Dyson-Selberg Integrals and aspects of quantum geometry

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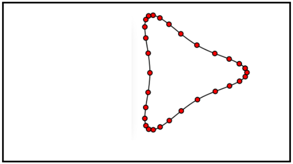
Fekete points (1923) - finite-dimensional approximation of conformal maps

- Riemann mapping $f(z)$: $\mathbb{C} \setminus \mathscr{D} \to \mathbb{C} \setminus \mathbb{D}$, $f(z) \to z$, $z \to \infty$ an exterior of a domain $\mathscr{D}$ to the exterior of a disk of a radius r,

$$f(z) = \lim_{N\to\infty} \left(\prod_{i=1}^{N} (z - \xi_i) \right)^{1/N}$$

- A set $\xi_1, \ldots, \xi_N$ are Fekete points, minimizing a Coulomb energy of a conductor

$$\min \sum_{i>j} [-\log |\xi_i - \xi_j|], \quad \xi_i \in \Gamma = \partial D$$



$$\sum_{j\neq i} \frac{1}{\xi_i - \xi_j} = 0, \quad \xi_i \in \Gamma$$

Density of Fekete points and Harmonic measure

- At large N images of Fekete points are uniformly distributed along a circle.
- Density of Fekete points approaches to **Harmonic measure**

$$\rho(z)|dz| \underset{N\to\infty}{\sim} \frac{1}{N}\sum_i \delta_\Gamma(z,\xi_i)|dz| = \frac{1}{2\pi}|f'(z)||dz|$$

- Loewner energy: $e^{-\phi} = |f'(z)|$, metric $|dz| = e^{\phi}|dw|$

$$E = -\sum_{i>j} \log|\xi_i - \xi_j| \underset{N\to\infty}{\to} \text{Loewner energy} = \frac{1}{8\pi}\oint_\Gamma \phi \hat{\mathcal{N}} \phi \, ds$$

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Neumann jump operator

$$\hat{\mathscr{N}} h = \mathrm{disc}_\Gamma\Big[\partial_n\Big(\text{Harmonic continuation of } h\Big)\Big].$$

Simple layer potential

$$\boxed{h(z) = -\frac{1}{2\pi}\oint_\Gamma \log|z-\xi|g(\xi)|d\xi|}$$

$$\hat{\mathscr{N}} h = g$$

Is there a version of "*quantum*" Fekete which approximates

Boundary CFT?

Dyson Diffusion

Stochastic quantization of Fekete condition

$$\sum_{j\neq i} \frac{dt}{\xi_i - \xi_j} = dB_i, \quad \xi_i \in \Gamma, \quad \mathbb{E}[dB_i dB_j] = 4\beta\,\delta_{ij} dt$$

The Gibbs distribution is the Dyson's measure

$$dP_N = Z_N^{-1} \prod_{i>j=1}^{N} |\xi_i - \xi_j|^{2\beta} d\mu \qquad d\mu = |d\xi_1| \dots |d\xi_N$$

$$\boxed{Z_{N,\beta}[\Gamma] = (2\pi)^{-N} \oint_\Gamma \prod_{i>j\geq 1}^{N} |\xi_i - \xi_j|^{2\beta} d\mu(\xi)}$$

Coulomb gas

Statistical mechanics of particles with repulsive log (Coulomb) interaction

$$Z_{N,\beta} = (2\pi)^{-N} \oint_\Gamma \prod_{i>j=1}^{N} |\xi_i - \xi_j|^{2\beta} d\mu(\xi), \qquad d\mu = |d\xi_1| \ldots |d\xi_N|$$

$$Z_N = \oint_\Gamma e^{-\beta E(\xi_1,\ldots,\xi_N)} d\mu(\xi)$$

$$E = -\sum_{i>j} \log|\xi_i - \xi_j|$$

Orthogonal polynomials

Determinantal point processes ($\beta = 1/2, 1, 2$) are related to orthogonal polynomials

Extension of Szego's theorem:

$$p_n(z) = \oint_\Gamma \prod_{i>j=1}^{n} |\xi_i - \xi_j|^2 \prod_{i=1}^{n} (z - \xi_i) d\mu(\xi_i)$$

are bi-orthogonal with the measure $d\mu(\xi)$

$$\oint_\Gamma p_n(z) \overline{p_m(z)} d\mu = h_n \delta_{nm}$$

Selberg Integral (1941 and 1944.)

