Gaudin Integrability of Conformal Blocks

Volker Schomerus (DESY HH), Les Diablerets, Feb 5, 2023

Based on joint work with I. Buric, M. Isachenkov, S. Harris, P. Liendo, Y. Linke, A. Kaviraj, S. Lacroix, J. Mann, L. Quintavalle, Y. Sobko …

HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

DESY.

Conformal Field Theory in d > 2

A modern challenge

Vast zoo of relevant models

3D critical Ising, O(N)-models,

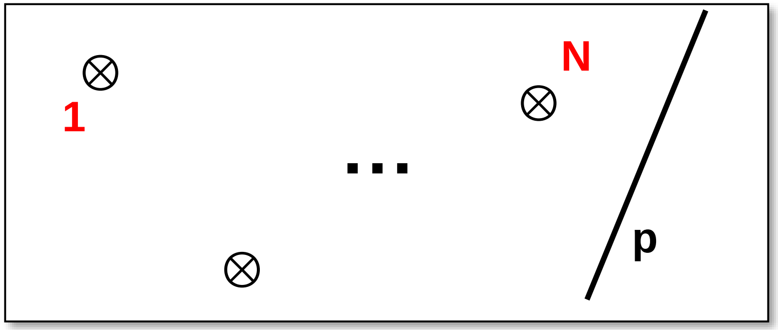
SUSY gauge theories in 3D, 4D …

One side of AdS_{d+1}/CFT_d-duality

learn about quantum gravity in D > 3

Rich set of non-local observables

boundaries, interfaces, Wilson lines …



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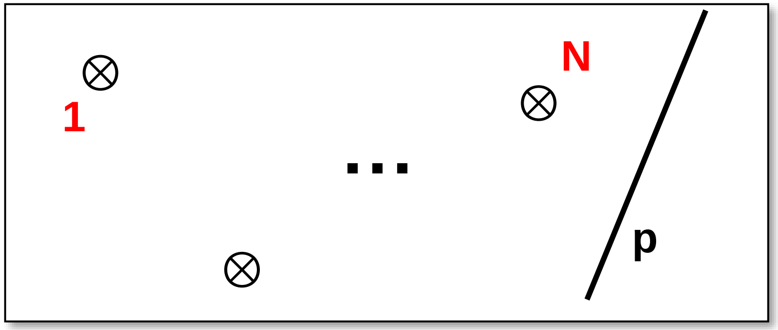

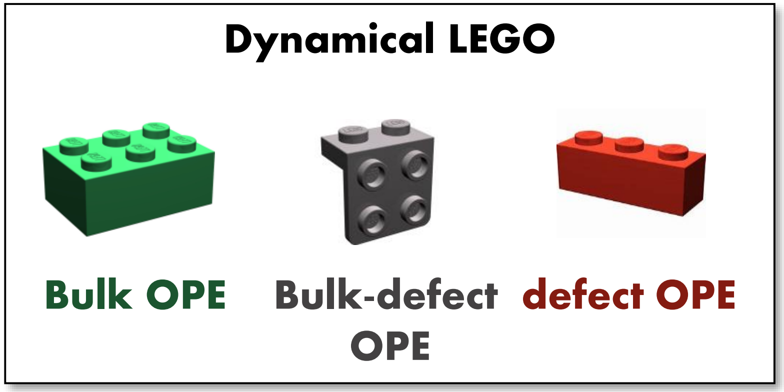

Conformal block expansions:

$$\langle \Phi_1(x_1) \cdots \Phi(D_p) \rangle \sim \sum_{\Lambda} P_\Lambda \Psi_\Lambda(U)$$

constant coefficients *P* factorize !

