# The q-deformed Haldane-Shastry model at q=i


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also J. Lamers, V. Pasquier, D.S., arXiv:2004.13210

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# Plan

- Introduction: the nearest-neighbour XXZ spin chain
- Long-range integrable models: the isotropic Haldane-Shastry model
- The q-deformed Haldane-Shastry model
- The q=i limit: definition and spectrum
- Conclusions and open questions

# The nearest-neighbour XXZ spin chain

spin 1/2 su(2) representation

Heisenberg model:

$$H_{\mathrm{XXZ}} = J \sum_{j=1}^{N} \frac{1}{2} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z - \Delta \right)$$

$$\Delta = \frac{\mathrm{q} + \mathrm{q}^{-1}}{2}$$

$\mathrm{q} \to 1 \quad \longrightarrow \quad \Delta \to 1$

isotropic Heisenberg model:

$$H_{\mathrm{XXX}} = J \sum_{j=1}^{N} (P_{j,j+1} - 1)$$

spin permutation

# The nearest-neighbour XXZ spin chain

Boundary conditions and symmetry:

the isotropic XXX model possesses su(2) symmetry $[H_{\mathrm{XXX}}, S^a] = 0\,, \quad a = x, y, z$

the periodic XXZ model has only u(1) symmetry $[H_{\mathrm{XXZ}}, S^z] = 0 \qquad S^a = \frac{1}{2}\sum_{j=1}^{N} \sigma^a$

for particular boundary conditions, the open XXZ chain has $U_q sl(2)$ symmetry

$$H_{\mathrm{XXZ}}^{\mathrm{open}} = \sum_{j=1}^{N-1} h_{[j,j+1]} + \frac{\mathrm{q}-\mathrm{q}^{-1}}{4}(\sigma_1^z - \sigma_N^z) = -\sum_{j=1}^{N-1} e_j$$

**[Pasquier, Saleur, 90]**

Temperley-Lieb (TL) generators: $e_j = -h_{[j,j+1]} - \frac{\mathrm{q}-\mathrm{q}^{-1}}{4}(\sigma_j^z - \sigma_{j+1}^z)$

TL algebra:

$$\begin{aligned} e_j^2 &= (\mathrm{q}+\mathrm{q}^{-1})\, e_j \\ e_j e_{j\pm 1} e_j &= e_j\,, \\ e_j e_k &= e_k e_j \qquad (\text{for } j \neq k,\ k \pm 1)\,, \end{aligned}$$

# The nearest-neighbour XXZ spin chain

Boundary conditions and symmetry:

The closed chain can have twisted b.c: $\sigma^z_{N+1} = \sigma^z_1$ and $\sigma^\pm_{N+1} = e^{\mp i\phi}\sigma^\pm_1$

For $|\mathrm{q}| = 1,\ \Delta \leq 1$ in the large size limit, close to the antiferromagnetic state, the system is approximated by a 2d CFT $\longrightarrow$ lattice regularisation of (non-unitary) CFTs

**[Koo, Saleur, 94]**

The (effective) central charge depends on the twist $$c = 1 - \frac{3}{2}\frac{\phi^2}{\pi(\pi-\gamma)} \qquad \mathrm{q} = e^{i\gamma}$$

**[Klumper, Batchelor, Pearce, 91]**

Special points: $\phi = 0 \qquad c = 1$ ,

$$\phi = \pm 2\gamma \qquad c = 1 - 6\frac{\gamma^2}{\pi(\pi-\gamma)}$$

When $\mathrm{q} = e^{i\frac{\pi}{k+1}}$ (q root of unity) we retrieve known CFTs, *e.g.*

$$\mathrm{q} = i \qquad c = 1 \quad \text{or} \quad -2$$

# The nearest-neighbour XXZ spin chain

$\mathrm{q} = i$ : $c = 1$ or $-2$

At $\Delta = 0$ the model can be solved by the Jordan-Wigner transformation:

$$c_j^+ = e^{\frac{i\pi}{2}\sum_{l=1}^{j-1}(\sigma_l^z+1)}\,\sigma_j^+ \,, \qquad c_j = e^{-\frac{i\pi}{2}\sum_{l=1}^{j-1}(\sigma_l^z+1)}\,\sigma_j^- \qquad\qquad \{c_j^+, c_k\} = \delta_{jk}$$

