Fusion products and finite-dimensional quotients for periodic Temperley-Lieb algebras

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Joint works with Y. Ikhlef and A. Langlois-Rémillard

Integrability in Condensed Matter Physics and Quantum Field Theory

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Outline

1) Affine and periodic Temperley-Lieb algebras
2) Uncoiled affine/periodic Temperley-Lieb algebras
3) Fusion of standard representations
4) Fusion of standard and irreducible representations

1) Affine and periodic Temperley-Lieb algebras

Definition of the algebras

- Three algebras:
  - Affine Temperley-Lieb algebra: $\mathsf{aTL}_N(\beta) = \langle \Omega^{\pm 1}, e_0, \ldots, e_{N-1} \rangle$
  - Periodic Temperley-Lieb algebra: $\mathsf{pTL}_N(\beta) = \langle \mathbf{1}, e_0, e_1, \ldots, e_{N-1} \rangle$
  - (Usual) Temperley-Lieb algebra: $\mathsf{TL}_N(\beta) = \langle \mathbf{1}, e_1, \ldots, e_{N-1} \rangle$
- Relations satisfied by the generators (with i, j taken mod N)

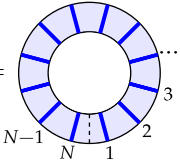
$$e_j^2 = \beta\, e_j \qquad e_j e_{j\pm 1} e_j = e_j \qquad e_i e_j = e_j e_i \quad \text{for } |i-j| > 1$$

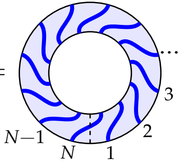
$$\Omega\, e_j\, \Omega^{-1} = e_{j-1} \qquad \Omega\, \Omega^{-1} = \Omega^{-1}\, \Omega = \mathbf{1} \qquad \Omega^2 e_1 = e_{N-1} e_{N-2} \cdots e_2 e_1$$

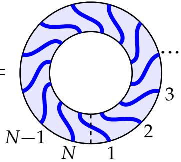
- Subalgebra structure: $\mathsf{TL}_N(\beta) \subset \mathsf{pTL}_N(\beta) \subset \mathsf{aTL}_N(\beta)$
- $\mathsf{TL}_N(\beta)$ is finite-dimensional
- $\mathsf{pTL}_N(\beta)$ and $\mathsf{aTL}_N(\beta)$ are infinite-dimensional

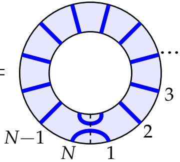
Generators and diagrams of $\mathsf{aTL}_N(\beta)$

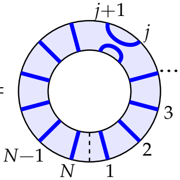
- **Connectivity diagrams** for the generators:

$\mathbf{1} =$ 

$\Omega =$ 

$\Omega^{-1} =$ 

$e_0 =$ 

$e_j =$ 

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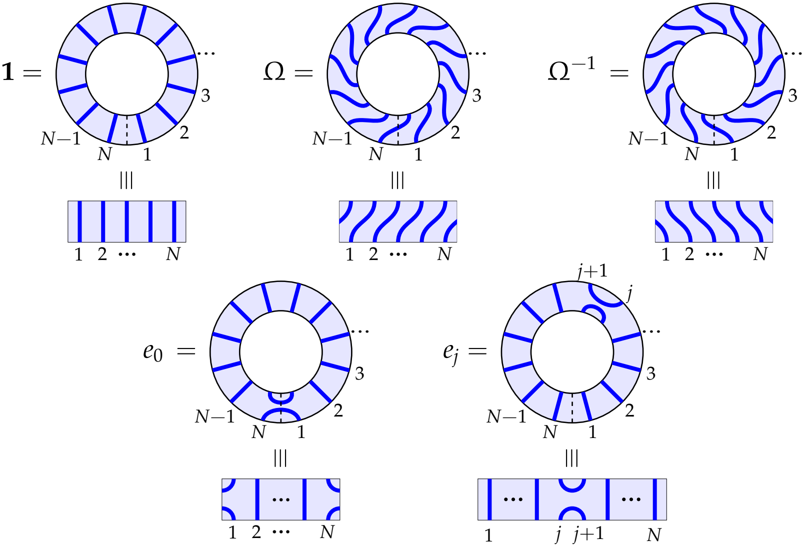
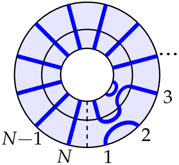
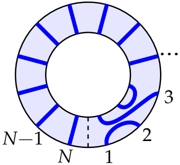
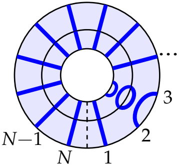


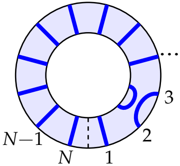
Product of diagrams

- Examples of products:

$e_1 e_2 =$ 

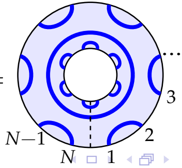
$=$ 

$(e_2)^2 =$ 

$= \beta$ 

$= \beta\, e_2$

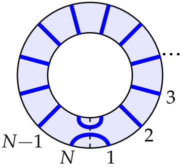
- Diagrams with arbitrary windings: $\Omega^k, k \in \mathbb{Z}$
- Non-contractible loops for N even:

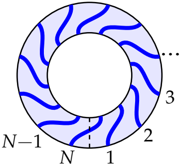
$(e_1 e_3 \cdots e_{N-1})(e_0 e_2 \cdots e_{N-2}) =$ 

Diagrammatic definitions

algebra	set of connectivity diagrams
$\mathsf{aTL}_N(\beta)$	all possible diagrams, with arbitrary windings and non-contractible loops
$\mathsf{pTL}_N(\beta)$	subset of even diagrams of $\mathsf{aTL}_N(\beta)$ excluding $\{\Omega^{2k}, k \in \mathbb{Z}\}$
$\mathsf{TL}_N(\beta)$	subset of diagrams of $\mathsf{aTL}_N(\beta)$ without arcs crossing the dashed line

- Parity of a diagram = number of arcs crossing the dashed line

even: 

odd: 

Various representations of $\mathsf{TL}_N(\beta)$

- XXZ representation: (defined on $(\mathbb{C}^2)^{\otimes N}$ with $\beta = -q - q^{-1}$)

$$\chi(e_j) = \underbrace{\mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_2}_{j-1} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & 1 & -q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \underbrace{\mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_2}_{N-j-1}$$

$$H_{XXZ} = -\sum_{j=1}^{N-1} \chi(e_j)$$

Pasquier, Saleur (1990)

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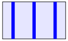
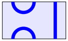
- Adjoint representation: (example for $\mathsf{TL}_3(\beta)$)

$v_1 =$  $v_2 =$  $v_3 =$  $v_4 =$  $v_5 =$ 

$$\mu(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & \beta & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mu(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & \beta \end{pmatrix}$$

Various representations of $\mathsf{TL}_N(\beta)$

- XXZ representation: (defined on $(\mathbb{C}^2)^{\otimes N}$ with $\beta = -q - q^{-1}$)

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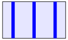
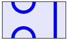
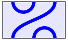
$v_1 =$  $v_2 =$  $v_3 =$  $v_4 =$  $v_5 =$ 

$$\mu(e_1) = \left(\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & 0 \\ \hline 1 & \beta & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \beta & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \mu(e_2) = \left(\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \beta & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & \beta \end{array}\right)$$

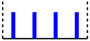
Standard modules $\mathsf{W}_{k,z}(N)$ over $\mathsf{aTL}_N(\beta)$

- Bases of link states of $\mathsf{W}_{k,z}(N)$ with $2N$ nodes and $2k$ defects:

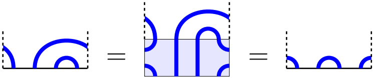
$\mathsf{W}_{0,z}(4)$:

$\mathsf{W}_{1,z}(4)$:

$\mathsf{W}_{2,z}(4)$:



- Action of the algebra: $e_0 \cdot$



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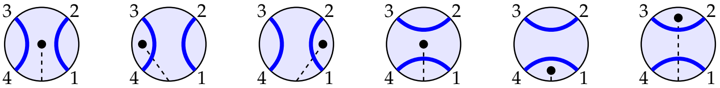
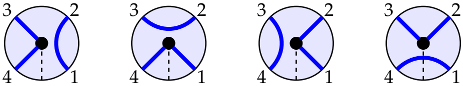
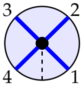


$\mathsf{W}_{1,z}(4)$:



$\mathsf{W}_{2,z}(4)$:



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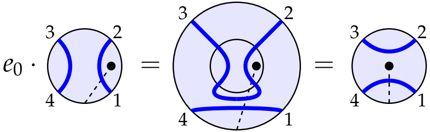

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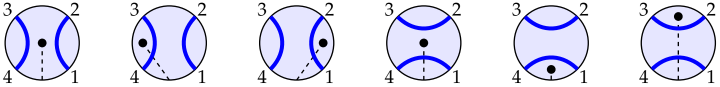

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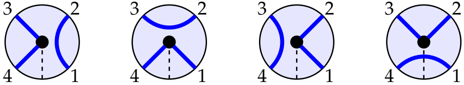
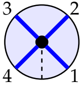


$\mathsf{W}_{2,z}(4)$:



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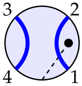
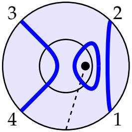
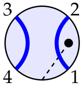
$e_1 \cdot$  $=$  $= \alpha$ 