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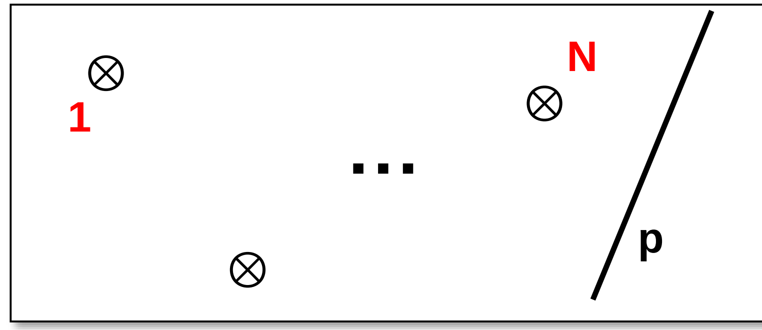

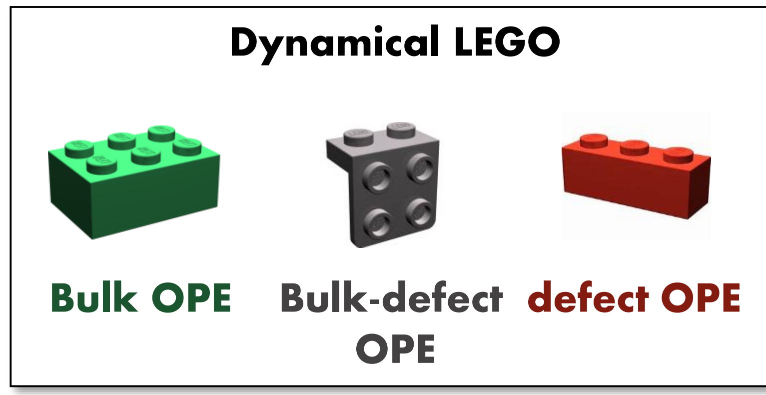

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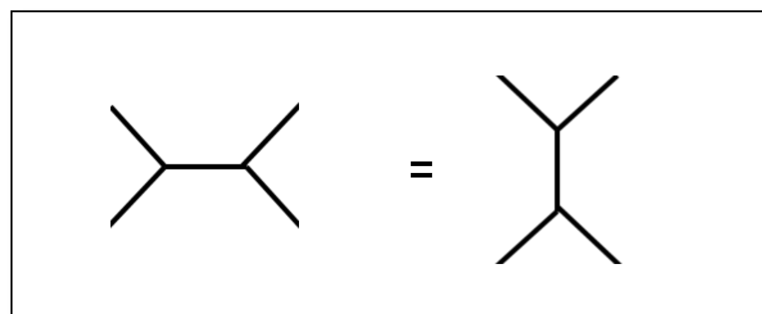

NP conformal bootstrap program

exploits crossing symmetry

Goals and Plan

I. What kind of functions are conformal blocks $\Psi_\Lambda(U)$ in d > 2 ?

any number N of insertion points, non-local operators …

II. Can one make conformal bootstrap program work in d > 2 ?

i.p. what so we need to know about the block ?

$\rightarrow$ Jeremy‘s talk

Plan:

1. **Conformal Field Theory 101 (d>2)**
2. **Conformal blocks: General Theory**
3. **Conformal blocks: Examples**

Conformal Field Theory 101

HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

DESY.

Conformal Symmetry

The Conformal Group and its subgroups

Euclidean conformal group $G_d = SO(1, d+1)$ with Lie algebra $so(1, d+1)$

Generators: Rotations so$(d) \ni L_{\mu\nu}$ Translations $R^d \ni P_\mu$

Dilations so$(1,1) \ni D$ Special CT $R^d \ni K_\mu$

There are a number of important <u>subgroups</u> and associated <u>quotients</u>

Stabilizer of a point:	$S_* \cong [SO(1,1) \times SO(d)] \times R^d$ (dilations, rotations, special CT)	$G_d/S_* \cong R^d$
Stabilizer of 2 points:	$S_0 \cong SO(1,1) \times SO(d) \cong: K_d$	$G_d/S_0 \cong R^{2d}$
p-dimensional defect:	$S_p \cong SO(1, p+1) \times SO(d-p)$ (parallel CTs, transverse rotations)	$G_d/S_p \cong \mathcal{M}_{d,p}$

dim$\mathcal{M}_{d,p}$ = (p+2)(d-p) parameters of conformal defect

Primary Fields

Local and non-local observables

A local primary (= non-derivative) field $\Phi(x)$ of weight Δ & spin λ satisfies

$$[D, \Phi(x)] = (x^\nu \partial_\nu + \Delta)\, \Phi(x) \qquad [L_{\mu\nu}, \Phi(x)] = \left(x_\nu \partial_\mu - x_\mu \partial_\nu + \Sigma^\lambda_{\mu\nu}\right) \Phi(x)$$

$$[K_\mu, \Phi(x)] = \left(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu - 2x_\mu \Delta + 2x^\nu \Sigma^\lambda_{\mu\nu}\right) \Phi(x)$$

