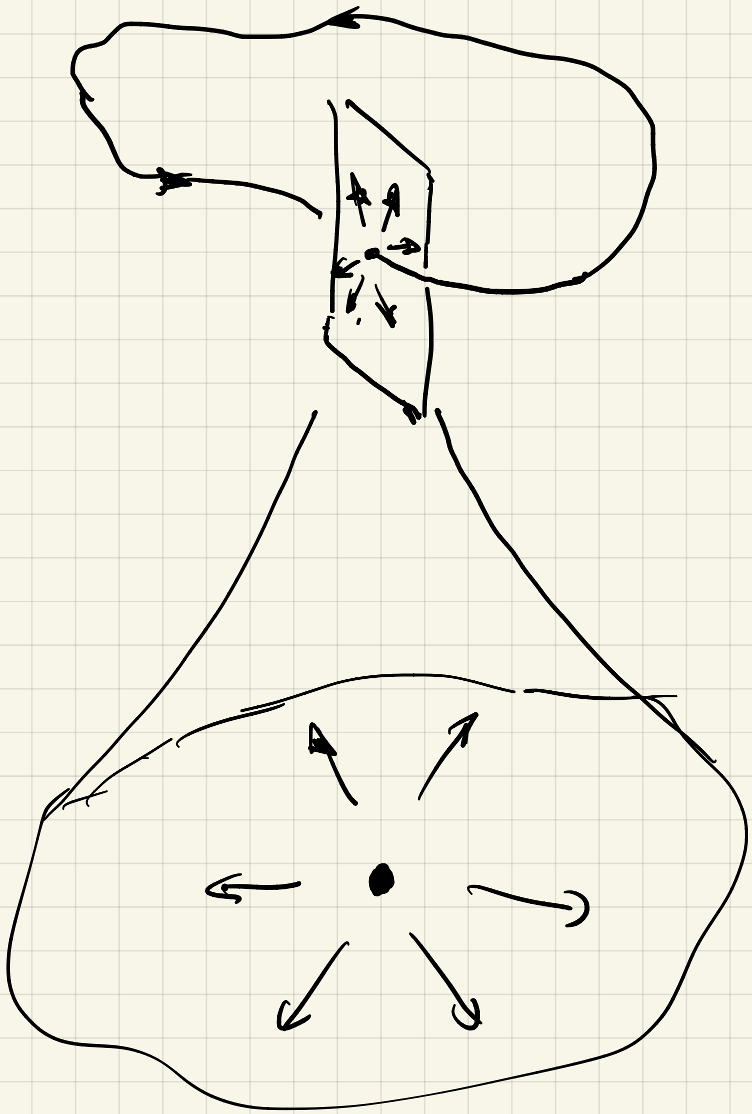
C. Kristjansen, K.Z. to appear

"Integrability in Condensed Matter Physics and Quantum Field Theory," SwissMAP Research Station, Les Diablerets, 6.02.23

't Hooft loops

't Hooft '78
Kapustin '05



Boundary conditions at $r \to 0$:

$$F_{ij} = \frac{B}{2} \varepsilon_{ijk} \frac{x_k}{r^3}$$

$$\Phi_I = \frac{B}{2} \frac{n_I}{r}$$

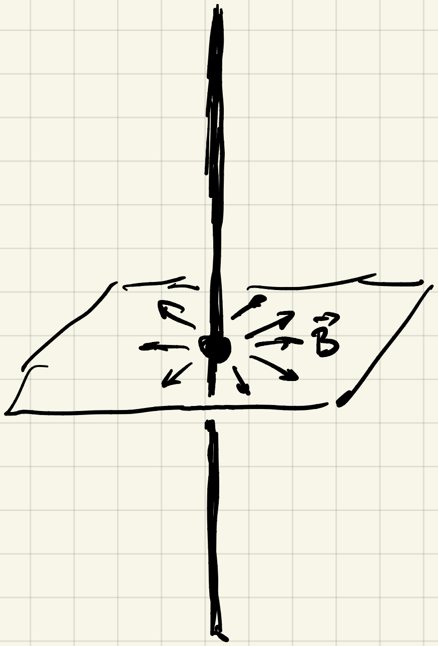
$B = \mathrm{diag}(B_1 \ldots B_N)$

$B_a \in \mathbb{Z}$

↑ Dirac quantization cond.

- magnetic monopole of charge $\frac{B}{2}$

't Hooft line in $\mathcal{N}=4$ SYM



- defines a dCFT
  $\uparrow$
  defect, codim = 2

- $PSU(2,2|4) \longrightarrow OSp(4^*|4)$

- This dCFT is expected to be integrable

Dekel, Oz '11

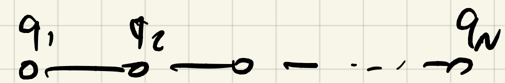
S-duality

$$g_{YM}^2 \longrightarrow \frac{16\pi^2}{g_{YM}^2}$$

't Hooft $\qquad\qquad$ Wilson

$$T_B(C) \longrightarrow W_R(C)$$

$B = (B_1, \ldots, B_N)$ $\qquad$ $R = (q_1 \ldots q_N)$:



$$B_{a+1} - B_a = q_a$$

- access to strong coupling "for free"

