Constructing perturbative long-range spin chain deformations

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Minahan, Zarembo, 2003

In planar $\mathcal{N}=4$ SYM at one-loop

$$\boxed{\text{Dilatation operator}} \longrightarrow \boxed{\mathcal{H}_{NN}}$$

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$$\Big\downarrow \ \text{su(2) sector}$$

$$\mathcal{H}_{i,i+1} \propto \mathbb{1}_{i,i+1} - P_{i,i+1} = \text{Heisenberg XXX spin chain}$$

$$\mathcal{H} = \overset{\text{range } 2}{\boxed{\mathcal{H}_{NN}}} +$$

Beisert, Kristjansen & Staudacher, 2003

Beisert, Dippel & Staudacher, 2004

$$\mathcal{H} = \overset{\text{range 2}}{\boxed{\mathcal{H}_{NN}}} + g^2 \overset{\text{range 3}}{\boxed{\mathcal{H}_{NNN}}} + $$

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$$\mathcal{H} = \overset{\text{range } 2}{\boxed{\mathcal{H}_{NN}}} + g^2 \overset{\text{range } 3}{\boxed{\mathcal{H}_{NNN}}} + g^4 \overset{\text{range } 4}{\boxed{\mathcal{H}_{NNNN}}} + \cdots$$

↳ full $\mathcal{H}$ has infinite range

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↳ full $\mathcal{H}$ has infinite range

At each order in g^2, $\mathcal{H}$ is integrable

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- A general framework for this type of spin chains:
  - Perturbative method;
  - Based on the charges;
  - Uses the boost operator;

deformation equation

Bargheer, Beisert, & Loebbert 2008 2009

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- How to discover if a Hamiltonian (long-range) is integrable?

- How to systematically construct the Lax & the R-matrix?

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  Klabbers & Lamers, 2020

Very Rich & interesting subject.

PLAN

- The deformation equation;
- The basics of our method: NN;
- Generalization for long-range spin chains (perturbatively);
- Conclusions & open questions;

Deformation equation:

Bargheer, Beisert, & Loebbert 2008 2009

$$Q_r(g) = Q_r^{(0)} + g\, Q_r^{(1)} + g^2\, Q_r^{(2)} + \cdots$$

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$\hookrightarrow$ usual charges

while $Q_r^{(n)}$ has range $r+n$

In order to have integrability one needs

$$[Q_r(g), Q_s(g)] = 0$$

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where $X(g) = \sum_{n=0}^{\infty} X^{(n)} g^n$ can correspond to

- local operators
- boosted charges
- bilocal charges

$$\Rightarrow \quad \boxed{Q_r^{(n+1)} = \sum_{m=0}^{n} \left[X^{(m)}, Q_r^{(n-m)} \right]}$$

Beisert, Fiévet, de Leeuw & Loebbert, 2013

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Are all these deformations generated by a Lax?

Lifting constant Hamiltonians (NN)

It is well known that for periodic chain

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$$\mathbb{Q}_{n+1} = \frac{d^{(n)}}{du^n} \log t(u) \Big|_{u=0}$$

$\mathcal{L}(u)$ satisfies

$$R_{ab}(u,v)\,\mathcal{L}_{aj}(u)\,\mathcal{L}_{bj}(v) = \mathcal{L}_{bj}(v)\,\mathcal{L}_{aj}(u)\,R_{ab}(u,v)$$

- Start with some ansatz Hamiltonian

- $$\mathcal{H} = h_1 \mathbb{I} + h_2 (\sigma^z \otimes 1 - 1 \otimes \sigma^z) + h_3 \sigma^+ \otimes \sigma^- + h_4 \sigma^- \otimes \sigma^z + h_5 (\sigma^z \otimes 1 + 1 \otimes \sigma^z)$$