Δ, λ determine representation $\pi^{\Delta,\lambda}$ of $S_* \cong [SO(1,1) \times SO(d)] \times R^d$ on V_π

Action $\mathcal{T}^{\Delta,\lambda}_\alpha$ of conformal generators on primary field $\Phi(x)$ coincides with action on

$$\Gamma^{\Delta,\lambda} \cong \{\, f : G_d \to V_\pi \mid f(gs) = \pi^{\Delta,\lambda}(s) f(g)\, , s \in S_* \,\}$$

$$f(De^{ix^\mu P_\mu}) = f(e^{ix^\mu P_\mu}(D + x^\mu P_\mu)) = (-ix^\mu \partial_\mu - i\Delta) f(e^{ix^\mu P_\mu}) \qquad f(e^{ix^\mu P_\mu} D) = -i\Delta f(e^{ix^\mu P_\mu})$$

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Action $\mathcal{T}^{D_p}_\alpha$ of conformal generators on defect field $\Phi(D_p)$ coincides with action on

$$\Gamma^{D_p} \cong \{ f : G_d \to \mathbb{C} \mid f(gs) = f(g) \,, s \in S_p \}$$

Correlation Functions 1

Ward identities & 3-point functions

Correlation functions of primary fields satisfy conformal Ward identities

$$\sum_{i=1} \mathcal{T}_\alpha^{\Delta_i,\lambda_i} \langle \Phi_1(x_1) \dots \Phi_N(x_N) \rangle = 0$$

dim $SO(1, d+1)$ first order diff. Eqs.
- not all independent in general -

Determine 3-point functions of scalar primary fields up to one constant

$$\langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_{12}} x_{23}^{\Delta_{23}} x_{13}^{\Delta_{13}}} =: C_{123}$$

$$\Delta_{12} = \Delta_1 + \Delta_2 - \Delta_3$$

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$$\langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x)\rangle = C_{12;(\Delta,l)}$$

$\phi_{\Delta,l}(x)$

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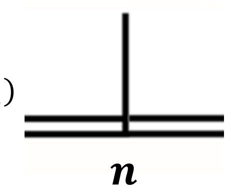
$$\phi_{\Delta,l}(x)$$

Note: STT-STT-scalar 3pt functions are not fixed by conformal symmetry

$$\langle \Phi_{\Delta_1,l_1}(x_1)\Phi_2(x_2)\Phi_{\Delta_3,l_3}(x_3\,)\rangle = \sum_{n=0}^{min(l_a,\,l_b)} C^{(n)}_{(\Delta_1,l_1)2(\Delta_3,l_3)}$$



Correlation Functions 2

Operator product expansions and conformal blocks

In a CFT the operator product of fields possesses a convergent expansion

e.g. scalar primaries:

$$\Phi_1(x)\,\Phi_2(y) \sim \sum_{\Phi_{\Delta,l}} C_{12\Phi}\, \mathcal{D}_{\Delta,l}(\mathrm{x}-\mathrm{y},\partial_y)\Phi_{\Delta,l}(y)$$

sum over primaries Φ (STT)

3-point couplings

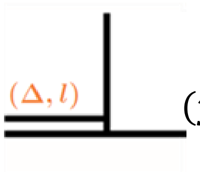
∞ order differential operators

Ferrara, Grillo,Gatto,
Nouvo Cim 2, eq (21)

$$\mathcal{D}_{\Delta,0}(x,\partial_y) = |x|^{\Delta-\Delta_1-\Delta_2}\left[1+\frac{1}{2}\,x^\mu\partial_\mu+\beta(\Delta)x^\mu x^\nu\partial_\mu\partial_\nu+\beta'(\Delta)x^2\partial^2+\cdots\right]$$