$$H_{\mathrm{XX}} = \sum_{j=1}^{N}\left(\sigma_{j+1}^+\sigma_j^- + \sigma_j^+\sigma_{j+1}^-\right) = \sum_{j=1}^{N}\left(c_j^+ c_{j+1} + c_{j+1}^+ c_j\right)$$

the boundary conditions for the chain influence the boundary conditions for the fermions hence the change in the central charge (Dirac versus symplectic fermions in the CFT limit)

for twisted boundary conditions the chain can be mapped to an alternating gl(1|1)-symmetric spin chain via the non-unitary transformation

$$f_j^+ = (-i)^j\, c_j^+ \,, \qquad f_j = (-i)^j\, c_j \,, \qquad \{f_j^+, f_k\} = (-1)^j\,\delta_{jk}$$

$\longrightarrow$ the spectrum contains indecomposable representations [Gainutdinov, Read, Saleur, 11]

# The isotropic Haldane Shastry Hamiltonian $\quad q \to 1$

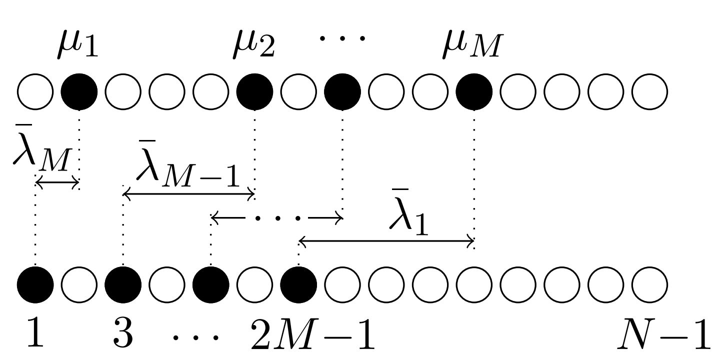
**[Haldane, 88; Shastry, 88]**

N spins 1/2 on a circle with periodic boundary conditions $\quad z_j \longmapsto \omega^j = \mathrm{e}^{2\pi \mathrm{i} j/N}$

$$H_{\mathrm{HS}} = -\sum_{i \neq j} V(z_i, z_j)\, P_{ij} \qquad V(z_i, z_j) = \frac{z_i z_j}{(z_i - z_j)^2} = \frac{1}{\sin^2 \pi(i-j)/N}$$

The model has much more symmetry (Yangian) compared to its nearest-neighbour cousin XXX

the spectrum is much simpler, in particular there are no bound states; encoded in motifs (collections of M integers between 1 and N-1)




M magnon motif

$$\bar\lambda_m = \mu_{M-m+1} - 2(M-m) - 1 \qquad \mu_{m+1} > \mu_m + 1$$

“vacuum” M magnon motif

$$E(\mu) - E_0 = \sum_{m=1}^{M} \varepsilon(\mu_m) = \sum_{m=1}^{M} \mu_m (N - \mu_m)$$

# The isotropic Haldane Shastry Hamiltonian

The solution of the Haldane-Shastry model in not obtained by Bethe Ansatz but from the link with the spin Calogero-Sutherland model:

**[Polychronakos 92, Bernard, Gaudin, Haldane, Pasquier, 93]**

Dynamical model:

$$\widetilde{H}^{\mathrm{CS}}_{\pm} = \frac{1}{2}\sum_{j=1}^{N} (z_j\,\partial_{z_j})^2 + \beta \sum_{i<j}^{N} \frac{z_i\, z_j}{z_{ij}\, z_{ji}}\,(\beta \mp P_{ij})$$

is diagonalised on functions completely (anti)symmetric by permutations of spins and coordinates

$$\Psi = \prod_{i<j} (z_i - z_j)^{\beta}\ \widetilde{\Psi}$$

$$\widetilde{\Psi} = \sum_{i_1<i_2<\ldots<i_M} \Psi\big(z_{\{i_1,i_2,\ldots,i_M\}}, \bar{z}_{\{i_1,i_2,\ldots,i_M\}}\big)\ |i_1, i_2, \ldots, i_M\rangle\rangle$$

partially (anti)symmetric in two complementary groups of variables

$$|i_1,\cdots,i_M\rangle\rangle := \sigma^-_{i_1}\cdots\sigma^-_{i_M}\,|\uparrow\cdots\uparrow\rangle$$

eigenfunctions of the CS model: partially symmetrised, non-symmetric Jack polynomials