Gomis, Okuda, Trancanelli '09

- does <u>not</u> commute w. large-N

Perturbation theory

$$A_\mu = A_\mu^{(cl)} + a_\mu$$

$$\Phi_I = \Phi_I^{(cl)} + \phi_I$$

$$S = S[A_\mu^{(cl)}, \Phi_I^{(cl)}] + \int d^4x \left\{ |D_\mu \phi|^2 + [\Phi^{(cl)}, \phi]^2 \right\} + \dots$$

$$D_\mu = \partial_\mu - i\,[A_\mu^{(cl)}, \cdot\,]$$

$$-D^2 + [\Phi^{(cl)}, \cdot\,]^2 = -\frac{\partial^2}{\partial t^2} - \vec{D}^2 + \frac{B^2}{4r^2}$$

Hamiltonian of

charged particle interacting w. Dirac monopole

Quantum mechanics of Dirac monopole

Dirac '31
Tamm '31
Fierz '44
Wu, Yang '76

$$H = \vec{p}^{\,2}$$

$$[p_i, p_j] = \frac{iB}{2} \varepsilon_{ijk} \frac{x_k}{r^3}$$

$$[p_i, x_j] = -i \delta_{ij}$$

Spherical symmetry:

$$[H, L_i] = 0$$

$$L_i = \varepsilon_{ijk} x_j p_k - \frac{B}{2} \frac{x_i}{r}$$

Fierz '44

$$[L_i, L_j] = i \varepsilon_{ijk} L_k$$

$$H = \frac{1}{r}\left(-\frac{\partial^2}{\partial r^2} + \frac{\vec{L}^2 - \frac{B^2}{4}}{r^2}\right) r$$

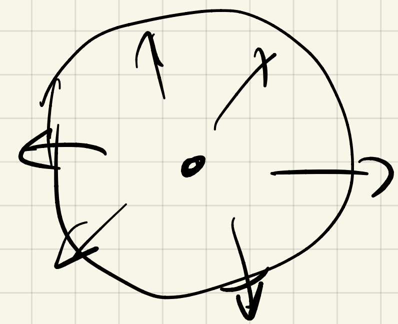
Monopole harmonics:

$$\vec{L}^2 \, Y_{lm} = l(l+1) \, Y_{lm}$$

$$L_z \, Y_{lm} = m \, Y_{lm}$$

Landau gauge:

$$A = \frac{B}{2}(1-\cos\theta)\,d\varphi \quad \leadsto \quad dA = \frac{B}{2}\sin\theta\, d\varphi \wedge d\theta$$



constant flux

$$L_z = p_\varphi - \frac{B}{2}\cos\theta$$

$$= -i\partial_\varphi - \frac{B}{2}$$

$$L_z Y_{lm} = m\, Y_{lm} \quad \Longrightarrow \quad Y_{lm} = e^{i(m+\frac{B}{2})\varphi}\, U_{lm}(\theta)$$

$$Y_{lm} = e^{i\left(m+\frac{B}{2}\right)\varphi}\, U_{lm}(\theta)$$

- Must be single-valued under $\varphi \to \varphi + 2\pi$:

$$m + \frac{B}{2} \in \mathbb{Z} \quad (1)$$

- SU(2) rep. theory:

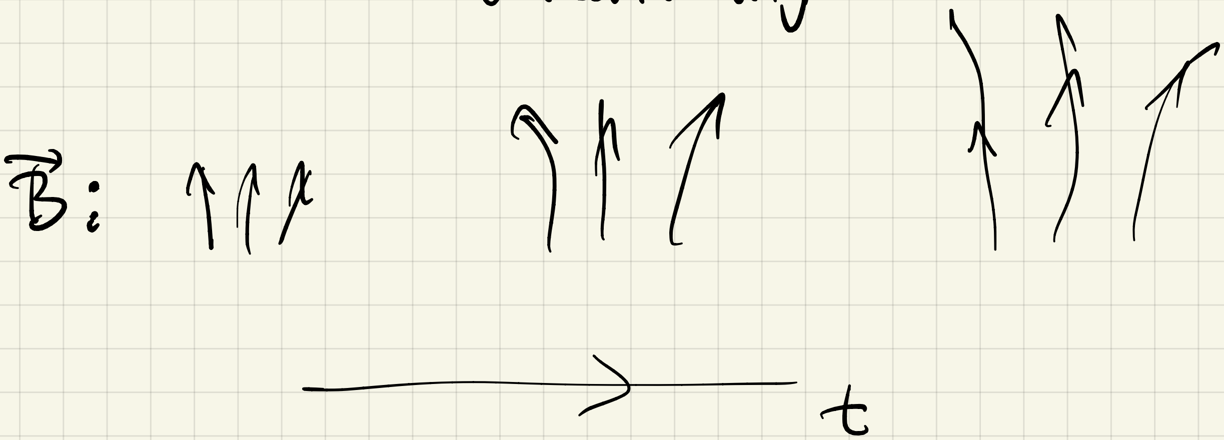
$$m \in \frac{\mathbb{Z}}{2} \quad (2)$$

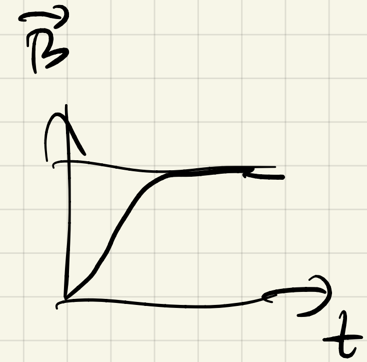
(1), (2)