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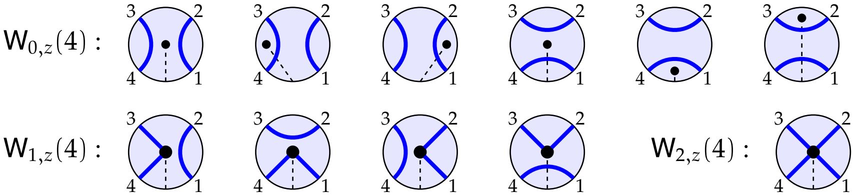

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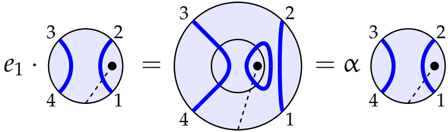
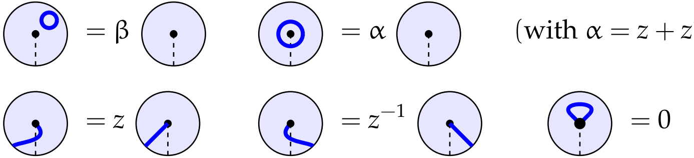


- Diagrammatic rules:



(with $\alpha = z + z^{-1}$)

2) Uncoiled affine/periodic Temperley-Lieb algebras

Joint work with A. Langlois-Rémillard

Uncoiled algebras for N odd

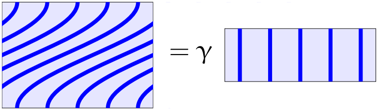
- Definition of the **uncoiled affine Temperley-Lieb algebra:**

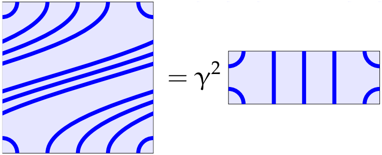
$$\mathsf{uaTL}_N(\beta,\gamma) = \frac{\mathsf{aTL}_N(\beta)}{\{\Omega^N = \gamma \mathbf{1}\}}$$

- Definition of the **uncoiled periodic Temperley-Lieb algebra:**

$$\mathsf{upTL}_N(\beta,\gamma) = \frac{\mathsf{pTL}_N(\beta)}{\{e_0(e_{N-1}e_{N-2}\cdots e_1 e_0)^{N-2} = \gamma^2 e_0\}}$$

- Extra relations in diagrams:



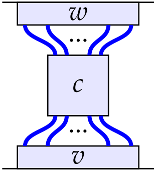


- Subalgebra structure: $\mathsf{upTL}_N(\beta,\gamma) \subset \mathsf{uaTL}_N(\beta,\gamma)$

Sandwich diagrams for N odd

- Sets of sandwich diagrams

$$\mathsf{S}_k(\mathsf{E}) = \Big\{ \quad$$



$$\Big|\ v, w \in \mathsf{W}_{k,z}(N),\ c \in \mathsf{E} \subset \mathsf{aTL}_{2k}(\beta) \Big\}$$

- Sandwich basis for $\mathsf{uaTL}_N(\beta,\gamma)$:

$$\mathsf{uaTL}_N(\beta,\gamma): \quad \bigcup_{k=\frac{1}{2},\frac{3}{2},\dots,\frac{N}{2}} \mathsf{S}_k(\{\Omega^r \,|\, 0 \leqslant r < 2k\})$$

- Dimension of the algebra:

$$\dim \mathsf{uaTL}_N(\beta,\gamma) = \sum_{k=\frac{1}{2},\frac{3}{2},\dots,\frac{N}{2}} 2k\,\big[\underbrace{\dim \mathsf{W}_{k,z}(N)}_{=\binom{N}{N/2-k}}\big]^2 = N\binom{N-1}{\frac{N-1}{2}}^2$$

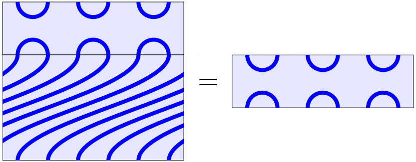
- $\mathsf{uaTL}_N(\beta,\gamma)$ is a **sandwich cellular algebra.** D. Tubbenhauer (2022)

Uncoiled algebras for N even

- Constraining relation in $\mathsf{aTL}_N(\beta)$ for N even:

$$\Omega^N E = E$$

$$(E = e_1 e_3 \cdots e_{N-1})$$



- Two families of uncoiled affine Temperley-Lieb algebras:

$$\mathsf{uaTL}_N^{(1)}(\beta, \alpha) = \frac{\mathsf{aTL}_N(\beta)}{\{\Omega^N = \mathbf{1}, E\Omega E = \alpha E\}}$$