Can be used to compute all N-point functions of primary fields, e.g. N = 4

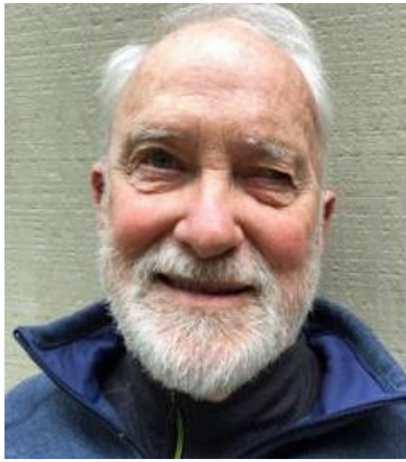
$$\langle\Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\Phi_4(x_4)\rangle \sim \sum_{\Phi_{\Delta,l}} C_{12\Phi}C_{34\Phi}\ \mathcal{D}_{\Delta,l}(x_{12},\partial_y)$$



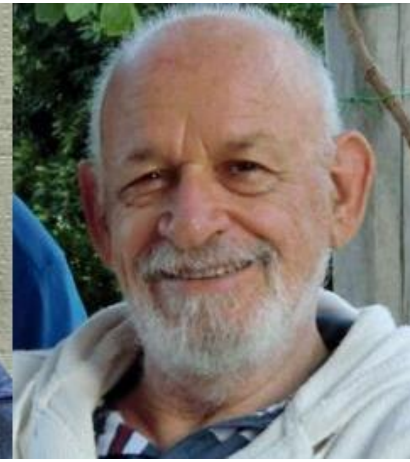
(y, x_3, x_4)

→ Conformal Block for scalar 4-point functions $\Psi^{12}_{\Delta,l}(u,v)$

$$u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2} \qquad v=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}$$



Bill Sutherland



Francesco Calogero

Michel Gaudin

Conformal Blocks as Integrable Wave Functions

Multipoint Conformal Blocks

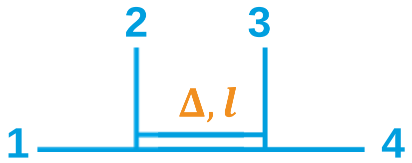
Cross ratios and quantum numbers

Conformal blocks form a `basis‘ in space of conformal multi-point invariants

OPE channels → $\Psi^{\mathcal{C}}_{\Lambda}(U)$ ← cross ratios

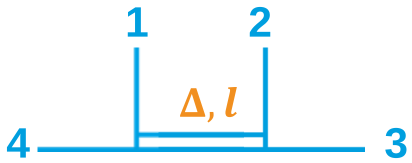
internal labels ↗

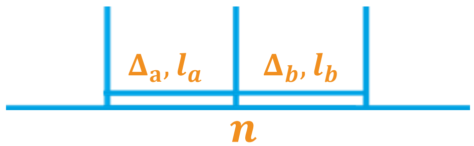
Examples (d>3):



2 cross ratios

2 labels: Δ, l

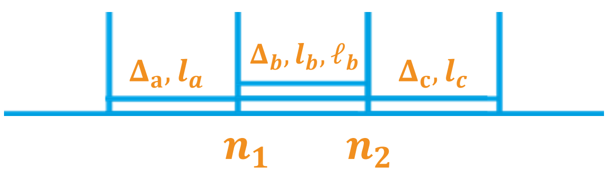


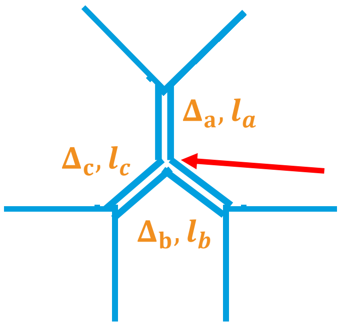


5 cross ratios

5 labels:

$\Delta_a, l_a, \Delta_b, l_b, n$





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$$\Psi^{c}_{\Lambda}(U)$$ ← cross ratios

internal labels → Λ

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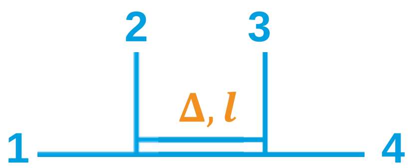

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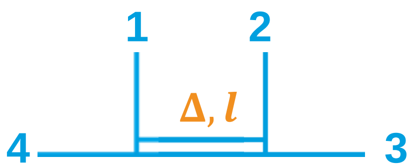