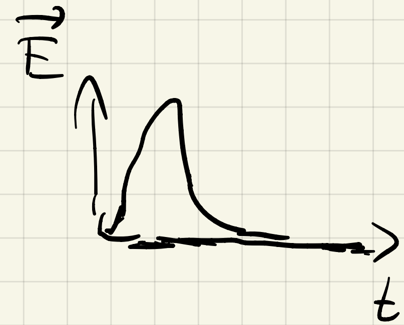
$\hookrightarrow$ B is integer

Dirac '31

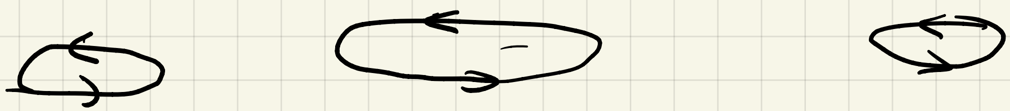
- Imagine the field of the monopole is switched on adiabatically:



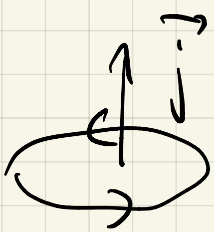




$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

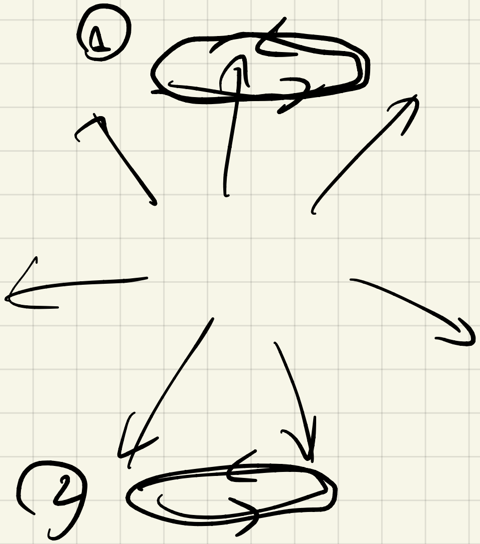


- charged particle is accelerated in the angular direction:



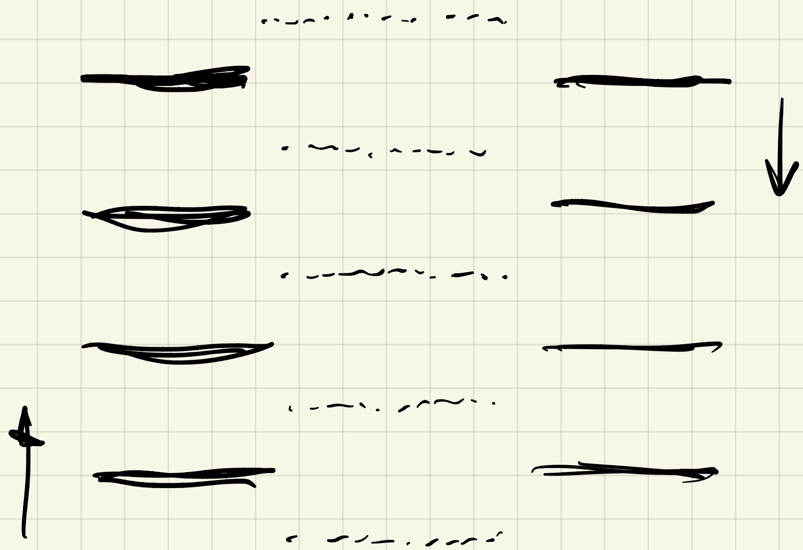
$$j \to j + \frac{B}{2}$$

Wilczek '82



$\rightsquigarrow \quad j + \frac{B}{2}$

$\rightsquigarrow \quad j - \frac{B}{2}$



Levels meet again when magnetic charge $\left(q_M = \frac{B}{2}\right)$

is <u>half-integer</u>

Wilczek '82

$$\left(-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{\alpha^2}{4\sin^2\frac{\theta}{2}} + \frac{\beta^2}{4\cos^2\frac{\theta}{2}}\right)U_{lm} = l(l+1)\,U_{lm}$$

$$\alpha = \left|m + \frac{B}{2}\right|$$

$$\beta = \left|m - \frac{B}{2}\right|$$

Solution:

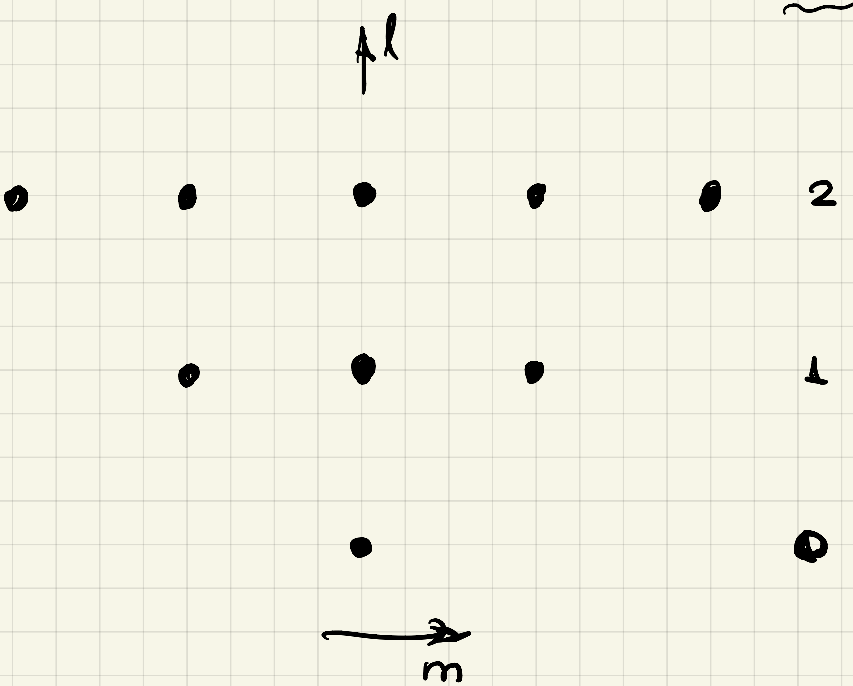
$$\boxed{U_{lm}(\theta) = \sin^{\alpha}\frac{\theta}{2}\,\cos^{\beta}\frac{\theta}{2}\;P^{(\alpha,\beta)}_{l-\frac{\alpha+\beta}{2}}(\cos\theta)}$$