# The isotropic Haldane Shastry Hamiltonian

in principle, all the eigenfunctions of Haldane-Shastry should be obtainable from those of the su(2) CS model by **freezing** i.e. by sending $\beta \to \infty$

this fixes the positions of the particles in the minima of the potential thus creating the regular periodic lattice

$$z_j \longmapsto \omega^j = \mathrm{e}^{2\pi \mathrm{i} j/N}$$

$$\Psi(z_{\{i_1,i_2,\ldots,i_M\}}, \bar{z}_{\{i_1,i_2,\ldots,i_M\}}) \longrightarrow \psi(z_{i_1}, z_{i_2}, \ldots, z_{i_M})$$ using $$\sum_{j=1}^{N} z_j^n = N\, \delta_{n,0 \bmod N}$$

For the states which are Yangian highest weights we have

**[BGHP, 93]**

**[Bernard, Pasquier, D.S., 93]**

**[Lamers, D.S, 22]**

$$\boxed{\psi_\lambda(z_1, z_2, \ldots, z_M) = \prod_{m<n} (z_m - z_n)^2 \; P_\lambda^{\beta=2}(z_1, z_2, \ldots, z_M)}$$

symmetric Jack polynomials labelled by partitions $N - 2M + 1 \geq \lambda_1 \geq \ldots \geq \lambda_M \geq 1$

# The q-Haldane-Shastry Hamiltonian (Uglov-Lamers)

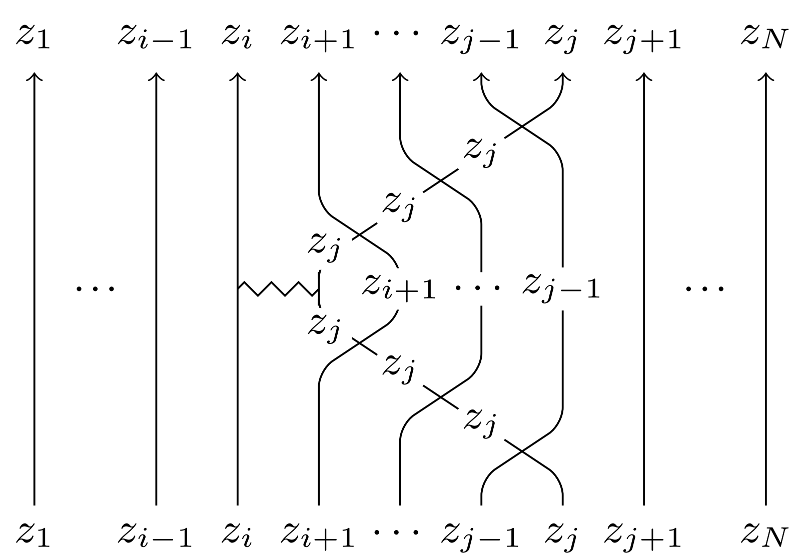
[Bernard, Gaudin, Haldane, Pasquier, 93; Uglov 95; Lamers 18]

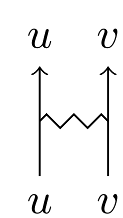
The XXZ model can also be deformed to accommodate for long-range interaction, at the price of introducing multi-spin interaction

$$\widetilde{\mathrm{H}}^{\mathrm{L}} = \frac{[N]}{N} \sum_{i<j} V(z_i, z_j)\ S^{\mathrm{L}}_{[i,j]}$$

$$V(z_i, z_j) = \frac{z_i\, z_j}{(\mathsf{q}\, z_i - \mathsf{q}^{-1} z_j)(\mathsf{q}^{-1} z_i - \mathsf{q}\, z_j)} \qquad [N] := \frac{\mathsf{q}^N - \mathsf{q}^{-N}}{\mathsf{q} - \mathsf{q}^{-1}}$$

