Reflection relations and singular vectors

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1. Reflection relations and fermionic basis in quotient space

Central charge

$$c = 1 - \frac{6}{p(p+1)}\,,$$

primary field Φ_α has dimension

$$\Delta_\alpha = \frac{\alpha(\alpha-2)}{4p(p+1)}\,.$$

Here we work modulo left action of local integrals of motion $\mathbf{i}_{2j-1}$.
Consider Verma module of degree L. Virasoro basis is generated by $\mathbf{l}_{-2k}$ ($p(L/2)$ of them), Heisenberg basis is generated by produced by $p(L/2)$ even monomials of a_{-j} (not unique).

The matrix $\overline{U}^{(L)}(\alpha)$ in the quotient space is defined by

$$\mathbf{v}_i\Phi_\alpha \equiv \overline{U}^{(L)}(\alpha)_{i,j}\mathbf{h}_j\Phi_\alpha\,,$$

Two reflections : $\alpha \to 2-\alpha$ and $\alpha \to -\alpha$.

Fermions. As usual for $I=\{2i_1-1,\cdots,2i_m-1\}$,

$$\boldsymbol{\beta}^*_I=\boldsymbol{\beta}^*_{2i_1-1}\cdots\boldsymbol{\beta}^*_{2i_m-1},\quad \boldsymbol{\gamma}^*_I=\boldsymbol{\gamma}^*_{2i_m-1}\cdots\boldsymbol{\gamma}^*_{2i_1-1}\,.$$

It is convenient to extract transcendental multipliers

$$\boldsymbol{\beta}^*_{2j-1}=D_{2j-1}(\alpha)\boldsymbol{\beta}^{\mathrm{CFT}\,*}_{2j-1}\,,\quad \boldsymbol{\gamma}^*_{2j-1}=D_{2j-1}(2-\alpha)\boldsymbol{\gamma}^{\mathrm{CFT}\,*}_{2j-1}\,,$$

where

$$D_l(\alpha)=\frac{\Gamma\left(\frac{1}{2}(\alpha+l(1+p))\right)}{\left(\frac{l-1}{2}\right)!\Gamma\left(\frac{1}{2}(\alpha+lp)\right)}\,,$$

and

$$\boldsymbol{\beta}^{\mathrm{CFT}\,*}_{I^+}\boldsymbol{\gamma}^{\mathrm{CFT}\,*}_{I^-}\Phi_\alpha=C_{I^+,I^-}\Big(P^{\mathrm{even}}_{I^+,I^-}(\{\mathbf{l}_{-2k}\})+d_\alpha P^{\mathrm{odd}}_{I^+,I^-}(\{\mathbf{l}_{-2k}\})\Big)\Phi_\alpha\,.$$

with

$$d_\alpha=\frac{(1-\alpha)(2p+1)}{p(p+1)},\quad C_{I^+,I^-}=\det\Big(\frac{2}{a+b}\Big)_{\substack{a\in I^+\\ b\in I^-}}\,.$$

The coefficients of the polynomials P^{even}, P^{odd} are symmetric under $\alpha\leftrightarrow 2-\alpha$, in other words, they depend on Δ_α only.

Later we shall use $\mathbf{i}_I=\mathbf{i}_{2i_1-1}\cdots\mathbf{i}_{2i_m-1}$.

Reflection relations

$$f_{I^+,I^-}(\alpha) = \prod_{a\in I^+} \frac{\alpha + a(p+1)}{\alpha + ap} \prod_{a\in I^-} \frac{\alpha - ap}{\alpha - a(p+1)} f_{I^+,I^-}(\alpha+2)\,\overline{S}^{(l)}(\alpha)\,,$$

where

$$\overline{S}^{(l)}(\alpha) = \overline{U}^{(l)}(-\alpha)\overline{U}^{(l)}(\alpha)^{-1},$$

Examples:

$$\overline{U}^{(2)}(\alpha) = \frac{(p-\alpha)(\alpha+p+1)}{4p(1+p)}\,,$$

$$\overline{U}^{(4)}(\alpha) = \frac{1}{288}\begin{pmatrix} 2\big(9 - \frac{\alpha^3(1+\alpha)}{p^2(1+p)^2} - \frac{3\alpha(2+\alpha)}{p(1+p)}\big) & 24\big(\frac{(1+\alpha)(3+4\alpha)}{p(1+p)} - 3\big) \\ \frac{\alpha^2}{p(p+1)}\big(6 - \frac{\alpha(3+\alpha)}{p(p+1)}\big) & \frac{12\alpha(3+\alpha)}{p(1+p)} \end{pmatrix}.$$