Jacobi polynomial

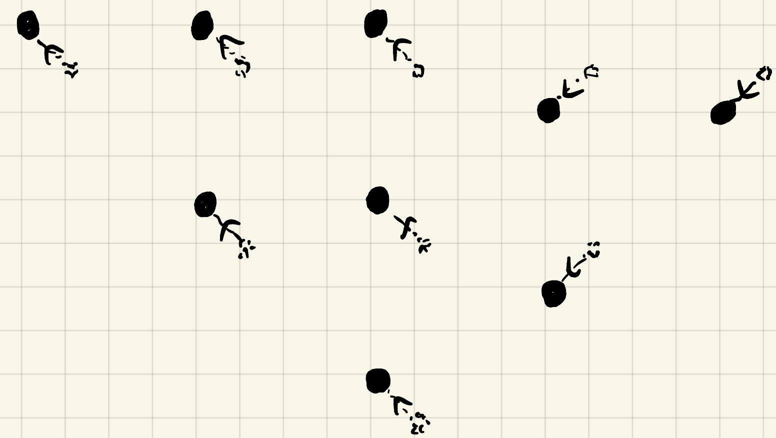
Tamm '31

Wu, Yang '76

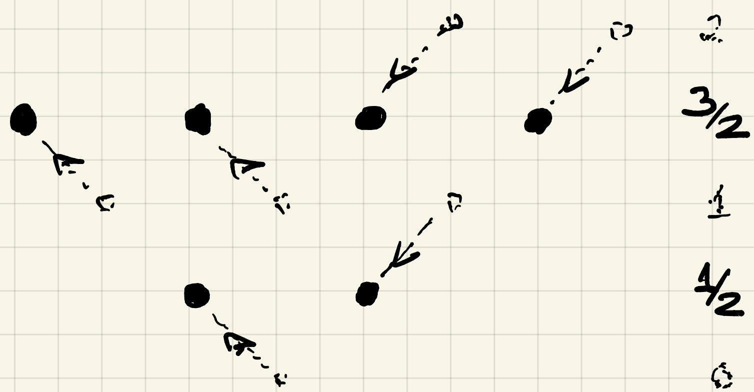
Spectral flow



$B = 0$



$B > 0$

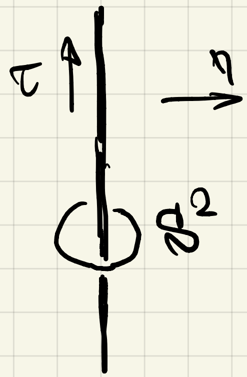


$B = 1$

Orbital momentum is quantized in half-integers!

Reduction to AdS_2

Natural coordinates for a line defect:



$$ds^2 = \eta^2 \Big(\underbrace{\frac{d\tau^2 + d\eta^2}{\eta^2}}_{AdS_2} + \underbrace{d\theta^2 + \sin^2\theta\, d\varphi^2}_{S^2} \Big)$$

$$AdS_2 \times S^2$$

KK reduction on S^2:

monopole harmonics

$$\phi(x) = \frac{1}{\eta} \sum_{lm} \chi_{lm}(\tau,\eta)\, Y_{lm}(\theta,\varphi)$$

fields on AdS_2

$$S = \int d^4x \left(|D_\mu^{(cl)} \phi|^2 + [\Phi^{(cl)}, \phi]^2 \right)$$

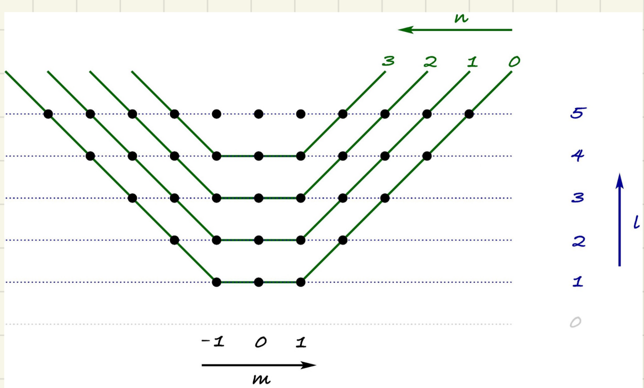
$$\downarrow$$

$$dt\, \eta^2 \times \frac{1}{\eta}\, \phi \left(-\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \eta^2} + \frac{\vec{L}^2}{\eta^2} \right) \eta \phi$$

- $\vec{L}^2$ is the mass operator on AdS_2

Spectrum: $m^2 = l(l+1)$ $\qquad l = \frac{B}{2}, \frac{B}{2}+1, \dots$

Ex ($B=2$)