- $h_i \to$ constants

- $\mathcal{L}'(u)\big|_{u=0} = P\mathcal{H}$ and $\mathcal{L}(0) = P$

$$\mathcal{L}(u) = P\left(1 + u\mathcal{R} + \sum_{i>1} \frac{\mathcal{L}^{(i)} u^i}{i!}\right)$$

$$\mathcal{L}(u) = P\left(1 + u\mathcal{K} + \sum_{i>1} \frac{\mathcal{L}^{(i)} u^i}{i!}\right)$$

where

$$\mathcal{L}^{(i)} = \begin{pmatrix} \tilde{\ell}_1^{(i)} & 0 & 0 & 0 \\ 0 & \tilde{\ell}_2^{(i)} & \tilde{\ell}_3^{(i)} & 0 \\ 0 & \tilde{\ell}_4^{(i)} & \tilde{\ell}_5^{(i)} & 0 \\ 0 & 0 & 0 & \tilde{\ell}_6^{(i)} \end{pmatrix}$$

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All $\underline{u}$ dependence is now explicit

$\mathcal{L}^{(i)} \rightarrow$ constant

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NOT RELEVANT FOR COMMUTATOR PURPOSES

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$$\Rightarrow \quad \tilde{\ell}_6^{(2)} = -8h_2^2 - 2h_3h_4 + 8h_5^2 - \tilde{\ell}_1^{(2)} + \tilde{\ell}_3^{(2)} + \tilde{\ell}_4^{(2)}$$

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$$Q_4 = t^{-1}(0)\, t'''(0) + \boxed{f(Q_2, Q_3)}$$

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$$\tilde{l}_6^{(3)} = \cdots$$

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Again only one condition and all other $\tilde{l}_i^{(3)}$ remain free

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$$\tilde{\ell}_6^{(3)} = \cdots$$

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$$\boxed{Q_n = t^{-1}(0)\, t^{(n-1)}(0)} + f(Q_2, \ldots Q_{n-1})$$

$[Q_2, Q_n] = 0 \Rightarrow$ fixes $\tilde{\ell}_6^{(n-1)}$

So, all $\tilde{\ell}_i^{(j)}$, $i = 1, \ldots, 5$ are free

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- Now we have a perturbative series in $\underline{u}$ and in principle we have to sum it $\boxed{\text{hard part}}$

- In this case it is easy:

- One can put all free $\tilde{l}_i^{(j)}$ to zero and actually do the sum to obtain

$$\mathcal{L}(u) = \begin{pmatrix} 1+2uh_5 & 0 & 0 & 0 \\ 0 & uh_4 & 1-2uh_2 & 0 \\ 0 & 1+2uh_2 & uh_3 & 0 \\ 0 & 0 & 0 & A(u) \end{pmatrix}$$

where $$A(u) = \frac{1-u^2(4h_2^2+h_3h_4)}{1+2uh_5}$$

- RLL $\Rightarrow$ R-matrix

There is an easier way

- Apply the method to understand which (how many) elements are independent,
- Put all the "free" ones to zero,
- Let us say you find

$$\mathcal{L}(u) = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & f_1(u) & * & 0 \\ 0 & * & f_2(u) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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$f_1(u)$, $f_2(u)$ are the complicated ones

Trick:

$$\boxed{[t(u), H] = 0}$$

$$\Downarrow$$

Immediately gives you $f_1(u)$ and $f_2(u)$

Once you have $\mathcal{L}$, substitute it in RLL to find the R-matrix

$$\boxed{R_{ab}(u,v)}\,\mathcal{L}_{aj}(u)\,\mathcal{L}_{bj}(v) = \mathcal{L}_{bj}(v)\,\mathcal{L}_{aj}(u)\,\boxed{R_{ab}(u,v)}$$

Notice: since

$$\mathcal{L}(u) = P\left(1 + u\mathcal{H} + \sum_{i>1} \frac{\mathcal{L}^{(i)} u^i}{i!}\right)$$

- lowest commutation relations give
  - If $\mathcal{H}$ is already integrable $\therefore$ it finds $\mathcal{L}$ elements

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  - If not yet, find conditions for both $\mathcal{H}$ and $\mathcal{L}$

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- Lowest commutation relations give
  - If $\mathcal{H}$ is already integrable $\therefore$ it finds $\mathcal{L}$ elements
  - If not yet, find conditions for both $\mathcal{H}$ and $\mathcal{L}$
- It can be used to classify integrable models as well as to find $\mathcal{L}$ and R.