2. Solution for quotient space

Introducing

$$\det\left(\overline{U}^{(L)}(\alpha)\right) = N^{(L)}(\alpha)\frac{D_V^{(L)}(\Delta_\alpha)}{D_H^{(L)}(\alpha^2)} \ ,$$

we look for the solution in the form

$$P^{\text{even}}_{I^+,I^-} = \mathbf{v}_1 + \frac{1}{D_V^{(k)}(\Delta, c)} \sum_{i=2}^{p(k/2)} X_{I^+,I^-,i}(\Delta, c)\mathbf{v}_i \ ,$$

$$P^{\text{odd}}_{I^+,I^-} = \frac{1}{D_V^{(k)}(\Delta, c)} \sum_{i=2}^{p(k/2)} Y_{I^+,I^-,i}(\Delta, c)\mathbf{v}_i \ .$$

The polynomials $X_{I^+,I^-,i}(\Delta, c)$, $Y_{I^+,I^-,i}(\Delta, c)$ have degree D in Δ, the degree D is a parameter of our construction.

We require consistency with the Heisenberg basis:

$$\boldsymbol{\beta}^{\mathrm{CFT}\,*}_{I^+}\boldsymbol{\gamma}^{\mathrm{CFT}\,*}_{I^-}\Phi_\alpha = C_{I^+,I^-}\prod_{a\in I^+}(\alpha+a(p+1))\prod_{a\in I^-}(\alpha-ap)$$
$$\times\Big(Q^{\mathrm{even}}_{I^+,I^-}(\{a_{-k}\})+g_\alpha Q^{\mathrm{odd}}_{I^+,I^-}(\{a_{-k}\})\Big)\Phi_\alpha\,,\quad g(\alpha)=\frac{2p+1}{2}\alpha\,.$$

The coefficients of the polynomials Q^{even}, Q^{odd} are even in α.
Introduce even and odd under $p\leftrightarrow -p-1$ parts

$$T^{+}_{I^+,I^-}(\alpha)=\frac{1}{2}\Big\{\prod_{a\in I^+}(\alpha-a(p+1))\prod_{a\in I^-}(\alpha+ap)$$
$$+\prod_{a\in I^+}(\alpha+ap)\prod_{a\in I^-}(\alpha-a(p+1))\Big\}\,,$$
$$T^{-}_{I^+,I^-}(\alpha)=\frac{1}{2(2p+1)}\Big\{\prod_{a\in I^+}(\alpha-a(p+1))\prod_{a\in I^-}(\alpha+ap)$$
$$-\prod_{a\in I^+}(\alpha+ap)\prod_{a\in I^-}(\alpha-a(p+1))\Big\}\,.$$

This consistency with the Heisenberg basis leads to the requirements

First, the polynomial

$$D_V^{(L)}(\Delta_{-\alpha})D_H^{(L)}(\alpha^2)$$
$$\times \Big\{T^{+}_{I^+,I^-}(\alpha)\Big(D_V^{(L)}(\Delta_\alpha)U^{(L)}_{1,j}(\alpha) + \sum_{i=2}^{p(L/2)} X_{I^+,I^-,i}(\Delta_\alpha)U^{(L)}_{i,j}(\alpha)\Big)$$
$$- \frac{(2p+1)(1-\alpha)}{p(p+1)}T^{-}_{I^+,I^-}(\alpha)\sum_{i=2}^{p(L/2)} Y_{I^+,I^-,i}(\Delta_\alpha)U^{(L)}_{i,j}(\alpha)\Big\} \quad \text{is even in } \alpha\,,$$

Second,

$$D_V^{(L)}(\Delta_{-\alpha})D_H^{(L)}(\alpha^2)$$
$$\times \Big\{T^{-}_{I^+,I^-}(\alpha)\Big(D_V^{(L)}(\Delta_\alpha)U^{(L)}_{1,j}(\alpha) + \sum_{i=2}^{p(L/2)} X_{I^+,I^-,i}(\Delta_\alpha)U^{(L)}_{i,j}(\alpha)\Big)$$
$$+ \frac{1-\alpha}{p(p+1)}T^{+}_{I^+,I^-}(\alpha)\sum_{i=2}^{p(L/2)} Y_{I^+,I^-,i}(\Delta_\alpha)U^{(L)}_{i,j}(\alpha)\Big\} \quad \text{is odd in } \alpha\,,$$