1944 paper in "Bemerkninger om et multipelt integral" (Remarks on a multiple integral) published in Norsk Matematisk Tidsskrift.

$$\int_0^1 \prod_{i=1}^{N} \xi_i^{a_1-1}(1-\xi_i)^{a_2-1} \prod_{i>j} |\xi_i - \xi_j|^{2\beta} d\xi_1 \ldots d\xi_N =$$

$$= \prod_{j=0}^{N-1} \frac{\Gamma(a_1+\beta j)\Gamma(a_2+\beta j)\Gamma(1+\beta+\beta j)}{\Gamma(a_1+a_2+(N+j-1)\beta)\Gamma(1+\beta)}$$

Atle Selberg: born in 1917, passed on August 6th 2007, age 90

A good review by Peter Forrester and Ole Warnaar 2007:

"The importance of Selberg Integrals"

In 1941 Selberg published the result (also in Norwegian), not a detailed proof.

"This paper was published with some hesitation, and in Norwegian, since I was rather doubtful that the results were new. The journal is read by mathematics teachers in the gymnasium, ... "

Unfortunately, I have been unable to find the formula in the literature. To present proof here, however, seems inappropriate, as it would make this paper significantly longer. If it turns out that the formula is new, I intend to publish a proof at a later date."

Random Matrix Theory : Wigner 1950, Dyson, Mehta 1963

$$\int |\mathrm{Det}\, M|^a \, |\mathrm{Det}\,(z - M)|^b \, D\mu(M), \quad M = N \times N \text{ unitary matrix}$$

$$M = U^{-1} \mathrm{diag}\,(\xi_1, \dots \xi_N)\, U$$

$$D\mu(M) = D\mu(U) \prod_{i>j} |\xi_i - \xi_j|^2 d\xi_1 \dots d\xi_N$$

$$\oint_{S^1} \prod_{i=1}^{N} \xi_i^a |z - \xi_i|^b \prod_{i>j} |\xi_i - \xi_j|^{2\beta} d\xi_1 \dots d\xi_N, \qquad \beta = 1,$$

Integrable structures:

Z_N is the τ-function of the Toda lattice integrable hierarchy.

Dyson integral (1961-64)

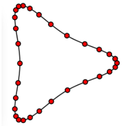
$$(2\pi)^{-N}\oint_{S^1}\prod_{N\geq i>j\geq 1}|\xi_i-\xi_j|^{2\beta}d|\xi_1|\ldots|d\xi_N| = \frac{\Gamma(1+N\beta)}{\Gamma^N(1+\beta)}$$

$$(2\pi)^{-N}\oint_{S^1}\prod_{i=1}^{N}\xi_i^{a_1-1}|1+\xi_i|^{2a_2}\prod_{i>j}|\xi_i-\xi_j|^{2\beta}|d\xi_1|\ldots|d\xi_N| =$$

$$\prod_{j=0}^{N-1}\frac{\Gamma(1+2a_2+\beta j)\Gamma(1+\beta+\beta j)}{\Gamma(1+a_1+a_2+\beta j)\Gamma(1+a_2-a_1+\beta j)\Gamma(1+\beta)}$$

Primary interest:

- Dyson integral on an arbitrary (not circular) simple closed contour at large $N \to \infty$.



$$Z_{N,\beta}[\Gamma] = (2\pi)^{-N} \oint_{\Gamma} \prod_{i>j\geq 1}^{N} |\xi_i - \xi_j|^{2\beta} |d\xi_1| \dots |d\xi_N|$$

Expectation values of operators (Dyson Integral)

$$\langle \mathcal{O} \rangle = Z^{-1} \int_\Gamma \prod_{i=1}^{N} \mathcal{O}(\xi_1, \ldots, \xi_N) \prod_{i>j} |\xi_i - \xi_j|^{2\beta} |d\xi_1| \ldots |d\xi_N|,$$

$$Z = \int_\Gamma \prod_{i>j} |\xi_i - \xi_j|^{2\beta} |d\xi_1| \ldots |d\xi_N|$$