Perturbative long-range spin chains

Basically the same method I just presented for NN

+ extra ingredient NNN

$$t(u) = \mathrm{tr}_A\, \mathcal{L}_{AN}(u) \cdots \mathcal{L}_{A1}(u)$$

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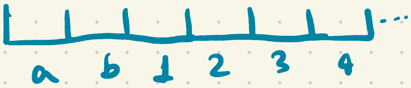
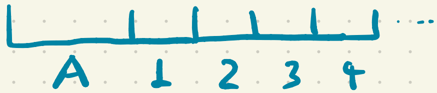
Gombor, Pozsgay, 2021

$$t(u) = \mathrm{tr}_A\, \mathcal{L}_{AN}(u) \cdots \mathcal{L}_{A1}(u)$$

$$\mathcal{L}_{Aj} : \underbrace{V \otimes V}_{A} \otimes \underbrace{V \otimes \cdots \otimes V}_{1 \cdots N} \mapsto V \otimes V \otimes \underbrace{V \otimes \cdots \otimes V}_{1 \cdots N}$$

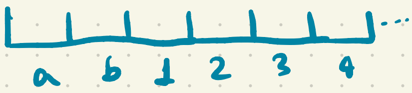
i.e. the auxilliary space was $\boxed{\text{doubled}}$

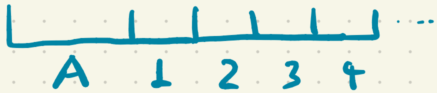




$\mathcal{O}_{a1}$ (NNN) $\qquad$ $\mathcal{O}_{A1}$ (NN)

$$\mathcal{L}_{A1}(u) = \mathcal{L}(u) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(N-1)}$$





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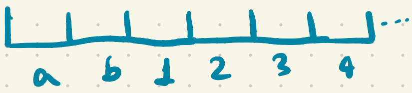
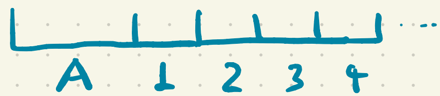
$$\boxed{\begin{aligned} \mathcal{L}_{A2}(u) &= P_{12}\, \mathcal{L}_{A1}(u)\, P_{12} \\ \mathcal{L}_{A3}(u) &= P_{23}\, \mathcal{L}_{A2}(u)\, P_{23} \\ &\vdots \end{aligned}}$$





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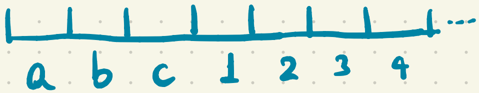
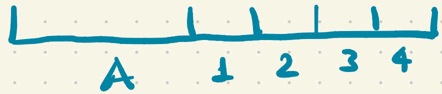
$$\mathcal{L}_{Aj} = \mathcal{L}_{abj}$$

$$\mathcal{L}_{Aj}(0) = P_{aj} P_{bj}$$

NNNN





$\vartheta_{a1}(NNNN)$ $\qquad$ $\vartheta_{A1}(NN)$

$\vdots$

In general, $\quad t(u) = \mathrm{tr}_A \, \mathcal{L}_{AN}(u) \cdots \mathcal{L}_{A1}(u)$

$$\dim\left(V^{(A)}\right) = \left(\dim V^{(i)}\right)^{r-1}$$

Method:

Ansatz for $\mathcal{H}$

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$\boxed{t^A(u) \text{ defined on larger space}}$

Method:

Ansatz for H

$\Rightarrow$

Expansion for $\mathcal{L}$
(now both on g and u)

$\Downarrow$

$t^A(u)$ defined on larger space

$\Leftarrow$

$$Q_{n+1}^{(j)} = \frac{d}{du} \log t^A(u)\Big|_{u=0}$$

$$[Q_2^A, Q_n^A] = 0$$

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Solve these assuming regularity