The unknown are the coefficients of the polynomials $X_{I^+,I^-,i}(\Delta)$ and $Y_{I^+,I^-,i}(\Delta)$. The number of unknowns is

$$\#(\text{unknowns}) = 2(p(L/2)-1)(D+1)\,.$$

The number of equations is

$$\#(\text{equations}) = p(L/2)$$
$$\times\left(2\text{deg}_\Delta(D_V^{(L)}(\Delta,c)) + \text{deg}_\alpha(D_H^{(L)}(\alpha^2,Q^2)U^{(L)}(a)) + 2\#(I^+) + 2D+1\right).$$

For sufficiently large D the system is overdetermined, and the very existence of solution is a miracle produced by our fermionic basis.

The solutions have been found up to level 12.

It seems for $L\geq 6$, $\text{deg}_\Delta(D_V^{(L)}(\Delta,c)) = (L/2-3)^2+1$ and the degree of the numerator is $\text{deg}_\Delta(D_V^{(L)}(\Delta,c))+1$.

Example

$$P^{\text{even}}_{\{1,3\},\{3,1\}} = \mathbf{l}^4_{-2} + \frac{4(c-22)}{3}\mathbf{l}_{-4}\mathbf{l}^2_{-2} - \frac{1}{9}\big(8(c-25)\Delta_\alpha + c^2 - 34c + 333\big)\mathbf{l}^2_{-4}$$
$$+ \frac{2}{15}\big(8(c-28)\Delta_\alpha + 5c^2 - 193c + 1544)\big)\mathbf{l}_{-6}\mathbf{l}_{-2} - \frac{4}{3}(24\Delta_\alpha + 11c - 71)\mathbf{l}_{-8}$$
$$- \frac{5c-122}{42(\Delta_\alpha+4)}\mathbf{w}^{(8)}_4 - \frac{1}{42(\Delta_\alpha+11)}(5c^2 - 526c + 8648)\mathbf{w}^{(8)}_{11}$$

where

$$\mathbf{w}^{(8)}_4 = 28\mathbf{l}_{-4}\mathbf{l}^2_{-2} - 3(c-36)\mathbf{l}^2_{-4} + 2(5c-12)\mathbf{l}_{-6}\mathbf{l}_{-2} - (5c^2 - 325c + 4128)\mathbf{l}_{-8}\,,$$
$$\mathbf{w}^{(8)}_{11} = 3\mathbf{l}^2_{-4} + 4\mathbf{l}_{-6}\mathbf{l}_{-2} + (5c-89)\mathbf{l}_{-8}\,.$$

3. Lifting the quotient space restriction

We have the congruency valid in the quotient space

$$\prod_{l\in I^+} D_l(\alpha) \prod_{l\in I^-} D_l(2-\alpha)\Big(P^{\rm even}_{I^+,I^-}(\{\mathbf{l}_{-2k}\}) + d_\alpha P^{\rm odd}_{I^+,I^-}(\{\mathbf{l}_{-2k}\})\Big)\Phi_\alpha$$

$$\equiv \prod_{l\in I^+}(\alpha+(p+1)l)D_l(\alpha)\prod_{l\in I^-}(\alpha-pl)D_l(2-\alpha)$$

$$\times\Big(Q^{\rm even}_{I^+,I^-}(\{a_{-k}\}) + g_\alpha Q^{\rm odd}_{I^+,I^-}(\{a_{-k}\})\Big)\Phi_\alpha\,.$$

In the complete space

$$\prod_{l\in I^+}(\alpha+(p+1)l)D_l(\alpha)\prod_{l\in I^-}(\alpha-pl)D_l(2-\alpha)$$

$$\times\Big(Q^{\rm even}_{I^+,I^-}(\{a_{-k}\}) + g_\alpha Q^{\rm odd}_{I^+,I^-}(\{a_{-k}\})\Big)\Phi_\alpha$$

$$=\prod_{l\in I^+} D_l(\alpha) \prod_{l\in I^-} D_l(2-\alpha)\Big(P^{\rm even}_{I^+,I^-}(\{\mathbf{l}_{-2k}\}) + d_\alpha P^{\rm odd}_{I^+,I^-}(\{\mathbf{l}_{-2k}\})\Big)\Phi_\alpha$$

$$+\sum_{\substack{|I^+|+|I^-|\\=|K|+|J^+|+|J^-|}} C^{K,J^+,J^-}_{I^+,I^-}(\alpha)\mathbf{i}_K\,\boldsymbol{\beta}^*_{J^+}\boldsymbol{\gamma}^*_{J^-}\,\Phi_\alpha\,,$$