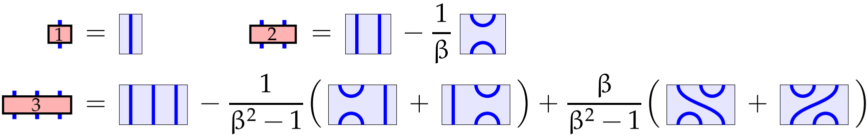
$$\mathsf{uaTL}_N^{(2)}(\beta, \gamma) = \frac{\mathsf{aTL}_N(\beta)}{\{\Omega^N = \gamma \mathbf{1}, E = 0\}}$$

- Similarly, there are two families of uncoiled algebras associated to $\mathsf{pTL}_N(\beta)$ for N even: $\mathsf{upTL}_N^{(1)}(\beta, \alpha)$ and $\mathsf{upTL}_N^{(2)}(\beta, \gamma)$.

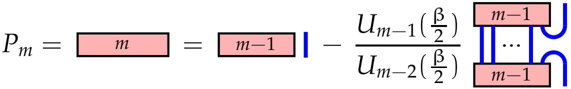
- We described these algebras with sandwich diagrams and computed their dimensions.

Wenzl-Jones projectors for $\mathsf{TL}_N(\beta)$

- The first three projectors P_1, P_2 and P_3 are



- Recursive definition:



- Properties:
  - satisfies the relations

$$P_m^2 = P_m \qquad e_i P_m = P_m e_i = 0 \qquad i = 1, \ldots, m-1$$

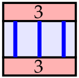
  - $P_N \in \mathsf{TL}_N(\beta)$ projects on the one-dimensional trivial representation τ

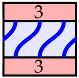
Wenzl-Jones projectors for $\mathsf{uaTL}_N(\beta)$

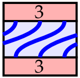
- Standard modules for $\mathsf{uaTL}_N(\beta)$ for N odd:

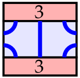
$$\left\{\mathsf{W}_{k,z}(N) \;\middle|\; z = \gamma^{1/2k}\mathrm{e}^{\pi \mathrm{i} r/k},\ k = \tfrac{1}{2}, \tfrac{3}{2}, \dots, \tfrac{N}{2},\ r = 0, 1, \dots, 2k-1\right\}$$

- We construct a projector $Q_{N,r}$ for each of the N one-dimensional modules.
- Example for $N = 3$: (with $\omega = \gamma^{1/3}\mathrm{e}^{2\pi \mathrm{i} r/3}$, $r = 0, 1, 2$)

$$Q_{3,r} = \frac{1}{3}\Bigg[$$ 

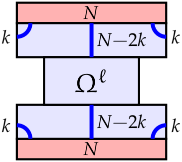
$$+\ \omega$$ 

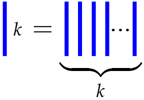
$$+\ \omega^2$$ 

$$-\ \frac{1}{\omega^2 + \omega^{-2} + \beta}$$ 

$$\Bigg]$$

- General construction for N odd:

$$Q_{N,r} = \sum_{k=0}^{\frac{N-1}{2}} \sum_{\ell=0}^{N-2k-1} \Gamma_{k,\ell}$$ 

where 

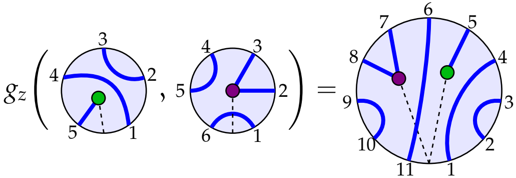
- Similar constructions for the other uncoiled algebras

3) Fusion of standard representations

Joint work with Y. Ikhlef – SciPost Phys. **12** (2022) 030

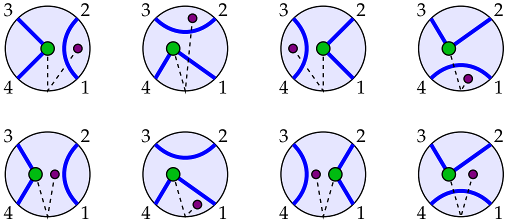
The representations $\mathsf{X}_{k,\ell,x,y,z}(N)$

- Gluing operator over $\mathsf{W}_{k,x}(N_a) \otimes \mathsf{W}_{\ell,y}(N_b)$:



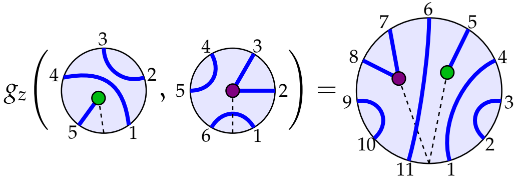
- Bases of link states with two marked points

for $\mathsf{X}_{1,0,x,y,z}(4)$:



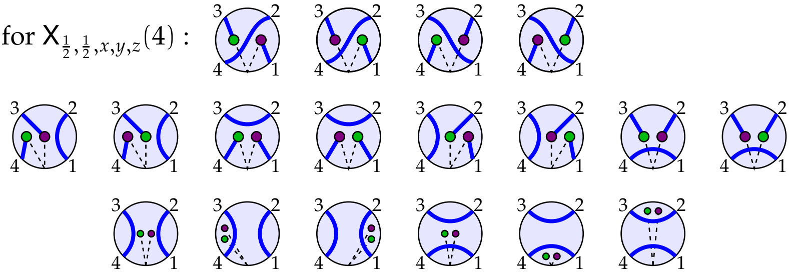
The representations $\mathsf{X}_{k,\ell,x,y,z}(N)$

- Gluing operator over $\mathsf{W}_{k,x}(N_{\mathsf{a}}) \otimes \mathsf{W}_{\ell,y}(N_{\mathsf{b}})$:

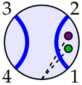


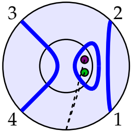
- Bases of link states with two marked points

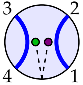
for $\mathsf{X}_{\frac{1}{2},\frac{1}{2},x,y,z}(4)$:



The representations $\mathsf{X}_{k,\ell,x,y,z}(N)$

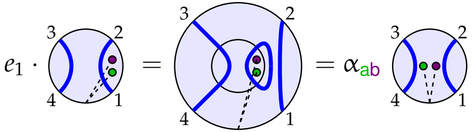
- Action of $\mathsf{aTL}_N(\beta)$: $e_1 \cdot$ 

$=$ 

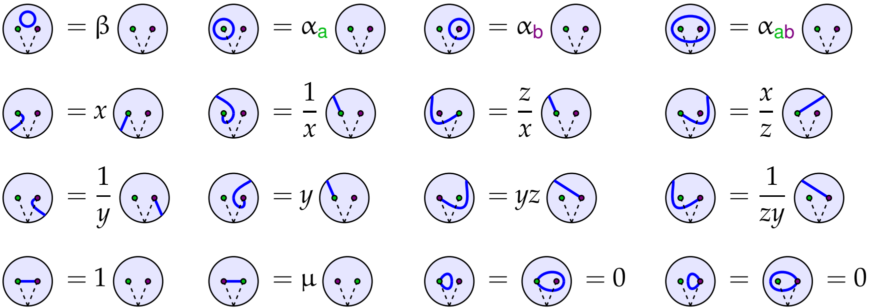
$= \alpha_{\mathsf{ab}}$ 

The representations $\mathsf{X}_{k,\ell,x,y,z}(N)$

- Action of $\mathsf{aTL}_N(\beta)$:



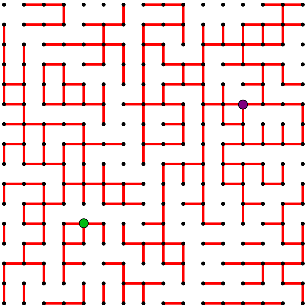
- Diagrammatic rules for the weights



with $\alpha_{\mathsf{a}} = x + x^{-1}$ $\alpha_{\mathsf{b}} = y + y^{-1}$ $\alpha_{\mathsf{ab}} = z + z^{-1}$ $\mu = \frac{zy}{x}$

Geometric correlation functions

- Correlations of connectivity operators $\mathcal{O}(z)$ in critical percolation



Geometric correlation functions

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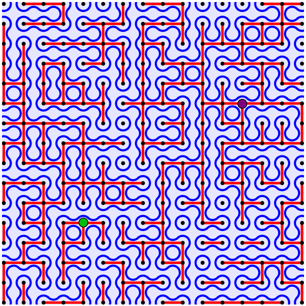

Geometric correlation functions

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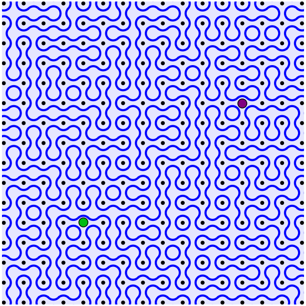

Geometric correlation functions

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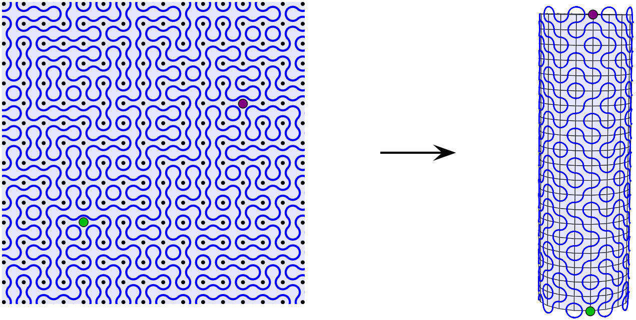
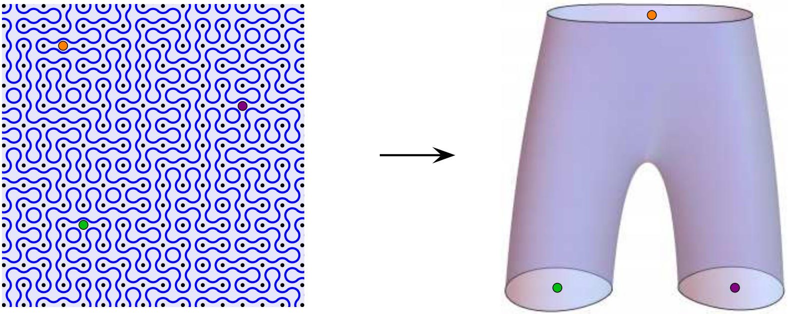