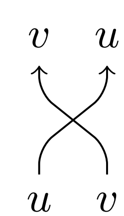
$S^{\mathrm{L}}_{[i,j]} :=$




$$:= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathrm{q}^{-1} & -1 & 0 \\ 0 & -1 & \mathrm{q} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = e_i$$

q-antisymmetriser
TL generator




$$:= \check{R}(u/v)$$

$$\check{\mathrm{R}}_{k,k+1}(u) = 1 - f(u)\ e_k\ , \qquad f(u) = \frac{u-1}{\mathrm{q}\, u - \mathrm{q}^{-1}}$$

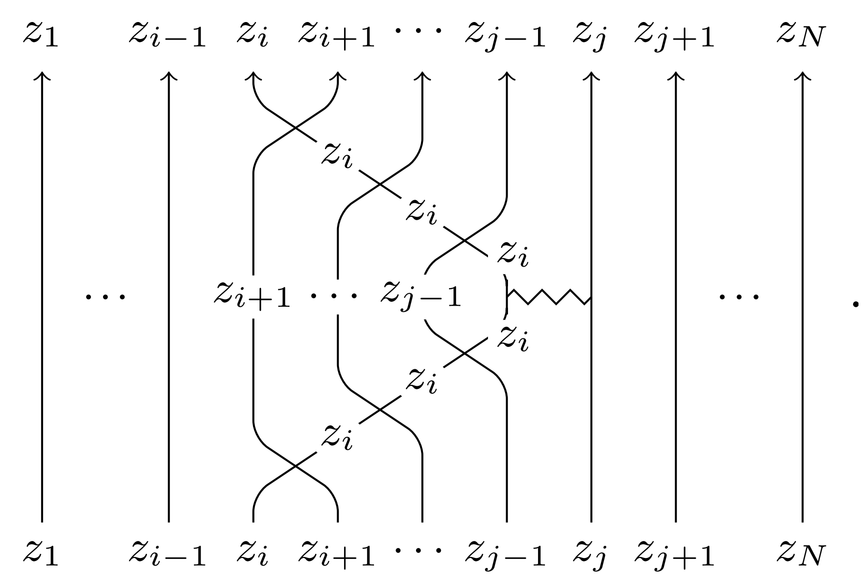
# The Uglov-Lamers Hamiltonian

There exists another Hamiltonian with the opposite "chirality"

$$\widetilde{\mathrm{H}}^{\mathrm{R}} = \frac{[N]}{N} \sum_{i<j} V(z_i, z_j)\ S^{\mathrm{R}}_{[i,j]}$$

$$[\widetilde{\mathrm{H}}^{\mathrm{L}}, \widetilde{\mathrm{H}}^{\mathrm{R}}] = 0$$

$$S^{\mathrm{R}}_{[i,j]} :=$$




Both Hamiltonians can be diagonalised simultaneously and the spectrum can be written in terms of motifs, with eigenvalues

$$\varepsilon^{\mathrm{L,R}}(\mu) = \sum_{m=1}^{M} \epsilon^{\mathrm{L,R}}(\mu_m) \qquad \varepsilon^{\mathrm{L}}(n) = \frac{1}{\mathrm{q}-\mathrm{q}^{-1}}\left(\mathrm{q}^{N-n}[n] - \frac{n}{N}[N]\right), \qquad \varepsilon^{\mathrm{R}}(n) = \frac{-1}{\mathrm{q}-\mathrm{q}^{-1}}\left(\mathrm{q}^{n-N}[n] - \frac{n}{N}[N]\right)$$

$\mathrm{H} = \frac{1}{2}(\widetilde{\mathrm{H}}^{\mathrm{L}} + \widetilde{\mathrm{H}}^{\mathrm{R}})$ has real spectrum both for q real and $|\mathrm{q}| = 1$

$$\varepsilon(n) = \frac{1}{2}\left(\varepsilon^{\mathrm{L}}(n) + \varepsilon^{\mathrm{R}}(n)\right) = \frac{1}{2}[n][N-n]$$

# The eigenvectors of the Uglov-Lamers model

[Lamers, Pasquier, D.S., 20]