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$\Rightarrow$

$$[t^A(u), \mathcal{H}^A] = 0$$

$\boxed{SU(2)}$

$$\mathcal{H} = (1 - P_{12}) + g^2(1 - P_{13}) + \cdots$$

$$\sim 1 - P_{12} - g^2 P_{13}$$

In order to avoid wrapping effects $\boxed{N = 2r}$

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NNN (range 3)

$$\boxed{\dim(V^{(A)}) = 4, \qquad N = 6}$$

Boundary Conditions: $\mathcal{L}'_{ab_j}(0) = P_{a_j} P_{b_j} \mathcal{H}_{ab_j}$

$$\mathcal{L}_{ab,j}(0) = P_{a_j} P_{b_j}$$

cf. Gombor 2022

$$\boxed{\mathcal{L}_{ab,j}(u) = P_{a_j} P_{b_j} \left(P_{ab}\,\mathcal{L}_{ab}(u) - \frac{2g^2 u}{(u-2)(u^2-1)} P_{a_j} \right)}$$

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RLL

$$R_{AB}(u,v)\,\mathcal{L}_{A1}(u)\,\mathcal{L}_{B1}(v) = \mathcal{L}_{B1}(v)\,\mathcal{L}_{A1}(u)\,R_{AB}(u,v)$$

$$\underset{4d}{V^{(A)}} \otimes \underset{4d}{V^{(B)}} \otimes \underset{2d}{V} \mapsto \underset{4d}{V^{(A)}} \otimes \underset{4d}{V^{(B)}} \otimes \underset{2d}{V}$$

$$\mathcal{L}_{B1} = 1_4 \otimes \mathcal{L}$$

$$\mathcal{L}_{A1} = P_{AB}\,\mathcal{L}_{B1}\,P_{AB}$$

$$P_{AB} = P \otimes 1_2$$

$$R_{AB} = R \otimes 1_2$$

$[\mathcal{L}]$ = 8x8 matrix

$[R]$ = 16x16 matrix

Remember it is all perturbative

$$R(u,v) = \boxed{R^{(0)}_{r=3}(u,v)} + g^2 R^{(2)}_{r=3}(u,v) + \cdots$$

$\downarrow$

RRRR (xxx)

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$$R(u,v) = R^{(0)}(u,v) + g^2 \boxed{R^{(2)}(u,v)} + \cdots$$

new

NNNN (range 4) $\dim(V^{(A)}) = 8$, $N = 8$

$$\mathcal{H}_{1234} = \mathcal{H}_{123} + g^4 \sum_i P_i$$

with

$$P_1 = A_1 \sum_{i=1}^{3} \sigma_i \otimes 1 \otimes 1 \otimes \sigma_i\,, \qquad P_2 = A_2 \sum_{i,j=1}^{3} \sigma_i \otimes \sigma_j \otimes \sigma_j \otimes \sigma_i$$

$$P_3 = A_3 \sum_{i=1}^{3} \sigma_i \otimes \sigma_j \otimes \sigma_i \otimes \sigma_j\,, \qquad P_4 = A_4 \sum_{i,j=1}^{3} \sigma_i \otimes \sigma_i \otimes \sigma_j \otimes \sigma_j$$

$$P_5 = A_5 \sum_{i,j,k=1}^{3} \varepsilon^{ijk} \sigma_i \otimes \sigma_j \otimes 1 \otimes \sigma_k\,, \qquad P_6 = A_6 \sum_{i,j,k=1}^{3} \varepsilon^{ijk} \sigma_i \otimes 1 \otimes \sigma_j \otimes \sigma_k$$

is $SU(2)$ invariant.

we find

$$\boxed{A_3 = \frac{1-2A_1-4A_2}{4},} \qquad \boxed{A_4 = \frac{1-2A_1}{4}} \qquad (*)$$

so anything NOT satisfying $(*)$ is not integrable

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so anything NOT satisfying $(*)$ is not integrable

If we assume H is symmetric only $(*)$ survives

$\Rightarrow$ we have two deformations;

However one of them is actually Q_4 for the original chain

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- It matches with the result by Mclaughin & Spiering 2022 which shows chaos if one stop at two loops.