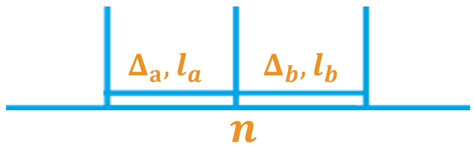

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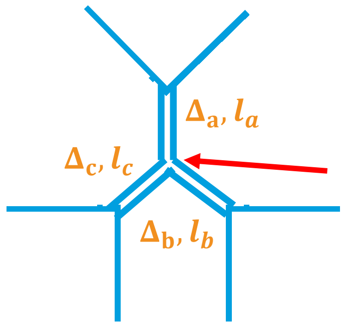

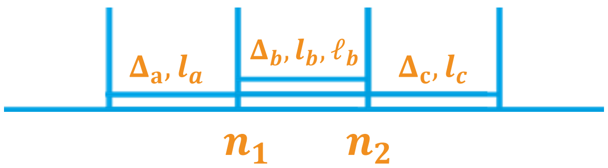

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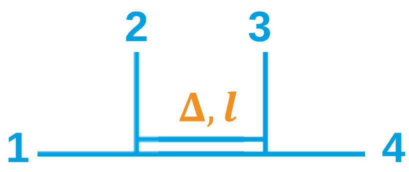

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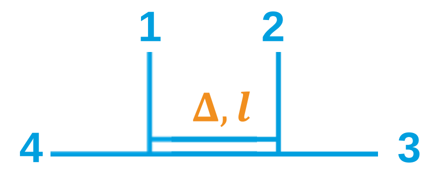

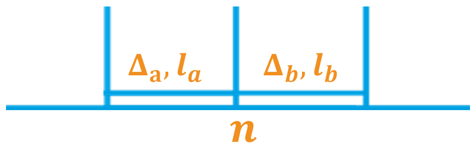

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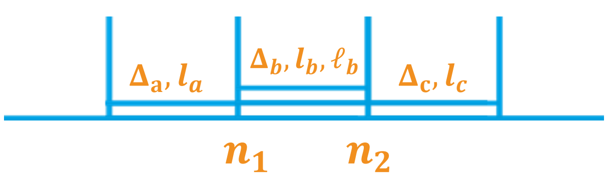

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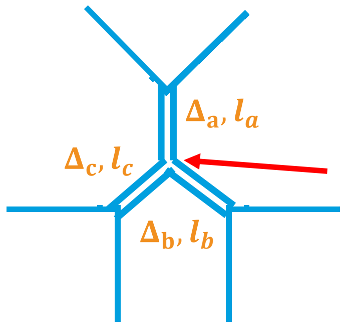

n_1, n_2, n_3

Gaudin integrable model

Associated w. CFT correlation function **[Buric,Lacroix,Mann,Quintavalle,VS]**

Lax connection: Introduce following family of matrix valued 1st order DOs

spectral parameter

$$\mathcal{L}(\omega;\omega_J) \equiv \sum_J \frac{T_\alpha \mathcal{T}_\alpha^{(J)}}{\omega - \omega_J} = \mathcal{L}_\alpha T_\alpha$$

complex parameters

$\mathcal{T}_\alpha^{(J)} \in \{\mathcal{T}_\alpha^{\Delta}, \mathcal{T}_\alpha^{D_p}\}$

generators of conformal algebra

$$\mathcal{H}_q(\omega;\omega_I) = \kappa_q^{\alpha_1\ldots\alpha_q}\mathcal{L}_{\alpha_1}\cdots\mathcal{L}_{\alpha_q} + lot$$

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Hamiltonians commute among each other & generate commutant of the generators $\mathcal{T}_\alpha = \Sigma\, \mathcal{T}_\alpha^{(J)}$ **of conf Ward identities** [Feigin,Frenkel,Reshetikhin]

→ **Quantum integrable system on reduced state space** $(\otimes_J \Gamma^{\Delta_J})^G$

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→ Quantum integrable system on reduced state space $(\otimes_J \Gamma^{\Delta_J})^G$

Remark: Also true in presence of defects, i.e. if $\mathcal{T}^{(J)}_\alpha = \mathcal{T}^{D_p}_\alpha$ for some J

OPE Channels and Gaudin Limits

Recovering Casimir operators

[Buric, Lacroix, Mann, Quintavalle, VS]