In order to satisfy the reflection $\alpha \to 2-\alpha$, $I^+ \leftrightarrow I^-$ we add the sum

$$\sum_{\substack{|I^+|+|I^-| \\ =|K|+|J^+|+|J^-|}} B^{K,J^+,J^-}_{I^+,I^-}(\alpha)\mathbf{i}_K \,\boldsymbol{\beta}^*_{J^+}\boldsymbol{\gamma}^*_{J^-}\,\Phi_\alpha\,.$$

Since we do not want to spoil the $\alpha \to -\alpha$ symmetry we require

$$B^{K,J^-,J^+}_{I^+,I^-}(\alpha) = B^{K,J^-,J^+}_{I^+,I^-}(-\alpha)\,.$$

Now we want to satisfy the $\alpha \to 2-\alpha$ symmetry. Since the first term in the right hand side respects this symmetry, the requirement reduces to

$$C^{K,J^+,J^-}_{I^+,I^-}(\alpha) + B^{K,J^+,J^-}_{I^+,I^-}(\alpha) = C^{K,J^-,J^+}_{I^-,I^+}(2-\alpha) + B^{K,J^-,J^+}_{I^-,I^+}(2-\alpha)\,.$$

All this leads to the linear difference equation for

$$B^{K,J^+,J^-}_{I^+,I^-}(\alpha) - B^{K,J^+,J^-}_{I^+,I^-}(\alpha-2) = C^{K,J^-,J^+}_{I^-,I^+}(2-\alpha) - C^{K,J^+,J^-}_{I^+,I^-}(\alpha)\,.$$

It is quite remarkable that the non-linearity of the problem is completely taken care of by out quotient space calculation.

Example of level 2

$$\boldsymbol{\beta}_1^*\boldsymbol{\gamma}_1^*\Phi_\alpha = \Big\{D_1(\alpha)D_1(2-\alpha)\mathbf{l}_{-2} + A(\alpha)\mathbf{i}_1^2\Big\}\Phi_\alpha \,.$$

where

$$A(\alpha) = \widetilde{A}(\alpha) - D_1(\alpha)D_1(2-\alpha)\frac{\alpha+1}{\alpha} + \frac{i}{4}\cot\frac{\pi p}{2}C_1(p)^2\,,$$

BLZ constants:

$$C_{2n-1}(p) = -\sqrt{\pi p(p+1)}\,\frac{\Gamma\left(\frac{1}{2}(2n-1)(1+p)\right)}{n!\Gamma\left(1+\frac{1}{2}(2n-1)p\right)}$$

The main part is

$$\widetilde{A}(\alpha) = \sin\tfrac{\pi}{2}(\alpha-p)\sin\tfrac{\pi}{2}(\alpha+p)$$
$$\times\frac{1}{\pi^2 i}\int\limits_{-\infty}^{\infty}\tanh\tfrac{\pi}{2}(t+i\alpha)\left|\Gamma\left(\frac{1-p+it}{2}\right)\Gamma\left(\frac{2+p-it}{2}\right)\right|^2\frac{t}{1+t^2}dt\,.$$

We require analyticity for $0<\alpha<2$ (natural), bounded for $\mathrm{Im}(\alpha)\to\pm\infty$ (not so obvious).

Our results must agree with the null-vectors.

The null-vectors allow rather simple description in the fermionic basis which goes back to the quantum Riemann bilinear relation.

Introduce

$$\mathcal{Q}_k = \operatorname*{res}_{z=\infty} \boldsymbol{\tau}^*(z)^{1/2}\boldsymbol{\gamma}(z)\frac{dz}{z^{k+1}}\,,$$

$$\mathcal{C}_k = \operatorname*{res}_{z=\infty}\Big(\boldsymbol{\beta}^*(z) + \operatorname*{res}_{w=\infty}\Big(t_k(z/w,\alpha)\boldsymbol{\tau}^*(z)^{\frac{1}{2}}\boldsymbol{\tau}^*(w)^{\frac{1}{2}}\boldsymbol{\gamma}(w)\frac{dw}{w}\Big)\Big)\boldsymbol{\gamma}(z)\frac{dz}{z^{2k+1}}\,.$$

where

$$\boldsymbol{\tau}^*(z) = \exp\Big(\sum_{j=1}^{\infty} C_{2j-1}(p)z^{-(2j-1)}\mathbf{i}_{2j-1}\Big)$$

and

$$t_k(z,\alpha) = \tfrac{1}{2}t_0(\alpha) + \sum_{j=1}^{k-1}(-1)^j\cot\tfrac{\pi}{2}(\alpha-(p+1)j)z^j\,.$$

This is a periodical function of α with period 2.