Explicitly,

$$
\begin{aligned}
\mathcal{L}^{(g^4)} =&\lambda_1 1+\\
&\lambda_2 \sum \sigma^i \otimes \sigma^i \otimes 1 \otimes 1 + \lambda_3 \sum 1 \otimes \sigma^i \otimes \sigma^i \otimes 1 + \lambda_4 \sum 1 \otimes 1 \otimes \sigma^i \otimes \sigma^i +\\
&\lambda_5 \sum \sigma^i \otimes 1 \otimes \sigma^i \otimes 1 + \lambda_6 \sum 1 \otimes \sigma^i \otimes 1 \otimes \sigma^i +\\
&\lambda_7 \epsilon_{ijk} \sigma^i \otimes \sigma^j \otimes \sigma^k \otimes 1 + \lambda_8 \epsilon_{ijk} 1 \otimes \sigma^i \otimes \sigma^j \otimes \sigma^k +\\
&\lambda_9 \sum \sigma^i \otimes 1 \otimes 1 \otimes \sigma^i + \lambda_{10} \sum \sigma^i \otimes \sigma^j \otimes \sigma^j \otimes \sigma^i + \lambda_{11} \sum \sigma^i \otimes \sigma^j \otimes \sigma^i \otimes \sigma^j +\\
&\lambda_{12} \sum \sigma^i \otimes \sigma^i \otimes \sigma^j \otimes \sigma^j + \lambda_{13}\, \epsilon_{ijk} \sigma^i \otimes \sigma^j \otimes 1 \otimes \sigma^k + \lambda_{14} \epsilon_{ijk} \sigma^i \otimes 1 \otimes \sigma^j \otimes \sigma^k
\end{aligned}
$$

$$\lambda_4 = \frac{(u-1)^3(3u-2)}{2(1-2u)^2u^2}(2A_1-1) + \left(\frac{2}{u}-1\right)\lambda_{10}$$
$$\lambda_6 = \frac{(u-1)^2}{2(2u-1)^2}\left(\frac{4(2u-1)}{u^2}A_1 - 2A_1 + 1\right) + \left(\frac{2}{u}-1\right)\lambda_9$$
$$\lambda_8 = \frac{i(u-1)^3}{(1-2u)^2(u-2)}\left(A_1 - \frac{(2A_1-1)(u+2)}{4u}\right) + \left(\frac{2}{u}-1\right)\lambda_{14}$$
$$\lambda_{11} = \frac{A_1(u(u(2u-5)+9)-4)(u-1)^2}{2(1-2u)^2u^2} + \frac{A_2(u-1)^2}{u(2u-1)} - i\lambda_{14} + \left(\frac{2}{u}-2\right)\lambda_9$$
$$- \frac{\left(u\left(u\left(u\left(4u^2-22u+31\right)-17\right)-4\right)+4\right)(u-1)^2}{4(u-2)^2u(2u-1)^3} - \frac{u}{2}\lambda_3 + \left(1-\frac{u}{2}\right)\lambda_5$$
$$\lambda_{12} = \frac{(1-2A_1)(u-1)^3}{4(1-2u)^2u} + \lambda_{10}$$
$$\lambda_{13} = \frac{iA_1(3u-2)(u-1)^2}{2(1-2u)^2u} - \frac{3iu(u-1)^2}{4(1-2u)^2(u-2)} + i\lambda_9$$

Because of Boundary conditions:

$$\lambda_i \sim \frac{a_i}{u} + b_i + O(u)$$

$$a_9 = A_1, \qquad a_{10} = \frac{1}{2} - A_1, \qquad b_9 = \frac{A_1}{2} \qquad b_{14} = \frac{i(2A_1+1)}{8}$$

the remaining a_i, b_i vanish.