Geometric correlation functions

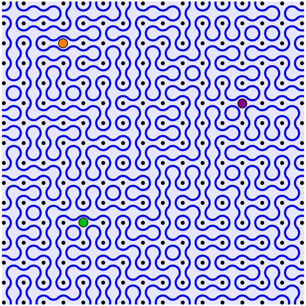
- Correlations of connectivity operators $\mathcal{O}(z)$ in critical percolation

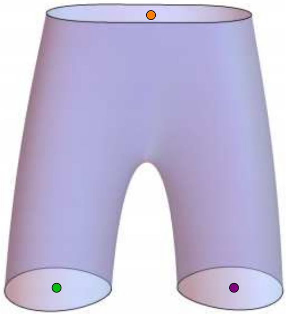


$\mathcal{O}(z_a) \times \mathcal{O}(z_b)$

Geometric correlation functions

- Correlations of connectivity operators $\mathcal{O}(z)$ in critical percolation

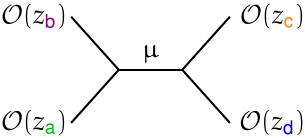




$\mathcal{O}(z_a) \times \mathcal{O}(z_b)$

- Four-point correlation functions:

$$\left\langle \mathcal{O}(z_a)\mathcal{O}(z_b)\mathcal{O}(z_c)\mathcal{O}(z_d) \right\rangle = \sum_{\mu}$$



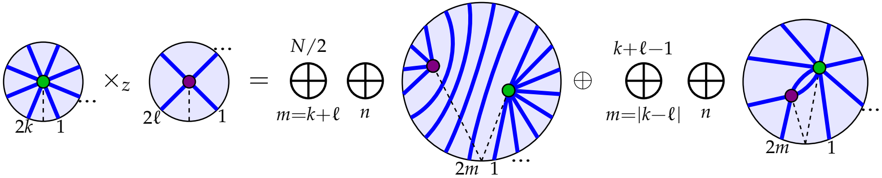
Properties of the representations $\mathsf{X}_{k,\ell,x,y,z}(N)$

- Dimensions: $\dim \mathsf{X}_{k,\ell,x,y,z}(N) = (\frac{N}{2} - |k-\ell| + 1)\binom{N}{\frac{N}{2} - |k-\ell|}$
- Decomposition over irreducible modules for β, z generic:

$$\mathsf{X}_{k,\ell,x,y,z}(N) \simeq \mathsf{W}_{|k-\ell|,z^\sigma}(N) \oplus \bigoplus_{m=|k-\ell|+1}^{N/2} \bigoplus_{n=0}^{2m-1} \mathsf{W}_{m,\omega_{m,n}}(N)$$

$$\omega_{m,n} = z^{(k-\ell)/m} \mathrm{e}^{\mathrm{i}n\pi/m} \qquad \sigma = \begin{cases} 1 & k \geqslant \ell \\ -1 & k < \ell \end{cases}$$

- Diagrammatic interpretation:



Fusion of representations

- Fusion channels and full fusion products:

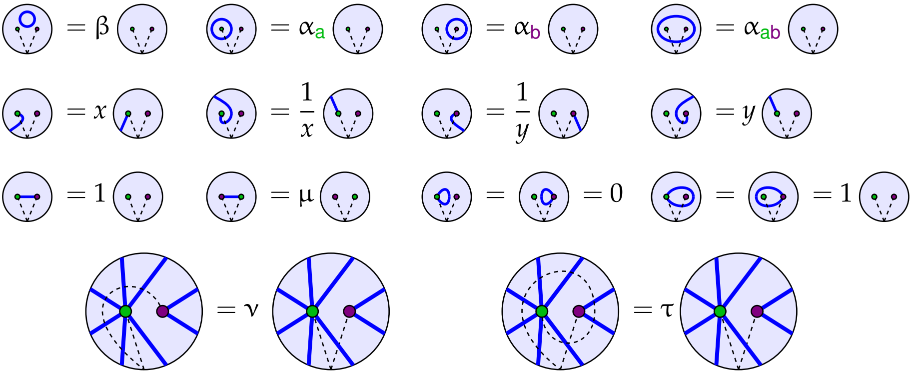
$$\mathsf{W}_{k,x}(N_\mathsf{a}) \times_z \mathsf{W}_{\ell,y}(N_\mathsf{b}) = \mathsf{X}_{k,\ell,x,y,z}(N)$$

$$\mathsf{W}_{k,x}(N_\mathsf{a}) \times \mathsf{W}_{\ell,y}(N_\mathsf{b}) = \bigoplus_{z\in\mathbb{C}^\times} \big[\mathsf{W}_{k,x}(N_\mathsf{a}) \times_z \mathsf{W}_{\ell,y}(N_\mathsf{b})\big]$$