- Vertex operator

$$\mathcal{O}_a(z|\xi) = \prod_{i=1}^{N} (z - \xi_i)^a$$

Elliptic operators and their determinants

Neumann jump operator

$$\hat{\mathscr{N}} h = \mathrm{disc}_\Gamma\Big[\partial_n\Big(\text{Harmonic continuation of } h\Big)\Big].$$

Simple layer potential

$$\boxed{h(z) = -\tfrac{1}{2\pi}\oint_\Gamma \log|z-\xi| g(\xi) |d\xi|}$$

$$\hat{\mathscr{N}} h = g$$

Double layer potential

$$\mathbb{V}g(z) = \frac{1}{\pi} \oint_{\Gamma} g(\xi)\, \partial_{n_\xi} \log|\xi - z| |d\xi|, \quad z \in \Gamma$$

Spectral determinants(quantum geometry)

- Neumann jump spectral determinant

$$\det{}' \hat{\mathscr{N}}$$

- Fredholm determinant

$$\det(\mathbb{I} + \mathbb{V})$$

- spectral determinant of Laplace operators

$$\det(-\Delta_{\text{int}}), \quad \det(-\Delta_{\text{ext}})$$

Relations between spectral determinants

- The determinant of the Neumann jump operator is the inverse of the Fredholm determinant

$$\log \det{}' \hat{\mathcal{N}} = -\log \det(\mathbb{I} + \mathbb{V}) + \log[\text{Perimeter}] .$$

- Surgery formula (S. Zetdich et al)

$$\boxed{\log \det(-\Delta_{\text{int}}) + \log \det(-\Delta_{\text{ext}}) + \log \det{}' \hat{\mathcal{N}} = \log P + \text{const}}$$

Polyakov's formula

Determinants are expressed through the harmonic measure of the curve

$$\phi_{\mathrm{int/ext}} = -\log |f'|_{\mathrm{int/ext}}$$

$$ds = e^{\phi} |dw|, \quad w := f(z) \in S^1$$

$$\boxed{\log \det \left(-\Delta_{\mathrm{int/ext}}\right) = \mp \frac{1}{12\pi} \oint_{|w|=1} \left(\phi_{\mathrm{int/ext}} \partial_n \phi_{\mathrm{int/ext}} + 2\phi_{\mathrm{int/ext}}\right) |dw|,}$$

Main result

$$Z_{N,\beta}[\Gamma] = (2\pi)^{-N} \oint_\Gamma \prod_{N \geq i > j \geq 1} |\xi_i - \xi_j|^{2\beta} d\mu(\xi) \underset{N\to\infty}{\rightarrow}$$

$$A_{N,\beta} \cdot e^{-\frac{(\beta-1)^2}{8\pi\beta}[\text{Loewner energy}]} \times \begin{cases} \frac{1}{\sqrt{P}} \cdot \sqrt{\det' \hat{\mathcal{N}}} \\ \frac{1}{\sqrt{\det'(\mathbb{1}+\mathbb{V})}} \\ \frac{1}{\sqrt{\det \Delta_{\text{int}} \cdot \det \Delta_{\text{ext}}}} \end{cases} + \mathcal{O}\left(\frac{1}{N}\right)$$

$$\phi = -\log|f'(z)|, \qquad \text{Loewner energy} = \frac{1}{8\pi} \oint_\Gamma \phi \hat{\mathcal{N}} \phi \, ds$$

$$Z_{N,\beta}[S^1] = \frac{\Gamma(1+N\beta)}{\Gamma^N(1+\beta)}$$

- A Field Theory defined in a bounded simply-connected domain $\mathscr{D}$ on a plane.
- There is a set of local operators called "**primary** " $\mathscr{O}_\alpha(z)$ which correlation functions are conformally covariant with respect to deformation of the boundary:

$$\langle \mathscr{O}_{h_1}(z_1)\mathscr{O}_{h_2}(z_2)\dots\rangle_{\mathscr{D}} = [f'(z_1)]^{h_1}[f'(z_2)]^{h_2}\dots\langle \mathscr{O}_{h_1}(f(z_1))\mathscr{O}_{h_2}(f(z_2))\dots\rangle_{\mathbb{D}}$$

$$f(z): \quad \mathscr{D} \to \mathbb{D}$$

- A set of h_k is called a set of **dimensions** of primary operators.