Choice of an OPE channel $\mathcal{C}$ determines a set $\text{Cas}_{\mathcal{C}}$ of Casimir operators

Measure weight and spin of intermediate fields

Demanding $\text{Cas}_{\mathcal{C}} \subset \text{Ham}(\omega_J)$ fixes ω_J to some limiting configuration $\omega_J^{\mathcal{C}} \in \{0, 1, \infty\}$

$$\mathcal{H}_q^{[\rho]}(\omega) = \lim_{\varepsilon \to 0} \varepsilon^{q n_\rho} \mathcal{H}_q\left(\omega = \varepsilon^{n_\rho}\omega + g_\rho(\varepsilon), \omega_j = f_J(\varepsilon)\right)$$

$n_\rho = n_\rho^{\mathcal{C}} \in \mathbb{N}$

$g_\rho(w) = g_\rho^{\mathcal{C}}(w)$ **Poly-**

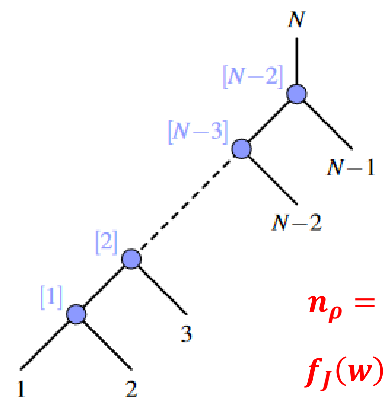
$f_J(w) = f_J^{\mathcal{C}}(w)$ **nomial**

The functions $\mathcal{H}_q^{[\rho]}(\omega)$, $\rho \in$ vertices of $\mathcal{C}$, contain a *complete* set of Hamiltonians.

$$\mathcal{H}_q^{[\rho]}(\omega) = \sum_{\nu=0}^{q} \binom{q}{\nu} \frac{D_\rho^{q,\nu}}{\omega^\nu (1-\omega)^{q-\nu}}$$

$D_\rho^{q,0}$ and $D_\rho^{q,q}$ are Casimir differential operators,

All others are new vertex differential operators.



$n_\rho = N - 2 - \rho$

$f_J(w) = w^{N-1-J}$

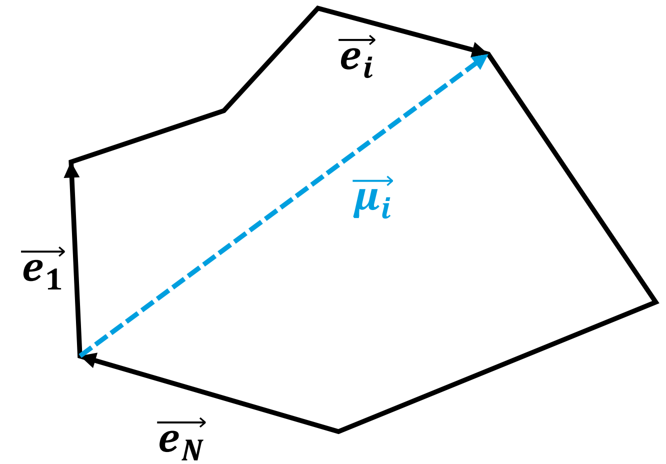
Relation with Bending FLows

Comb channel limit

Comb channel Gaudin limit was considered

before in context of `bending flows'

[Kapovich, Millson]



Consider N-gons in 3-dimensional space,

with edge lengths fixed to be $r_i, i = 1, \dots, N$

$$\mathcal{M}_r = \{ \vec{e}_i \in S^2_{r_i} \text{ with constraint } \textstyle\sum_i \overrightarrow{e_i} = 0 \}$$

Moduli space $\mathcal{M}_r$ is symplectic $\quad \omega_r = \sum_i \omega(S^2_{r_i})$

Claims: $h_i := \vec{\mu}_i^2$ **form a complete set of Poisson commuting Hamiltonians.**

They co-incide with the Hamiltonians of SU(2) Gaudin model in the comb channel (homogeneous, caterpillar) limit.

[Falqui, Musso]

[Chervov, Falqui, Rybnikov]

Conformal Blocks: Examples

Emergence of Calogero-Sutherland models

HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

DESY.