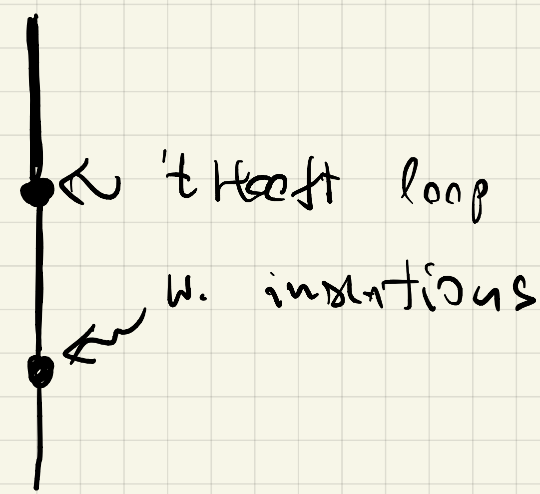
Effective action on AdS_2:

$$S = \sum_{lm} \int_{AdS_2} d^2\sigma \sqrt{g} \left[g^{\mu\nu} \partial_\mu \chi_{lm} \partial_\nu \chi_{lm} + l(l+1) \chi_{lm}^2 \right]$$

$$m^2 = \Delta(\Delta - 1) \iff \Delta = l + 1$$

Scaling dim's of defect operators:

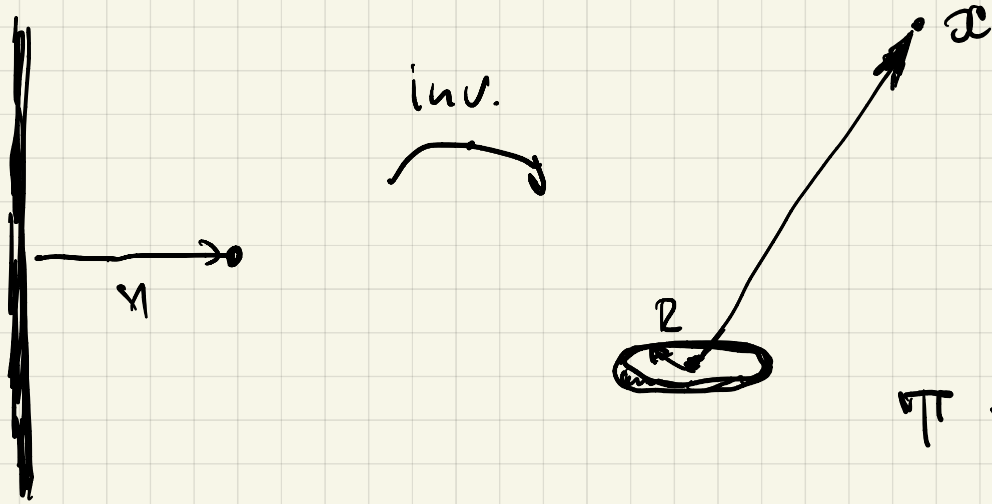
$$\Delta = \frac{B}{2} + 1, \ \frac{B}{2} + 2, \ \ldots$$



Correlation functions

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle_{\mathbb{T}} \equiv \frac{\langle \mathcal{O}_1 \ldots \mathcal{O}_n \mathbb{T} \rangle}{\langle \mathbb{T} \rangle}$$

$$\boxed{\langle \mathcal{O}_\Delta(x) \rangle_{\mathbb{T}} = \frac{\mathbb{C}}{(2y)^\Delta}}$$

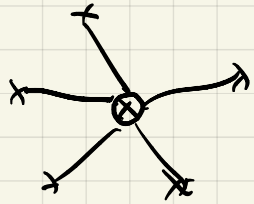


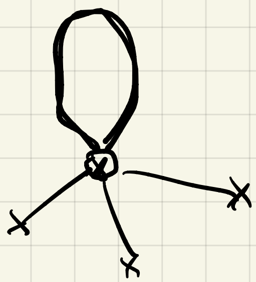
$$\langle \mathcal{O}_\Delta(x) \mathbb{T} \rangle \overset{x \to \infty}{\simeq} \frac{\mathbb{C} R^\Delta}{|x|^{2\Delta}}$$

$$\mathbb{T} = \sum_a \mathbb{C}_a R^{\Delta_a} \mathcal{O}_a(0)$$

$\mathbb{C}_a$ is the OPE coeff.

Weak coupling:

$$\langle \mathcal{O}(x) \rangle_\tau =$$ 

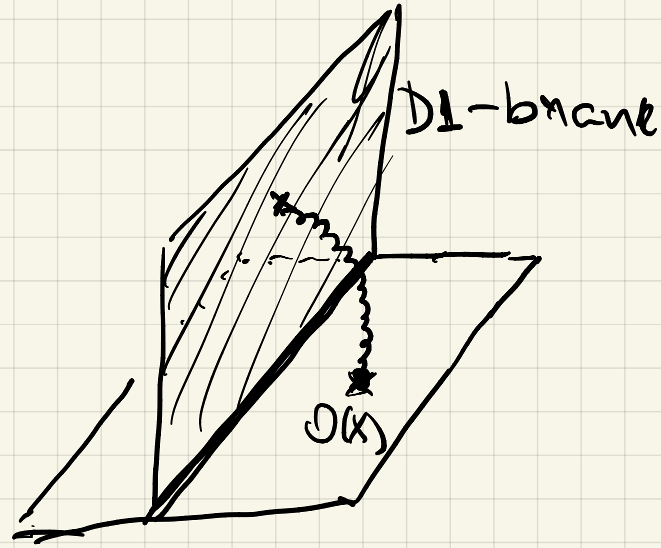
$+$  $+ \ \dots$

$$\langle \varphi^n(x) \rangle_\tau = \varphi_{cl}^{\,n}(x)$$

$$+ \ \varphi_{cl}^{\,n-2}(x)\, G(x;x) + \dots$$

propagator in the monopole bckd

Strong coupling:



or

- boundary conditions on D1 preserve integrability of string theory on $AdS_5 \times S^5$

