CHALMERS UNIVERSITY OF TECHNOLOGY | UNIVERSITY OF GOTHENBURG

Correlations for the XYZ spin chain and Painlevé VI

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A frog's perspective

XYZ spin chain

Chain of L spin $1/2$ particles.

Hamiltonian

$$H^{\mathsf{XYZ}} = -\frac{1}{2}\sum_{j=1}^{L}\left(J_x\,\sigma_j^x\sigma_{j+1}^x + J_y\,\sigma_j^y\sigma_{j+1}^y + J_z\,\sigma_j^z\sigma_{j+1}^z\right).$$

J_x, J_y, J_z (real) anisotropy parameters.

Periodic boundary conditions: $\sigma_{L+1}^x = \sigma_1^x$ etc.

Combinatorial/supersymmetric case

We consider the combinatorial or supersymmetric case

$$J_x J_y + J_x J_z + J_y J_z = 0.$$

Generalizes XXZ spin chain with $\Delta = -1/2$, $q = e^{2\pi i/3}$:

$$J_x = J_y = 1, \qquad J_z = -\frac{1}{2} = \Delta = \frac{q + q^{-1}}{2}.$$

Very special case — solvability already at finite lattice.

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Why combinatorial?

- XXZ model with $\Delta = -1/2$ has deep connections to combinatorics of alternating sign matrices and plane partitions (Razumov–Stroganov etc.).
- General XYZ case has connections to three-colourings (R. 2011, Hietala 2020).

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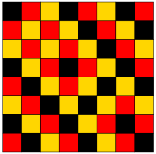

Supersymmetry

$$J_x J_y + J_x J_z + J_y J_z = 0$$

Why supersymmetric?

- Scaling limit to massive sine-Gordon theory.
  Under condition above it has $\mathcal{N} = 2$ supersymmetry (Saleur & Warner 1993).
- Supersymmetry on finite lattice (Fendley & Hagendorf 2012):

$$H^{\text{XYZ}} = \text{Const} + QQ^\dagger + Q^\dagger Q$$

(on subspace). $Q : V^{\otimes L} \to V^{\otimes (L+1)}$, $Q^2 = 0$.

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The importance of being odd

Baxter (1972) computed the ground state energy (lowest eigenvalue of H^{XYZ}) as $L \to \infty$.

When $J_xJ_y + J_xJ_z + J_yJ_z = 0$ and $J_x + J_y + J_z > 0$,

$$E_0 \sim -\frac{L}{2}(J_x + J_y + J_z), \qquad L \to \infty.$$

Stroganov (2001) conjectured that if L is odd then

$$E_0 = -\frac{L}{2}(J_x + J_y + J_z).$$

Proved by Hagendorf and Liénardy (2018) using supersymmetry.

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Correlation functions

We will assume

- Periodic boundary
- $J_x J_y + J_x J_z + J_y J_z = 0, \qquad J_x + J_y + J_z > 0,$
- $L = 2n + 1$ odd

$|\Psi\rangle$ ground state with even number of up spins.
Nearest neighbour correlations (for ground state)

$$C^x = \frac{\langle \Psi | \sigma^x_j \sigma^x_{j+1} | \Psi \rangle}{\langle \Psi | \Psi \rangle}, \quad C^y = \cdots, \quad C^z = \cdots .$$

Computed for XXZ chain ($J_x = J_y = 1$, $J_z = -1/2$) by Stroganov (2001):

$$C^x = C^y = \frac{5}{8} + \frac{3}{8L^2}, \qquad C^z = -\frac{1}{2} + \frac{3}{2L^2}.$$

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Preliminary result

For $a \in \{x, y, z\}$ we can write

$$C^a = 1 + \frac{J_x J_y J_z}{J_a^2 (J_x + J_y + J_z)} f_n$$

where f_n is a rational function of $Z = (J_x + J_y + J_z)^3 / J_x J_y J_z$.

$$f_0 = 0, \qquad f_1 = 1, \qquad f_2 = \frac{Z+27}{Z+25},$$

$$f_3 = \frac{(Z+24)(Z+27)}{(Z+21)(Z+28)},$$

$$f_4 = \frac{Z^3 + 74Z^2 + 1807Z + 14520}{Z^3 + 72Z^2 + 1701Z + 13068}, \ \ldots$$

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Polynomials s_n and $\bar{s}_n$

We express f_n in terms of the polynomials s_n and $\bar{s}_n$ of Bazhanov and Mangazeev (2005, 2010).