Spinning 4-point correlators

as K-spherical functions on the conformal group

$\pi_i = \pi^{\Delta_i, l_i}$

Recall: Reduced 4-site Gaudin is realized on space $(\Gamma^{\pi_1} \otimes \Gamma^{\pi_2} \otimes \Gamma^{\pi_3} \otimes \Gamma^{\pi_4})^G$

Prop: Tensor product $\Gamma^{\Delta_1} \otimes \Gamma^{\Delta_2}$ **can be realized on the space [Dobrev et al]**

$$\Gamma^{\pi_1} \otimes \Gamma^{\pi_2} \cong \Gamma^{\pi_a}_{G/K} \equiv \{ f: G_d \to V^{(2)} \mid f(gk) = \pi_a(k^{-1})\, f(g) \}$$

$$\pi_a = \pi_1 \otimes \pi_2 \qquad (\Delta_a, \lambda_a) = (\frac{\Delta_2 - \Delta_1}{2}, \bar{\lambda}_2 \times \lambda_1)$$

Isomorphism is known explicitly [Buric,Isachenkov,VS]

Space of spinning 4-point functions realized on double coset $K_d \backslash G_d / K_d$ **:**

$$(\Gamma^{\pi_1} \otimes \Gamma^{\pi_2} \otimes \Gamma^{\pi_3} \otimes \Gamma^{\pi_4})^G \cong \left(\Gamma^{\pi_a}_{G/K} \otimes \Gamma^{\pi_b}_{G/K}\right)^G \cong \Gamma^{\pi_a,\pi_b}_{K\backslash G/K} \longleftarrow \textbf{2-dimensional}$$

$$\equiv \{ f: G_d \to V^{(4)} \mid f(k_l g k_r) = \bar{\pi}^a(k_l)\pi^b(k_r^{-1})\, f(g) \}$$

The space $\Gamma^{\pi_a,\pi_b}_{K\backslash G/K}$ **is known as space of K-spherical functions**

The Casimir equations

Calogero-Sutherland models for roots system BC_2

Casimir operators on G_d descend to double coset and can be computed, using the Harish-Chandra radial component map.

universal in spin [Buric, VS]

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For scalar fields, the Casimir $D_2^{2,0}$ coincides w. Hamiltonian H_{CS} of hyperbolic Calogero-Sutherland model for root system BC_2

[Isachenkov, VS]

$$H_{\mathrm{CS}} = -\sum_{i=1}^{2} \frac{\partial^2}{\partial \tau_i^2} + \frac{k_3(k_3-1)}{2}\left[\sinh^{-2}\left(\frac{\tau_1+\tau_2}{2}\right) + \sinh^{-2}\left(\frac{\tau_1-\tau_2}{2}\right)\right] + \sum_{i=1}^{2}\left[k_2(k_2-1)\sinh^{-2}(\tau_i) - \frac{k_1(k_1+2k_2-1)}{4}\cosh^{-2}\left(\frac{\tau_i}{2}\right)\right]$$

$$k_1 = \Delta_4 - \Delta_3 \ , \quad 2k_2 = \Delta_2 - \Delta_1 + \Delta_3 - \Delta_4 + 1, \quad 2k_3 = d-2$$

Coordinates: $z_i = \cosh^{-2}\left(\frac{\tau_i}{2}\right),$ $\quad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z_1 z_2 \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z_1)(1-z_2)$

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Remark: 4-site Gaudin = elliptic BC_2 Calogero-Sutherland = Inozemtsev degenerates to hyperbolic BC_2 Calogero-Sutherland in all OPE channels.