The null-vectors for $\Phi_{m,n}$, $n \geq m$:

- Even n: $\mathcal{C}_{n-m}^{\frac{n}{2}} \mathcal{H}_{-n}$
- Odd mn: $\mathcal{Q}_{n-m} \mathcal{C}_{n-m}^{\frac{n-1}{2}}$, and $\mathcal{C}_{n-m}^{\frac{n+1}{2}} \mathcal{H}_{-n-1}$

Singular vector on level 2:

$$\mathcal{C}_1 \boldsymbol{\gamma}_3^* \boldsymbol{\gamma}_1^* \Phi_{1,2} = \boldsymbol{\beta}_1^* \boldsymbol{\gamma}_1^* \Phi_{1,2} = D_1(p) D_1(2-p) \Big(\mathbf{l}_{-2} - \frac{p+1}{p} \mathbf{l}_{-1}^2 \Big)$$

$$\lim_{\alpha \to p} \frac{\sin \frac{\pi}{2}(\alpha - p) \sin \frac{\pi}{2}(\alpha + p)}{\pi^2 i} \int\limits_{-\infty}^{\infty} \frac{t \tanh \frac{\pi}{2}(t + i\alpha) \left| \Gamma\left(\frac{1-p+it}{2}\right) \Gamma\left(\frac{2+p-it}{2}\right) \right|^2}{1+t^2} dt$$

$$= -\frac{i}{4} \cot \frac{\pi p}{2} C_1(p)^2 \,.$$

This is not very impressive.

On the level 3 here is one more identity which includes the fermions $\boldsymbol{\beta}_1^*\boldsymbol{\gamma}_1^*$ only:

$$\mathcal{Q}_2\mathcal{C}_2\boldsymbol{\gamma}_5^*\boldsymbol{\gamma}_3^*\boldsymbol{\gamma}_1^*\Phi_{1,3}$$
$$=\Bigl(C_1(p)\mathbf{i}_1\boldsymbol{\beta}_1^*\boldsymbol{\gamma}_1^* - \frac{1}{24}\bigl((3\cot(\pi p)+2\tan(\tfrac{\pi}{2}p))C_1(p)^3\mathbf{i}_1^3 - 24\tan(\tfrac{\pi}{2}p)C_3(p)\mathbf{i}_3\Bigr)\Phi_{1,3}$$
$$=C_1(p)D_1(2p)D_1(2-2p)\Bigl(\mathbf{i}_1\mathbf{l}_{-2}+\frac{1+3p}{6(1-p)}\mathbf{i}_3-\frac{1+p}{3p(1-p)}\mathbf{i}_1^3\Bigr)\Phi_{1,3}\,.$$

- The coefficient of $\mathbf{i}_1\mathbf{l}_{-2}$ is automatically correct.
- The coefficient of $\mathbf{i}_3$ is non-trivial, but it follows from gamma-function identity

$$C_3(p)=\frac{1+3p}{6(1-p)}C_1(p)D_1(2p)D_1(2-2p)\cot(\tfrac{\pi}{2}p)\,.$$

- The coefficient of $\mathbf{i}_1^3$ leads to quite non-trivial integral identity.

Lemma. The following integral can be evaluated exactly

$$\int\limits_{-\infty}^{\infty} \tanh \tfrac{\pi}{2}(t+2pi)\left|\Gamma\big(\tfrac{1}{2}(1-p+it)\big)\Gamma\big(\tfrac{1}{2}(2+p+it)\big)\right|^2 \frac{t}{t^2+1}dt$$

$$= \frac{\pi p(p+1)}{12\sin(\pi p)}\Big(\Gamma\big(-\tfrac{1}{2}p\big)\Gamma\big(\tfrac{1}{2}(p+1)\big)\Big)^2$$

$$+\frac{1}{4}(2p-1)(p+1)\Gamma\big(-\tfrac{1}{2}(p+1)\big)\Gamma\big(\tfrac{3}{2}(p+1)\big)\Gamma\big(\tfrac{1}{2}p\big)\Gamma\big(-\tfrac{3}{2}p\big)\,.$$

Here we imply $|p| < 1/2$, for other values the integral is to be continued analytically since the poles start to cross the real line.