- Commutativity: $\mathsf{W}_{k,x}(N_\mathsf{a}) \times \mathsf{W}_{\ell,y}(N_\mathsf{b}) = \mathsf{W}_{\ell,y}(N_\mathsf{b}) \times \mathsf{W}_{k,x}(N_\mathsf{a})$
- Remaining questions:
  - Is this the only way of defining representations with two marked points?
  - Can we give an algebraic definition of fusion for $\mathsf{aTL}_N(\beta)$?
  - Can we define fusion for other representations?
  - Are these fusion products associative?

The representations $\mathsf{Y}_{k,\ell,x,y,z,w}(N)$

- Also defined with link states with two marked points
- Different diagrammatic rules for the weights:



with $\quad \alpha_{a} = x + x^{-1} \qquad \alpha_{b} = y + y^{-1} \qquad \alpha_{ab} = w + w^{-1} \qquad \mu = \frac{zy}{x}$

- Dimensions: $\quad \dim \mathsf{Y}_{k,\ell,x,y,z,w}(N) = \lfloor \frac{N+1}{2} \rfloor \binom{N}{\lfloor \frac{N+1}{2} \rfloor}$

Algebraic definitions of fusion

- Definition of fusion for the Temperley-Lieb algebra:
  Read, Saleur (2007) Gainutdinov, Vasseur (2016)

$$\mathsf{V}(N_{\mathsf{a}}) \times \mathsf{W}(N_{\mathsf{b}}) = \mathsf{TL}_{N_{\mathsf{a}}+N_{\mathsf{b}}}(\beta) \otimes_{\mathsf{TL}_{N_{\mathsf{a}}}(\beta)\otimes\mathsf{TL}_{N_{\mathsf{b}}}(\beta)} \big(\mathsf{V}(N_{\mathsf{a}}) \otimes \mathsf{W}(N_{\mathsf{b}})\big)$$

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- Two previous attempts at defining fusion for $\mathsf{aTL}_N(\beta)$
  Gainutdinov, Saleur (2016) Gainutdinov, Jacobsen, Saleur (2018)
  Belletête, Saint-Aubin (2018)

- New ideas for a general definition of fusion for $\mathsf{aTL}_N(\beta)$:

$$\mathsf{V}(N_{\mathsf{a}}) \times \mathsf{W}(N_{\mathsf{b}}) \stackrel{?}{=} \mathsf{aTL}_{N_{\mathsf{a}}+N_{\mathsf{b}}}(\beta) \otimes_{\mathsf{TL}_{N_{\mathsf{a}}}(\beta)\otimes\mathsf{TL}_{N_{\mathsf{b}}}(\beta)} \big(\mathsf{V}(N_{\mathsf{a}}) \otimes \mathsf{W}(N_{\mathsf{b}})\big)$$

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  Gainutdinov, Saleur (2016)   Gainutdinov, Jacobsen, Saleur (2018)
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- New ideas for a general definition of fusion for $\mathsf{aTL}_N(\beta)$:

$$\mathsf{V}(N_\mathsf{a}) \times \mathsf{W}(N_\mathsf{b}) \neq \mathsf{aTL}_{N_\mathsf{a}+N_\mathsf{b}}(\beta) \otimes_{\mathsf{TL}_{N_\mathsf{a}}(\beta)\otimes\mathsf{TL}_{N_\mathsf{b}}(\beta)} \big(\mathsf{V}(N_\mathsf{a}) \otimes \mathsf{W}(N_\mathsf{b})\big)$$

Algebraic definitions of fusion

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$$\mathsf{V}(N_\mathsf{a}) \times \mathsf{W}(N_\mathsf{b}) \stackrel{?}{=} \mathsf{uaTL}_{N_\mathsf{a}+N_\mathsf{b}}(\beta,\gamma) \otimes_{\mathsf{TL}_{N_\mathsf{a}}(\beta)\otimes\mathsf{TL}_{N_\mathsf{b}}(\beta)} \big(\mathsf{V}(N_\mathsf{a}) \otimes \mathsf{W}(N_\mathsf{b})\big)$$