There exists an explicit expression of the (equivalent of) highest weight vectors

one for each multiplet of the quantum affine algebra (for each motif)

$$|\mu\rangle = \sum_{i_1<\cdots<i_M}^{N} \Psi_\mu(i_1,\cdots,i_M)\,|i_1,\cdots,i_M\rangle\!\rangle\,,$$

the component where all the reversed spins are at the left is particularly simple

$$\Psi_\mu(1,\cdots,M) = \langle\!\langle 1,\cdots,M|\mu\rangle = \mathrm{ev}_\omega\,\widetilde{\Psi}_{\lambda(\mu)}(z_1,\cdots,z_M)$$

$$\widetilde{\Psi}_\lambda(z_1,\cdots,z_M) \coloneqq \left(\prod_{m<n}^{M}(\mathsf{q}\,z_m-\mathsf{q}^{-1}z_n)\,(\mathsf{q}^{-1}z_m-\mathsf{q}\,z_n)\right)P^\star_\lambda(z_1,\cdots,z_M)$$

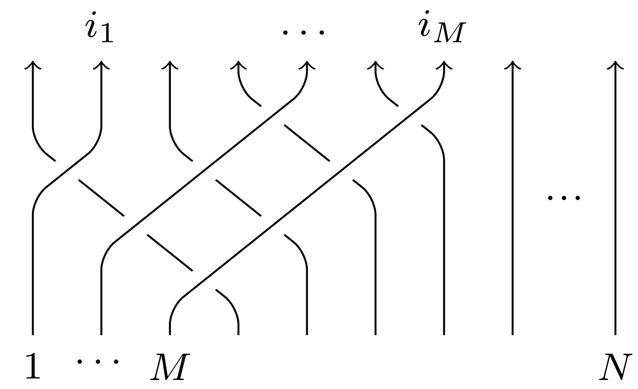
$P^\star_\lambda$ is a Macdonald polynomial with parameters $q^* = (t^*)^{1/2} = \mathsf{q}$

# The eigenvectors of the Uglov-Lamers model

The other components are slightly more involved and they correspond to transport of the indices using the Hecke generators (generalising the permutations)

$$\Psi_\mu(i_1,\cdots,i_M) = \langle\langle i_1,\cdots,i_M|\mu\rangle = \mathrm{ev}_\omega\Big(T^{\mathrm{pol}}_{\{i_1,\cdots,i_M\}}\widetilde{\Psi}_{\lambda(\mu)}(z_1,\cdots,z_M)\Big)$$

$$T_{\{i_1,\cdots,i_M\}} = T_{(i_1,\cdots,1)}\cdots T_{(i_M,\cdots,M)} =$$




$$T_{(j,j-1,\cdots,i)} = T_{j-1}\cdots T_i$$

This result nicely generalises the construction of the eigenvectors of the ordinary Haldane-Shastry model

# Th Uglov-Lamers model at q=i

The integrable structure of the isotropic Haldane-Shastry model can be retrieved in the c=1 CFT with affine su(2) symmetry **[Haldane et al. 92, Bernard, Pasquier, D.S., 94]**

What about the low energy limit of the q-HS model at $|\mathrm{q}| = 1$ ? What is the symmetry of the model for q root of unity?

The simplest case to look at is q=i, because of the link with the free fermions

$$\check{\mathrm{R}}_{k,k+1}(u) = 1 - f(u)\, e_k \ , \qquad f(u) = \frac{u-1}{\mathrm{q}\, u - \mathrm{q}^{-1}}$$

Since $$f(u) + f(u^{-1}) = (\mathrm{q}+\mathrm{q}^{-1}) f(u) f(u^{-1}) \quad \xrightarrow[\mathrm{q=i}]{} \quad f(u^{-1}) = -f(u)$$

in particular $$f(\omega^j) = -f(\omega^{-j}) = \tan(\pi j/N)$$

# Th Uglov-Lamers model at q=i

in this case the spin interaction can be written exclusively with in terms of nested commutators of the TL generators

$$e_{[l,m+1]} := [e_l, [e_{l+1}, \ldots [e_{m-1}, e_m] \ldots]] = [[\ldots [e_l, e_{l+1}], \ldots e_{m-1}], e_m]$$