Single Variable Vertex Systems

Lemniscatic elliptic Calogero-Moser-Sutherland model

For vertices contributing a single cross ratio we identified the associated vertex differential operators with the 4th order Hamiltonian of an integrable lemniscatic elliptic Calogero-Moser-Sutherland model. [Etingof, Felder, Ma, Veselov]

$d \geq 3$

$d \geq 4$

$d = 4$

Calogero-Sutherland wave functions

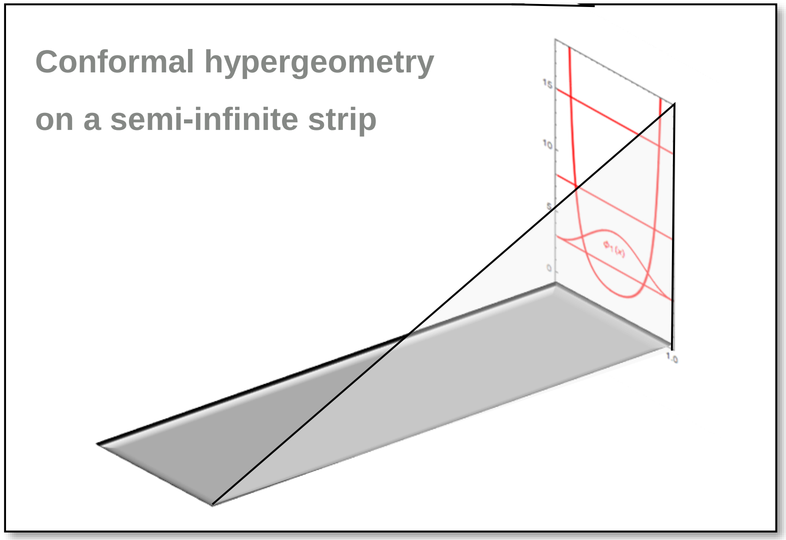
Scattering problem for CS particles in a Weyl chamber solved [Heckman, Opdam]

→ Heckman-Opdam hypergeometry

Conformal partial waves live on a strip

→ Conformal hypergeometry

[Isachenkov, VS]



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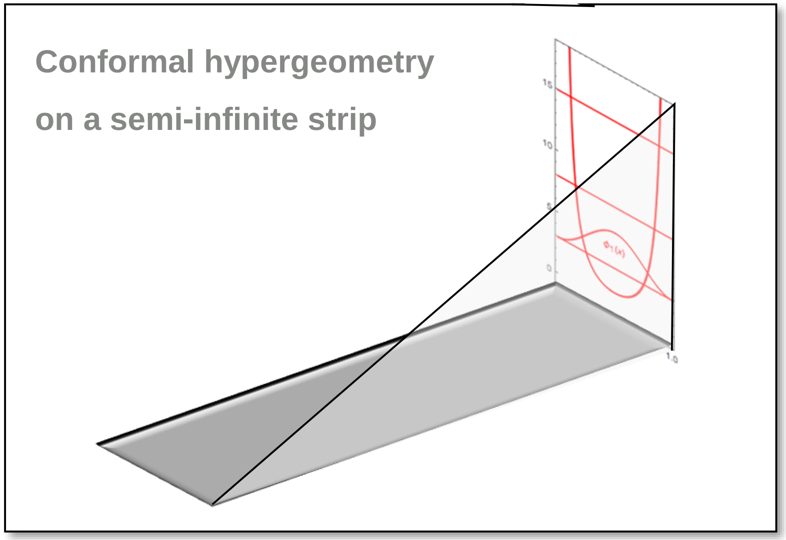
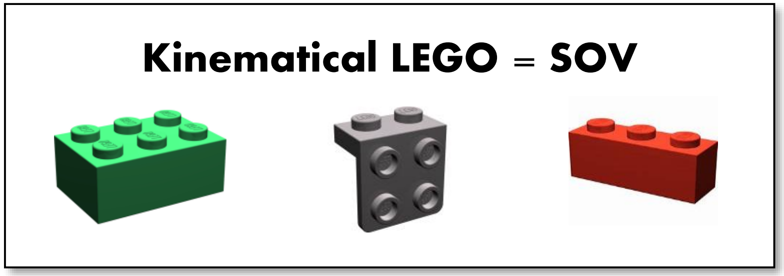


No solution theory for vertex systems

Separation of variables [SOV] for our Gaudin limits → factorization of blocks



SOV for 1D CFT blocks well understood & completely explicit in OPE limit

separated variable: momentum; OPE limit of Gaudin: $\phi^N(p) \sim {}_2F_1\,(p)$

Conclusion and Outlook

Conformal Blocks are wave functions of quantum integrable systems

of Gaudin-type, for limiting configuration of ω_I

Even if wave functions (= blocks) have not been constructed beyond some special cases (yet → SOV), one can do analytical bootstrap ….