Dekel, Oz '11

Localization and S-duality

Wilson:

$$\langle W \rangle = e^{\frac{\lambda}{8N}} \underbrace{L^{1}_{N-1}}_{\text{Laguerre}}\left(-\frac{\lambda}{4N}\right)$$

Drukker, Gross '00
Erickson, Semenoff, Z '00
Pestun '07

$N \to \infty$

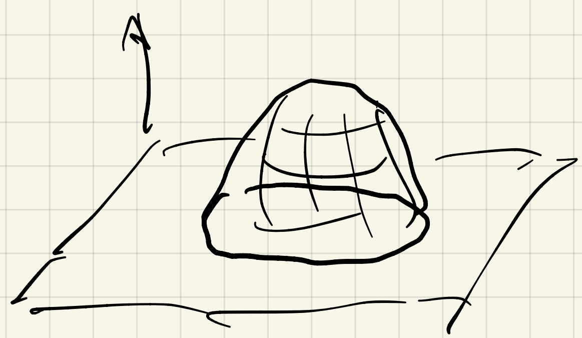
$$\langle W \rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$$

$\lambda \to 0$

$$1 + \frac{\lambda}{8} + \frac{\lambda}{256} + \dots$$

$\lambda \to \infty$

$$\sqrt{\frac{2}{\pi}}\, \lambda^{-3/4}\, e^{\sqrt{\lambda}}$$



$$e^{-T_{F1} \times \text{Area}}$$
$$\parallel$$
$$e^{-\frac{\sqrt{\lambda}}{2\pi} \times (-2\pi)}$$

't Hooft:

- apply S-duality: $\lambda \to \frac{16\pi^2 N^2}{\lambda}$

$$\langle T \rangle = e^{\frac{2\pi^2 N}{\lambda}} L^1_{N-1}\left(-\frac{4\pi^2 N^2}{\lambda}\right)$$

$$\Big\downarrow N \to \infty$$

$$\langle T \rangle = e^{N F_T(\lambda)} \qquad F_T(\lambda) = \frac{2\pi}{\sqrt{\lambda}}\sqrt{1+\frac{\pi^2}{\lambda}} + 2\ln\left(\sqrt{1+\frac{\pi^2}{\lambda}} + \frac{\pi}{\sqrt{\lambda}}\right)$$

- exponentiates at any λ

$\lambda \to 0$

$$F_T \simeq \frac{2\pi^2}{\lambda} - \ln\lambda$$

cf. action of Dirac monopole

$\lambda \to \infty$

$$F_T \simeq \frac{4\pi}{\sqrt{\lambda}}$$

$$T_{D1} \times \text{Area} = \frac{2N}{\sqrt{\lambda}} \times (-2\pi) = -\frac{4\pi N}{\sqrt{\lambda}}$$

1pt functions

$$\mathcal{O}_L = \mathrm{tr}\, Z^L$$

$$\langle \mathcal{O}_L W \rangle = \frac{1}{\sqrt{L}} \left(\frac{\sqrt{\lambda}}{2N x^2} \right)^L e^{\frac{\lambda}{8N}} \sum_{k=1}^{L} L_{N-k}^{L} \left(-\frac{\lambda}{4N} \right)$$

Okuyama, Semenoff '06
Semenoff, Z '01

$$\Big\downarrow \; S\text{-duality}$$

$$\langle \mathcal{O}_L T \rangle = \frac{1}{\sqrt{L}} \left(\frac{2\pi}{\sqrt{\lambda}\, x^2} \right)^L e^{\frac{2\pi^2 N}{\lambda}} \sum_{k=1}^{L} L_{N-k}^{L} \left(-\frac{4\pi^2 N}{\lambda} \right)$$

$$\Big\downarrow \; N \to \infty$$

$$\boxed{\langle \mathcal{O}_L \rangle_T = \frac{2\pi}{\sqrt{\lambda L}\, x^{2L}} \frac{\sinh L\xi}{\sinh \xi} \qquad \xi = \operatorname{arcsinh} \frac{\pi}{\sqrt{\lambda}}}$$

OPE coefficient: $\langle \mathcal{O}_L \rangle_{\tau} \overset{x\to\infty}{=} \frac{\mathbb{C}_L}{x^{2L}}$

$$\boxed{\mathbb{C}_L = \frac{\left(\sqrt{1+\frac{\pi^2}{\lambda}} + \frac{\pi}{\sqrt{\lambda}}\right)^L - \left(\sqrt{1+\frac{\pi^2}{\lambda}} - \frac{\pi}{\sqrt{\lambda}}\right)^L}{\sqrt{L}}}$$