Jacobi identity and TL algebra

for example:

$$\mathrm{S}^{\mathrm{L}}_{[i,i+2]} = e_{[i,i+1]} - f^2(\omega) e_{[i+1,i+2]} + f(\omega) e_{[i,i+2]}\,, \qquad e_{[k,k+1]} \equiv e_k$$

$$\begin{aligned}\mathrm{S}^{\mathrm{L}}_{[i,i+3]} &= e_{[i,i+1]} - f^2(\omega^2) e_{[i+1,i+2]} + f^2(\omega^2) f^2(\omega) e_{[i+2,i+3]} \\ &+ f(\omega^2) e_{[i,i+2]} - f^2(\omega^2) f(\omega) e_{[i+1,i+3]} + f(\omega^2) f(\omega) e_{[i,i+3]}\end{aligned}$$

and in general:

$$\mathrm{S}^{\mathrm{L}}_{[j,j+k]} = \sum_{l=0}^{k-1} \sum_{m=1}^{k-l} (-1)^l \prod_{i=1}^{l} f^2(\omega^{k-i}) \prod_{n=1}^{m-1} f(\omega^{k-l-n}) \; e_{[j+l,j+l+m]}$$

$$\mathrm{S}^{\mathrm{R}}_{[j,j+k]} = \sum_{l=0}^{k-1} \sum_{m=1}^{k-l} (-1)^{l+m-1} \prod_{i=1}^{l} f^2(\omega^{k-i}) \prod_{n=1}^{m-1} f(\omega^{k-l-n}) \; e_{[j+k-l-m,j+k-l]}$$

# Th Uglov-Lamers model at q=i

**There are two subtleties in defining the Hamiltonian at q=i :**

since $[2k] = 0$ and $[2k+1] = (-1)^k$

at $N = 2L+1:$ $\varepsilon(n) = \frac{1}{2}[n][N-n] = 0$ the total energy is identically zero

but $\varepsilon^{\mathrm{L}}(n) = -\varepsilon^{\mathrm{R}}(n) = \frac{(-1)^L}{2i}\begin{cases} -\frac{n}{N}\,, & n = 2k \\ \frac{N-n}{N}\,, & n = 2k+1 \end{cases}$

the total Hamiltonian is also zero for odd number of sites:

$$\mathrm{H} = \sum_{1\le p\le q<N} \left(h^{\mathrm{L}}_{p,q} + h^{\mathrm{R}}_{p,q}\right)\, e_{[p,q+1]}$$

$h^{\mathrm{L}}_{p,q} = -h^{\mathrm{R}}_{p,q}\,, \qquad 1 \le p \le q < N$ explicit but tedious expressions/proof

TO DO: fermionic expression: $e_j = \left(f^+_j + f^+_{j+1}\right)(f_j + f_{j+1})$

the nested TL commutators are quadratic in fermions $\longrightarrow$ long-range free fermionic model

# Th Uglov-Lamers model at q=i

One can get a non vanishing non-chiral Hamiltonian by expanding to the next order in $\mathrm{q}+\mathrm{q}^{-1}$

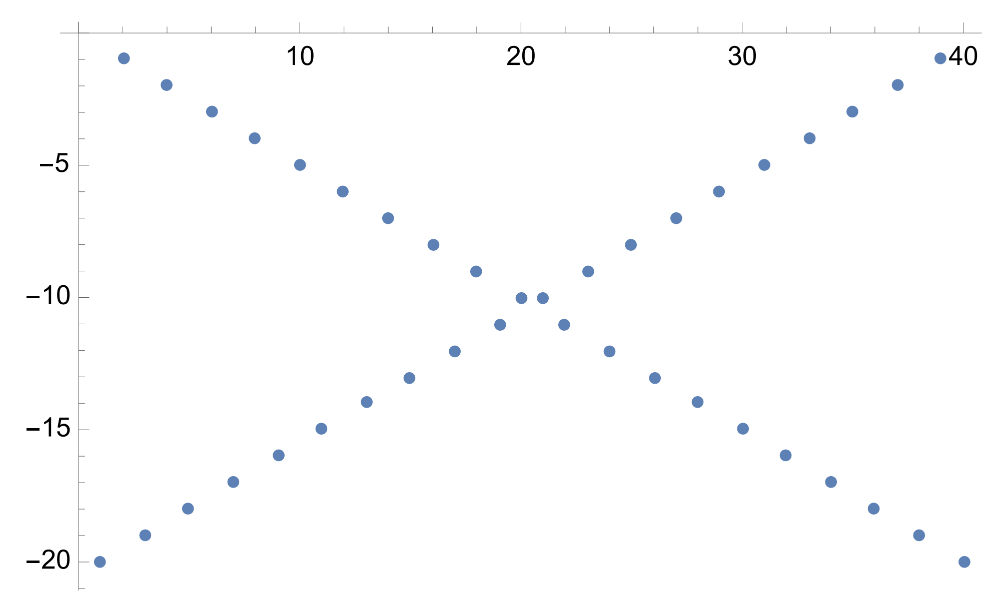
$$\widetilde{\mathrm{H}} := \lim_{\mathrm{q}\to i} \frac{\mathrm{H}}{\mathrm{q}+\mathrm{q}^{-1}}$$