Tau functions of Painlevé VI, related to Q-operator eigenvalue (see below).

Toda-type recursions

$$8(2n+1)^2 s_{n+1}s_{n-1}+2z(z-1)(9z-1)^2(s_n''s_n-(s_n')^2)+2(3z-1)^2(9z-1)s_ns_n' - \big(4(3n+1)(3n+2)+n(5n+3)(9z-1)\big)s_n^2=0.$$

$$\dots,\ s_{-2}=\frac{3+9z}{4},\quad s_{-1}=1,\quad s_0=0,$$
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Main result

Parametrize the chain as

$$J_x = 1 + \zeta, \qquad J_y = 1 - \zeta, \qquad J_z = \frac{\zeta^2 - 1}{2}.$$

Then

$$f_n = \frac{(\zeta^2+3)(\zeta^2-3)}{(\zeta^2-1)^2} - \frac{2\zeta^2(\zeta^2+3)}{(2n+1)^2(\zeta^2-1)^2} \frac{\bar{s}_n(\zeta^{-2})\bar{s}_{-n-1}(\zeta^{-2})}{s_n(\zeta^{-2})s_{-n-1}(\zeta^{-2})}.$$

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Infinite lattice limit

Baxter's formula for the free energy implies

$$f_\infty = \lim_{n\to\infty} f_n = \begin{cases} \dfrac{(\zeta^2+3)(\zeta^2-3)}{(\zeta^2-1)^2}, & |\zeta| \geq 3, \\ -\dfrac{(\zeta^2+3)(\zeta^2+6\zeta-3)}{8(\zeta-1)^2}, & -3 < \zeta < 0, \\ -\dfrac{(\zeta^2+3)(\zeta^2-6\zeta-3)}{8(\zeta+1)^2}, & 0 < \zeta < 3. \end{cases}$$

The term with tau functions gives finite length correction.

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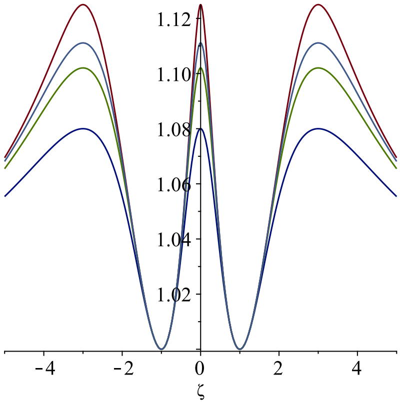
Finite versus infinite chain



From bottom to top: f_2, f_3, f_4 (length 5, 7, 9), f_∞.

Painlevé VI: elliptic form

PVI is the most general 2nd order ODE, all of whose movable singularities are poles.

Elliptic form:

$$\frac{d^2q}{dt^2} = \sum_{j=0}^{3} \alpha_j \wp'(q - \gamma_j | 2\pi \mathrm{i} t),$$

α_j parameters, $\wp$ Weierstrass' function with periods 1, $2\pi\mathrm{i}t$ and half-periods γ_j.