Finding R is again straightforward once $\mathcal{L}$ is known.

$$R(u,v) = \boxed{R^{(0)}(u,v)}$$

64 x 64

product of nine XXX R's

$$+ \; g^2 R^{(2)}(u,v)$$

$$+ \; g^4 R^{(4)}(u,v) + \cdots$$

$$R(u,v) = \boxed{R^{(0)}_{r=4}(u,v)}$$

range 4

64 x 64

product of nine ~~XXX~~ R's

$$+ \; g^2 \boxed{R^{(2)}_{r=4}(u,v)}$$

probably some

product of $R^{(0)}_{r=2}$ with $R^{(2)}_{r=3}$

$$+ \; g^4 \, R^{(4)}_{r=4}(u,v) + \dots$$

↳ new

$$R: V^{(A)} \otimes V^{(B)} \mapsto V^{(A)} \otimes V^{(B)}$$

range	$[L]$	$[R]$
2	4×4	4×4

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range	$[\mathcal{L}]$	$[R]$
2	4×4	4×4
3	8×8	16×16
4	16×16	64×64

- It grows fast
- It is of non-difference form

6 vertex model (NNN)

$$\mathcal{H}_{12} = h_1 \mathbb{I} + h_2 (\sigma^z \otimes 1 - 1 \otimes \sigma^z)$$
$$+ h_3 \, \sigma^+ \otimes \sigma^- + h_4 \, \sigma^- \otimes \sigma^+$$
$$+ h_5 (\sigma^z \otimes 1 + 1 \otimes \sigma^z) + \boxed{h_6 (\sigma^z \otimes \sigma^z)}$$

h_i's are constant

periodic

$$\mathcal{H}_{123} = H_{12} + g^2 \boxed{\widetilde{\mathcal{H}}_{123}}$$

$$\rightarrow \sum_{a,b,c} h_{a,b,c}\, \sigma^a \otimes \sigma^b \otimes \sigma^c$$

$$a,b,c = 0, \pm, z$$

$$\mathcal{H}_{123} = H_{12} + g^2 \boxed{\tilde{\mathcal{H}}_{123}}$$

$$\to \sum_{a,b,c} h_{a,b,c}\, \sigma^a \otimes \sigma^b \otimes \sigma^c$$

$$a, b, c = 0, \pm, z$$

$$\boxed{\left[\mathcal{H}, \sum_i \sigma_i^3\right] = 0}$$

$$\mathcal{L}^{(r=3)}_{ab,j}(u) = P_{aj} P_{bj} \left(P_{ab} \mathcal{L}_{ab}(u) + g^2 \sum_{i \geq 1} \frac{\tilde{\mathcal{L}}^{(i)}_{ab,j} u^i}{i!} \right)$$

$\mathcal{L}^{(i)}_{abij}$ has twenty non-zero entries

$N=6$

Used the expansion + $[Q_2, Q_j]=0$, $j=3\ldots,$

$$+\ [t^A(u), H]=0$$

$\mathcal{L}$ is very complicated

but still one can compute R using RLL

All Hamiltonian deformations (range 3) computed in

Beisert, Fiévet, de Leeuw & Loebbert, 2013

are generated by a Lax operator.

Advantages

- Start with a constant H

  $\Rightarrow$ difference-form models

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- It gives one element of each equivalence class.

- It can be used to systematically classify long-range deformations of spin chains and find $\mathcal{L}$ and $\mathcal{R}$

  (for regular $\mathcal{L}$ and $\mathcal{R}$)

For the future:

- Understand how to do the Algebraic Bethe Ansatz for systems with NNN interaction
  (ongoing project with M. de Leeuw)

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- For higher algebras: to classify all

$$\boxed{R(u) = R^{(0)}(u) + g^2 R^{(2)}(u) + \cdots}$$

for $su(n)$, $so(2n)$, $so(2n+1)$, ...

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- Apply to other sectors like $su(1|1)$ and sl_2
- Generalization for open chains
  (ongoing project with Chiara Paletta)

THANK YOU!