Proof We shall proceed as when proving the Barnes lemmas: computing certain integral by shift of variables ones derives the functional equation with respect to the parameter. Introduce the notation

$$g(t,p) = \tanh \tfrac{\pi}{2}(t{+}2pi)\Gamma\bigl(\tfrac{1}{2}(1{-}p{+}it)\bigr)\Gamma\bigl(\tfrac{1}{2}(1{-}p{-}it)\bigr)\Gamma\bigl(\tfrac{1}{2}(p{+}it)\bigr)\Gamma\bigl(\tfrac{1}{2}(p{-}it)\bigr)\,,$$

where we avoid using the absolute values since $g(t,p)$ will be needed as an analytical function. Set

$$f(t,p) = \tfrac{i}{2}(1-p+it)(p+it)g(t,p)\,.$$

We have

$$tg(t,p) = f(t+2i) - f(t)\,,$$

which allows to compute

$$\int_{-\infty}^{\infty} tg(t,p)dt = -2\pi i \operatorname*{res}_{t=i(1-2p)} f(t,p)dt\,,$$

obviously the simple pole at $t = i(1-2p)$ is the only singularity of $f(t,p)$ in the strip $0 \leq \mathrm{Im}(t) \leq 2$.

Denote the integral in the left hand side of Lemma by $I(p)$, its integrand is

$$\mathcal{I}(t,p) = t\frac{t^2+p^2}{4(t^2+1)}g(t,p)\,.$$

The function $I(p-2)$ is obtained by the analytical continuation

$$I(p-2) = \Big(\int\limits_{-\infty}^{\infty} - 2\pi i\,\big(\operatorname*{res}_{t=i(1-2p)} + \operatorname*{res}_{t=i(3-2p)}\big)\Big)\frac{t^2+(p-1)^2}{4(t^2+1)}tg(t,p)dt\,.$$

We find the combination of $I(p)$ and $I(p-2)$ in which the integrand with the denominator t^2+1 cancels:

$$p(p-2)I(p) - (p^2-1)I(p-2) = \frac{1-2p}{4}\int\limits_{-\infty}^{\infty} tg(t,p)dt$$

$$+\,2\pi i(p^2-1)\big(\operatorname*{res}_{t=i(1-2p)} + \operatorname*{res}_{t=i(3-2p)}\big)\frac{t^2+(p-1)^2}{4(t^2+1)}tg(t,p)dt$$

$$= \frac{p(1+p)(9p^3-34p^2+47p-18)}{2(-2+p)(-1+3p)(1+3p)}\Gamma\big(-\tfrac{1}{2}(p+1)\big)\Gamma\big(\tfrac{1}{2}(p+1)\big)\Gamma\big(\tfrac{1}{2}p\big)\Gamma\big(-\tfrac{3}{2}p\big)\,.$$

Form this identity it is easy to conclude conclude that the function

$$G(p) = \Big(\Gamma\big(-\tfrac{1}{2}p\big)\Gamma\big(\tfrac{1}{2}(p+1)\big)\Big)^{-2}$$
$$\times \Big(I(p) - \frac{1}{4}(2p-1)(p+1)\Gamma\big(-\tfrac{1}{2}(p+1)\big)\Gamma\big(\tfrac{3}{2}(p+1)\big)\Gamma\big(\tfrac{1}{2}p\big)\Gamma\big(-\tfrac{3}{2}p\big)\Big)\Big),$$

is periodic:

$$G(p) = G(p-2)\,.$$

This function decreases as $O(e^{-\pi|t|})$ as $t \to \pm i\infty$. To see this it is convenient to symmetrise the integrand in $I(p)$ which amounts to replacing $\tanh\frac{\pi}{2}(t+2pi)$ by $\sinh(\pi t)/|\cosh\frac{\pi}{2}(t+2pi)|^2$. Then even if the first multiplier in the right hand side grows, the expression in brackets decays faster. It remains to figure out the singularities. They are simple poles at integer p, in particular at this points the contour of integration in $I(p)$ is pinched by singularities of the integrand. So the periodic function in question is proportional to $1/\sinh\pi p$, the coefficient $\frac{\pi}{12}$ can be easily found by evaluation at the point $p = 0$.

QED