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$$q \in \mathbb{C}/(\mathbb{Z} + 2\pi \mathrm{i} t \mathbb{Z}) \simeq \mathbb{C}[x, y]/(y^2 = x(x-1)(x-s)).$$

Change of variable $(q, t) \mapsto (x, s)$ gives algebraic form of PVI:

$$\frac{d^2x}{ds^2} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-s}\right)\left(\frac{dx}{ds}\right)^2 - \left(\frac{1}{s} + \frac{1}{s-1} + \frac{1}{x-s}\right)\frac{dx}{ds}$$
$$+ \frac{x(x-1)(x-s)}{s^2(s-1)^2}\left(\alpha_0 - \alpha_1 \frac{s}{x^2} + \alpha_2 \frac{s-1}{(x-1)^2} + \left(\frac{1}{2} - \alpha_3\right)\frac{s(s-1)}{(x-s)^2}\right).$$

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Hamiltonian form of PVI

Painlevé VI is equivalent to non-stationary Hamiltonian system (Malmquist 1923, Manin 1998).

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q},$$

where

$$H(p, q, t) = \frac{p^2}{2} - V(q, t)$$

with

$$V(q, t) = \sum_{j=1}^{4} \alpha_j \wp(q - \gamma_j | 2\pi \mathrm{i} t)$$

Darboux–Inozemtsev–Treibich–Verdier-... potential.

Picard solutions

$$\frac{d^2q}{dt^2} = \sum_{j=0}^{3} \alpha_j \wp'(q - \gamma_j | 2\pi \mathrm{i} t).$$

When $\alpha_0 = \cdots = \alpha_3 = 0$, the solution is ...

Applying Bäcklund transformations

$$(q, p, \alpha_0, \ldots, \alpha_3) \mapsto (\tilde{q}, \tilde{p}, \tilde{\alpha}_0, \ldots, \tilde{\alpha}_3)$$

gives Picard class solutions with $\tilde{\alpha}_j = n_j^2/2$, $n_j \in \mathbb{Z}$.

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Tau functions

A tau function is a solution to

$$\frac{\tau'(t)}{\tau(t)} = H(p(t), q(t), t)$$

where (p, q) solves PVI.

The polynomials s_n and $\bar{s}_n$ can be identified with tau functions, corresponding to algebraic solutions in the Picard class.

The parameters α_j are

$$\left(\frac{n^2}{2}, \frac{n^2}{2}, 0, 0\right) \quad \text{for } s_n, \qquad \left(\frac{n^2}{2}, \frac{n^2}{2}, \frac{1}{2}, \frac{1}{2}\right) \quad \text{for } \bar{s}_n.$$

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Correlation functions and Painlevé VI

What does

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mean for PVI?

It is essentially a PVI Hamiltonian with parameters

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Transfer matrix

Parametrize $H^{\mathsf{XYZ}} = H^{\mathsf{XYZ}}(\eta, \tau)$ by elliptic functions.
Supersymmetric case is $\eta = \pi/3$.

Transfer matrices of eight-vertex model.
One-parameter family $\mathbf{T}(u) = \mathbf{T}(u, \eta, \tau)$ commuting with H^{XYZ}.
H^{XYZ} is essentially $\mathbf{T}^{-1}(u)\mathbf{T}'(u)\big|_{u=\eta}$.

Can extend Ψ to η near $\pi/3$ and write

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Correlation functions and transfer matrix eigenvalue

$$H^{\mathsf{XYZ}} = -\frac{1}{2}\sum_{j=1}^{L}\Big(J_x\,\sigma_j^x\sigma_{j+1}^x + J_y\,\sigma_j^y\sigma_{j+1}^y + J_z\,\sigma_j^z\sigma_{j+1}^z\Big)\,.$$

Gives

$$\varepsilon = \langle H^{\mathsf{XYZ}}\rangle = -\frac{L}{2}\left(J_xC^x + J_yC^y + J_zC^z\right).$$

Taking derivatives in η and τ, these will not hit C^x, C^y, C^z (Hellmann–Feynman theorem).
Gives a system of three equations for the three correlations.

Solution can be expressed in terms of the quantity $t\,t_{u\eta} - t_u t_\eta\big|_{u=\eta=\pi/3}$.