Central Charge, Background charge β, charges

Infinitesimal version:

$$f(z) = z + \epsilon(z)$$

$$\langle \delta_{\epsilon(z)} \mathcal{O}_h(z) \ldots \rangle \equiv \langle T(z) \mathcal{O}_h(z) \ldots \rangle \epsilon = (\epsilon \partial_z + h \partial_z \epsilon) \langle \mathcal{O}_h(z) \ldots \rangle$$

- Operator T is called stress energy tensor.
- CFT is characterized by a central charge.

$$\langle T(z) \rangle = \frac{c}{6}\{f, z\} = \frac{c}{6} \times \text{Schwarz derivative}$$

or parameter β such that

$$c = 1 - 6\left(\sqrt{\beta} - 1/\sqrt{\beta}\right)^2$$

- One-parametric family β.
- Customary to characterize operators by their charge:

$$h = \alpha(\alpha - \sqrt{\beta} + 1/\sqrt{\beta}), \quad a = \alpha\sqrt{\beta}$$

Emergent conformal symmetry

- Define

$$\mathcal{O}_h(z) = (f(z))^{-aN} \prod_i (z - \xi_i)^a, \quad z \notin \mathcal{D}$$

$$h = \alpha(\alpha - \sqrt{\beta} + 1/\sqrt{\beta}), \quad \alpha = a/\sqrt{\beta}$$

- Then

$$\langle \mathcal{O}(z) \rangle = \left(f'(z)\right)^{h/2}$$

$$\langle \mathcal{O}_{h_1}(z_1) \mathcal{O}_{h_2}(z_2) \dots \rangle_{\mathcal{D}} \approx (f'(z_1))^{h_1} (f'(z_2))^{h_2} \dots \left(\frac{f(z_k) - f(z_l)}{z_k - z_l} \right)^{\alpha_k \alpha_l} =$$

$$= (f'(z_1))^{h_1} (f'(z_2))^{h_2} \dots \langle \mathcal{O}_{h_1}(f(z)_1) \mathcal{O}_{h_2}(f(z)_2) \dots \rangle_{\mathbb{D}}$$

Comments on derivation

Gaussian field

- Gaussian Field

$$\varphi(z) = -\frac{2}{N} \mathrm{Im} \sum_i \beta \log(z - \xi_i)$$

- Expectation value

$$\langle \varphi(z) \rangle = \mathrm{Im}\left(-2\beta \log f(z) + \frac{2}{N}(\beta - 1) \log f' \right),$$

$$\partial_s \langle \varphi(z) \rangle |_\Gamma = (\beta - 1) \cdot \text{curvature}$$

- Correlation function

$$\beta^{-1} N^2 \langle \varphi(z) \varphi(\zeta) \rangle_c = G(z, \zeta) - \log |z - \zeta|$$

$G(z, \zeta)$ - Dirichlet Green function

- Schwarz derivative

$$\langle (\partial \varphi(z))^2 \rangle_c = \frac{\beta}{6} \{f, z\}$$

Stress Energy Tensor

- Holomorphic component of s.e. tensor

$$T(z) = -(\partial\varphi(z))^2 - \frac{i}{N}(1-\beta)\,\partial^2\varphi(z)$$

- Boundary components of s.e. tensor

$$2T_{sn} = \mathrm{Im}(\nu^2 T), \quad z \in \Gamma$$

$$2T_{nn} = \mathrm{Re}(\nu^2 T), \quad z \in \Gamma$$

ν is a normal vector to the boundary

Conformal Boundary Conditions and Ward Identity

At large N CFT conditions emerge

- Conformal Boundary conditions: sn-component continuous through the boundary

$$\boxed{\mathrm{disc}\,\langle [T_{sn}(z)]_\Gamma \rangle = 0}$$

- Conformal Ward Id: nn component generates a deformation of the boundary

$$\boxed{N^{-2}\delta \log Z_N = -\frac{1}{2\pi\beta}\oint_\Gamma \langle T_{nn}(z)\rangle \delta n(z)\,|dz|}$$

- A "quantum" version of Hadamard formula for variation of conformal maps.