Zhukowski variables: $x(u) + \frac{1}{x(u)} = \frac{4\pi u}{\sqrt{\lambda}}$

$$x\left(\tfrac{ia}{2}\right) \equiv i\, x_a$$

asymptotic

wrapping correction

$$\mathbb{C}_L = \frac{x_1^{L} - \frac{1}{x_1^L}}{\sqrt{L}}$$

- very similar to 1pt function in D1-D5 dCFT

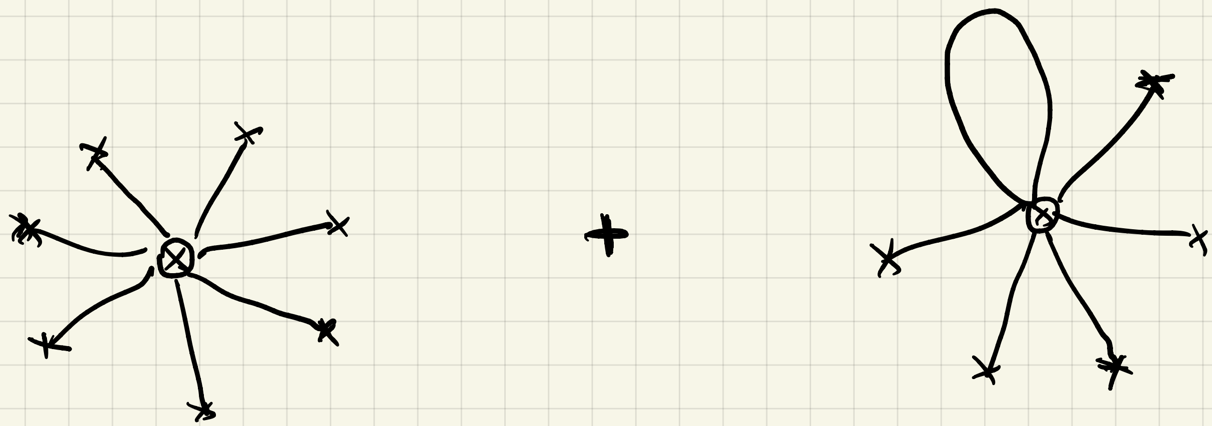
Komatsu, Wang '20

albeit much simpler:

no dependence on x_a w. $a>1$ (no bound states)

Weak coupling

$$\langle \mathcal{O}_L \rangle_\tau = \frac{1}{\sqrt{L}} \left(\frac{\pi}{\sqrt{\lambda}\, \tau} \right)^L \left(1 + \frac{\lambda}{4\pi^2} L + \dots \right)$$



$$\mathcal{O}_L = \frac{1}{\sqrt{L}} \left(\frac{2\pi}{\sqrt{\lambda}} \right)^L \operatorname{tr} Z^L$$

$$\langle Z \rangle_\tau = \frac{1}{2\tau}$$ ($\frac{1}{2}$ is the monopole charge)

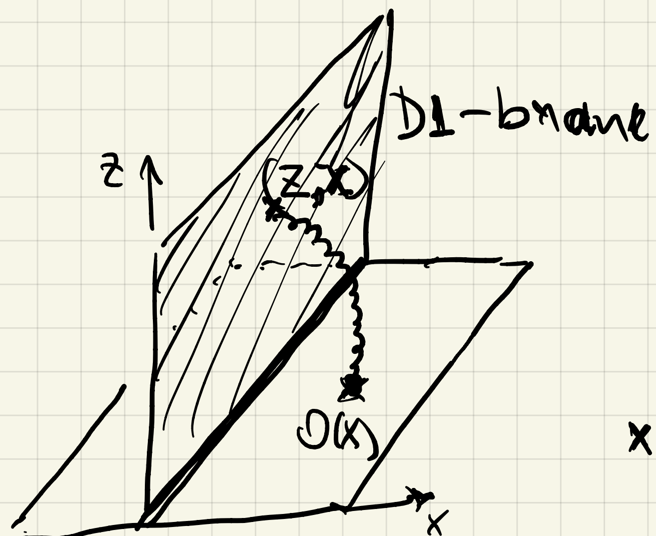
$$\lim_{x \to y} \langle Z(x)\, Z(y) \rangle_\tau = \frac{\lambda}{4\pi^2}$$

Strong coupling

Supergravity: $\lambda \to \infty \quad L\text{-fixed}$

$$\langle \mathcal{O}_L \rangle_\tau = \sqrt{\frac{\lambda}{L}} \, \frac{2\pi}{(2\pi)^L}$$

agree

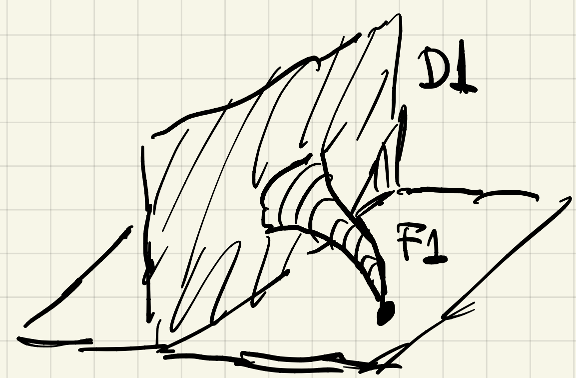


$$\langle \mathcal{O}_L \rangle_\tau = -\frac{1}{2\sqrt{\lambda L}} \int_{D1} d^2\sigma \sqrt{h}\, h^{ab} \partial_a X^m \partial_b X^n$$

$$\times \left[2 \nabla_m \nabla_n - L(L-1) g_{mn} \right] \underbrace{\frac{Z^L}{Z^2 + (X-x)^2}}_{\text{bulk-to-boundary prop}}$$

String theory: $\lambda \to \infty, \quad \frac{\lambda}{L^2}\text{ - fixed}$

$$\langle \mathcal{O}_L \rangle_\tau = \frac{1}{(2\pi)^L} \frac{2}{\sqrt{L}} \sinh \frac{\pi L}{\sqrt{\lambda}} \quad \overset{???}{\longleftrightarrow}$$



Unprotected operators and integrability

$$\mathcal{O} = \Psi^{I_1 \dots I_L} \, \mathrm{tr}\, \Phi_{I_1} \dots \Phi_{I_L} \qquad I_l = 1 \dots 6$$