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Q-operator

A Q-operator $\mathbf{Q}(u) = \mathbf{Q}(u, \eta, \tau)$ should satisfy $[\mathbf{Q}(u), \mathbf{T}(v)] = 0$,

$$\mathbf{T}(u)\mathbf{Q}(u) = \phi(u-\eta)\mathbf{Q}(u+2\eta) + \phi(u+\eta)\mathbf{Q}(u-2\eta),$$

where $\phi(u) = \theta_1(u|\tau)^L$ (Jacobi theta function).

If $\mathbf{Q}(u)\Psi = q(u)\Psi$ we get

$$t(u)q(u) = \phi(u-\eta)q(u+2\eta) + \phi(u+\eta)q(u-2\eta).$$

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in η at $\eta = \pi/3$.

Problem: Known constructions of Q-operators either work for $\eta \neq \pi/3$ (Baxter) or $\eta = \pi/3$ (Fabricius, Roan).

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Connection to s_n, $\bar{s}_n$

After computations, η-derivatives cancel and we can express correlations in terms of

$$q(0)\left(q''\left(\frac{\pi}{3}\right)+\frac{\phi'}{\phi}\left(\frac{\pi}{3}\right)q'\left(\frac{\pi}{3}\right)\right)-q''(0)q\left(\frac{\pi}{3}\right)$$

at $\eta=\pi/3$.

The polynomials s_n and $\bar{s}_n$ can also be expressed in terms of $q(u)$, but at different points!
For instance, $\bar{s}_n$ is essentally $q'(\pi+\pi\tau/2)$.

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Difference-differential equation

Let

$$\psi = \frac{\theta_1(u|\tau)^n}{\theta_1(3u|3\tau)\theta_3(3u/2|3\tau/2)}\, q(u).$$

Then

$$\psi_{uu} - V\psi = \alpha\psi(u) + \beta\frac{\theta_4(3u/2|3\tau/2)^2}{\theta_3(3u/2|3\tau/2)^2}\,\psi(u+\pi),$$

α and β are independent of u,
V is Darboux potential with parameters

$$\left(\frac{n(n+1)}{2}, \frac{n(n+1)}{2}, 1, 1\right).$$

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Three special cases

Manin (1998):

respectively twistor geometry and Frobenius manifolds.

Our last remark concerns some similarity between the (generalized) Lamé potentials in the theory of KdV–type equations and our classically integrable potentials of the non–linear equation (2.2). According to [TV], the former are of the form

$$\sum_{j=0}^{3} \frac{n_j(n_j+1)}{2} \wp(z + \frac{T_j}{2}, \tau),$$

whereas according to our discussion the latter have coefficients (proportional to) $(n_j^2)/2$ or $(n_j + \frac{1}{2})^2/2$. Is there a direct connection between the two phenomena?

References

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Half squares

Our expression for correlation functions can be interpreted as the Hamiltonian of PVI with parameters

$$\left(\frac{(n+1/2)^2}{2}, \frac{(n+1/2)^2}{2}, 0, 0\right)$$

evaluated at a solution to PVI with parameters

$$\left(\frac{n^2}{2}, \frac{n^2}{2}, 0, 0\right).$$

Triangular numbers

Q-operator eigenvalue satisfies differential-difference equation with parameters $\left(\frac{n(n+1)}{2}, \frac{n(n+1)}{2}, 1, 1\right)$.

Bazhanov and Mangazeev found that it satisfies the non-stationary Lamé equation (quantum Painlevé VI)

$$\psi_t = \frac{1}{2}\psi_{xx} - V\psi$$

with parameters $\left(\frac{n(n+1)}{2}, \frac{n(n+1)}{2}, 0, 0\right)$.

Manin's three cases appear together!

Questions

- Other correlations? One-point correlations, emptiness formation probability.
- Scaling limit $n \to \infty$, $\tau \to \mathrm{i}\infty$, $e^{\pi \mathrm{i} \tau} \sim n^{-2/3}$.
  Should be related to SUSY sine-Gordon and Painlevé III.
- Conceptual explanation for relations between XYZ model and PVI equation?
- What is the eigenvector Ψ? For the XXZ model, given by explicit integral formulas (Razumov, Stroganov & Zinn-Justin 2007).
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