$$[\widetilde{\mathrm{H}}, \widetilde{\mathrm{H}}^{\mathrm{R}}] = -[\widetilde{\mathrm{H}}, \widetilde{\mathrm{H}}^{\mathrm{L}}] = 0$$

the eigenstates of the chiral and the rescaled Hamiltonians are the same

result for the one-magnon dispersion relation:

$$\tilde{\varepsilon}(n) = \lim_{q\to i} \frac{\varepsilon(n)}{\mathrm{q}+\mathrm{q}^{-1}} = (-1)^{L-1} \begin{cases} \frac{n}{2}\,, & n = 2k \\ \frac{N-n}{2}\,, & n = 2k+1 \end{cases}$$




it might be possible, though hard, to get an explicit expression in terms of fermions

# Th Uglov-Lamers model at q=i

The second subtlety appears at *N* even $N = 2L$

in this case the dispersion relation is regular, but there are divergences (double poles) in the matrix elements of the Hamiltonians, since:

$$\mathrm{V}_{j,j+L} = \frac{1}{(\mathrm{q}+\mathrm{q}^{-1})^2} \qquad\qquad f(\omega^L) = f(-1) = \frac{2}{\mathrm{q}+\mathrm{q}^{-1}}$$

One of the poles is killed by the factor [N] in the Hamiltonian, but the second has to be killed "by hand" by multiplication with $\mathrm{q}+\mathrm{q}^{-1}$

Result: a Hamiltonian with finite matrix elements but with identically zero eigenvalues!

Example: for *N=2* $$2\mathrm{H} = \frac{1}{\mathrm{q}+\mathrm{q}^{-1}} e_1$$ is a projector with eigenvalues 0^3, 1

after rescaling, $$2\mathrm{H}(\mathrm{q}+\mathrm{q}^{-1}) = e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathrm{q}^{-1} & -1 & 0 \\ 0 & -1 & \mathrm{q} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow$$ Jordan block at q=i

# Th Uglov-Lamers model at q=i

Algebraic origin of the Jordan blocks at N even: **gl(1|1) symmetry**

at each site we have gl(1|1) representation with alternating central charge $E_j$

**[Gainutdinov, Read, Saleur, 11]**

$$\{f_j^+, f_j\} = (-1)^j \equiv E_j \ , \qquad N_j = (-1)^j f_j^+ f_j \ ,$$
$$[\mathrm{N}_j, f_j] = -f_j \ , \qquad [\mathrm{N}_j, f_j^+] = f_j^+ \ .$$

global generators: $F_1^+ = \sum_{j=1}^{N} f_j^+ \ , \qquad F_1 = \sum_{j=1}^{N} f_j \ , \qquad \mathrm{N} = \sum_{j=1}^{N} (-1)^j f_j^+ f_j - L \ , \qquad E = \sum_{j=1}^{N} E_j$

central element

Jordan blocks $\longleftarrow$ indecomposable representations of gl(1|1), at E=0

Experimentally, at larger lengths N=2L, the largest Jordan cell has size L+1

extended symmetry?

# Conclusions and open questions

- New fermionic long-range integrable model with extended (super)symmetry
- The odd and even lengths have very different properties (linear dispersion relation vs. Jordan blocks)
- Closed form expressions for the (regularised) matrix elements

- Identify the extended symmetry of the model
- Relation with non-unitary CFTs; Vertex operator construction
- The q=i limit of the eigenfunctions
- Other roots of unity: q^3=1 and c=0 CFT; gl(2|1) symmetry
- Higher rank?
- q-Inozemtsev at q=i?