$\downarrow$

wavefunction of length-L $SO(6)$ spin chain

One-loop dilatation op:

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_l \left(\underset{\text{perm.}}{2 - 2 P_{l,l+1}} + \underset{\text{trace}}{K_{l,l+1}} \right)$$

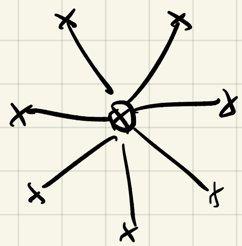
$$\Gamma | \Psi_n \rangle = \gamma_n | \Psi_n \rangle$$

$$\Delta = L + \gamma_n + \mathcal{O}(\lambda^2)$$

Boundary state

To the leading order:

$$\langle \mathcal{O} \rangle_\tau =$$



$$= \left(\frac{2\pi^2}{\lambda\tau^2}\right)^{\frac{L}{2}} \frac{1}{\sqrt{L}} \frac{\langle Bst | \Psi \rangle}{\langle \Psi | \Psi \rangle^{1/2}}$$

$$\Phi_I \longrightarrow \Phi_I^{(cl)} = \frac{B\, n_I}{2\tau}$$

$$Bst_{I_1 \ldots I_L} = n_{I_1} \ldots n_{I_L}$$

- projects all spins on a given axis

Bethe Ansatz

$$\left(\frac{u_{aj} - \frac{i q_a}{2}}{u_{aj} + \frac{i q_a}{2}}\right)^L \prod_{bk} \frac{u_{aj} - u_{bk} + \frac{i M_{ab}}{2}}{u_{aj} - u_{bk} - \frac{i M_{ab}}{2}} = -1$$

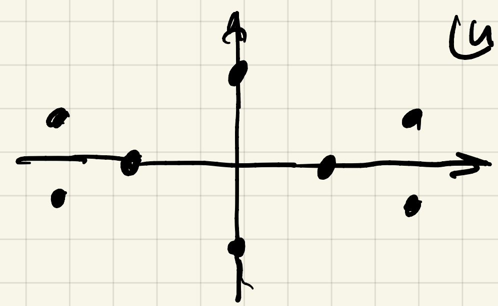
$$\circ \;-\!\!-\!\!-\; \overset{1}{\circ} \;-\!\!-\!\!-\; \circ$$

$$\{u_{1j}\}_{j=1}^{K_1} \qquad \{u_{2j}\}_{j=1}^{K_2} \qquad \{u_{3j}\}_{j=1}^{K_3}$$

$$\mu_a = L q_a - M_{ab} K_b$$

$$\gamma = \frac{\lambda}{8\pi^2} \sum_{aj} \frac{q_a^2}{u_{aj}^2 + \frac{q_a^2}{4}}$$

$\langle Bst|$ is integrable $\Longleftrightarrow$ $\mathbb{Z}_2$ symmetry: $\{u_{aj}\} = \{-u_{aj}\}$



$$\{u_{aj}\} = \{u_{aj}, -u_{aj}\}\Big|_{j=1}^{\frac{K_a}{2}}$$

Otherwise $\langle Bst|\{u_{aj}\}\rangle = 0$

Overlap formula

$$\langle \mathcal{O} \rangle_\tau = \left(\frac{\pi}{\sqrt{a}^{\,}\,4} \right)^{L} \sqrt{\frac{\prod_j u_{2j}^2 \left(u_{2j}^2 + \frac{1}{4} \right)}{L \prod_j u_{1j}^2 \left(u_{1j}^2 + \frac{1}{4} \right) \prod_j u_{3j}^2 \left(u_{3j}^2 + \frac{1}{4} \right)} \frac{\det G^+}{\det G^-}}$$

de Leeuw, Gombor, Kristjansen, Linardopoulos, Pozsgay '19

Gaudin superdeterminant:

$$G^{\pm}_{aj,bk} = \left(\frac{L q_a}{u_{aj}^2 + \frac{q_a^2}{4}} - \sum_{cl} K^{+}_{aj,cl} \right) \delta_{ab} \delta_{jk} + K^{\pm}_{aj,bk}$$

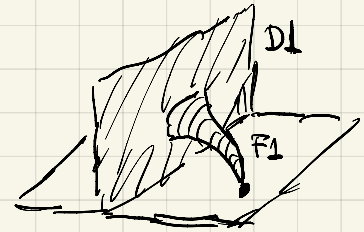
$$K^{\pm}_{aj,bk} = \frac{M_{ab}}{(u_{aj} - u_{bk})^2 + \frac{M_{ab}^2}{4}} \pm \frac{M_{ab}}{(u_{aj} + u_{bk})^2 + \frac{M_{ab}^2}{4}}$$

Brockmann, De Nardis, Wouters, Caux '14

Conclusions

- straight or circular 't Hooft loop in $\mathcal{N}=4$ SYM defines an integrable dCFT dual to D1 in $AdS_5 \times S^5$

- 1pt functions $\langle \mathcal{O}(x) \rangle_T$ can (in principle) be computed using integrability



- This talk:
  - protected op's
  - SO(6) sector at LO

- Larger sectors? NLO? Buhl-Mortensen, de Leeuw, Ipsen, Kristjansen, Wilhelm '16

All orders? Gombor, Bajnok '20

Komatsu, Wang '20

Buhl-Mortensen, de Leeuw, Ipsen, Kristjansen, Wilhelm '17

Exact, incl. wrapping corrections:

$$\langle \mathcal{O} \rangle_T = g\text{-function}$$

Jiang, Komatsu, Vescovi '19