4) Fusion of standard and irreducible representations

Joint work with Y. Ikhlef

Structure of $\mathsf{W}_{k,z}(N)$ for q generic

- For z generic, $\mathsf{W}_{k,z}(N)$ is **irreducible** (with $\beta = -q - q^{-1}$).
  Graham, Lehrer (2008)

- For $z = \varepsilon q^{\pm m}$ non-generic, there are non-zero homomorphisms between standard modules:

$$\mathsf{W}_{m,\,\varepsilon q^{\pm k}}(N) \longrightarrow \mathsf{W}_{k,\,\varepsilon q^{\pm m}}(N) \qquad k < m \leqslant \tfrac{N}{2} \qquad \varepsilon \in \{+1, -1\}$$

- The module is **indecomposable yet reducible** and its structure is read from its **Loewy diagram:**

$$\mathsf{W}_{k,\,\varepsilon q^{\pm m}}(N) \simeq \begin{array}{ccc} \mathsf{Q}_{k,\,\varepsilon q^{\pm m}}(N) & & \\ & \searrow & \\ & & \mathsf{R}_{k,\,\varepsilon q^{\pm m}}(N) \end{array}$$

(quotient module) — $\mathsf{Q}_{k,\,\varepsilon q^{\pm m}}(N)$

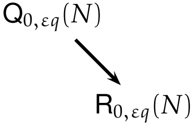
(radical submodule) — $\mathsf{R}_{k,\,\varepsilon q^{\pm m}}(N)$

- **Next goal:** study the fusion products $\mathsf{W}_{k,x}(N_\mathsf{a}) \times_z \mathsf{R}_{0,\,\varepsilon q^{\pm m}}(N_\mathsf{b})$ and $\mathsf{W}_{k,x}(N_\mathsf{a}) \times_z \mathsf{Q}_{0,\,\varepsilon q^{\pm m}}(N_\mathsf{b})$

Case $m = 1$: Basis for $\mathsf{R}_{0,\varepsilon q}$

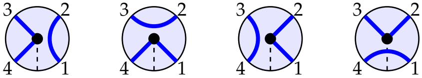
- Homomorphism and Loewy diagram:

$$\mathsf{W}_{1,1}(N) \to \mathsf{W}_{0,\varepsilon q}(N) \qquad \mathsf{W}_{0,\varepsilon q}(N) \simeq$$

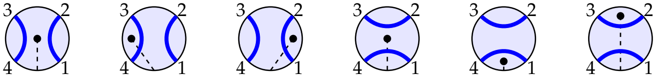


- Standard bases for $N = 4$:

$\mathsf{W}_{1,z}(4)$:

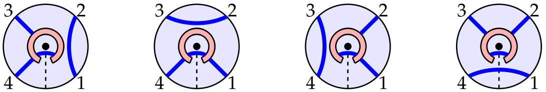


$\mathsf{W}_{0,z}(4)$:

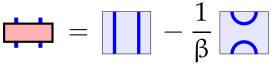


- A basis of $\mathsf{R}_{0,\varepsilon q}(N)$ is obtained using the projectors P_2:

$\mathsf{R}_{0,\varepsilon q}(4)$:



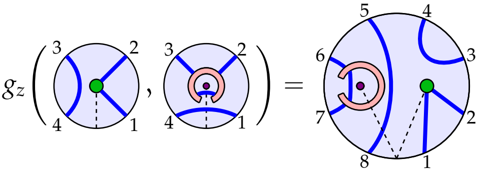
where



$\alpha = -\varepsilon\beta$

Case $m = 1$: The fusion $\mathsf{W}_{k,x} \times \mathsf{R}_{0,\varepsilon q}$

- Gluing of states in $\mathsf{W}_{k,x} \times \mathsf{R}_{0,\varepsilon q}$:



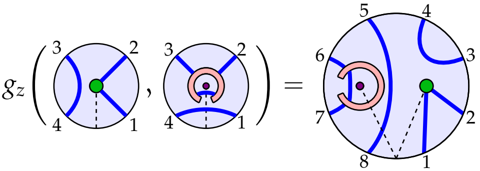
- The action of $\mathsf{aTL}_N(\beta)$ is defined in the same way for $\mathsf{W}_{k,x} \times \mathsf{R}_{0,\varepsilon q}$ as for $\mathsf{W}_{k,x} \times \mathsf{W}_{0,\varepsilon q}$, with $\alpha_{\mathsf{b}} = -\varepsilon\beta$.

- Submodule structure:

$$\mathsf{R}_{0,\varepsilon q} \subset \mathsf{W}_{0,\varepsilon q} \quad \longrightarrow \quad \mathsf{W}_{k,x} \times \mathsf{R}_{0,\varepsilon q} \subset \mathsf{W}_{k,x} \times \mathsf{W}_{0,\varepsilon q}$$

Case $m = 1$: The fusion $\mathsf{W}_{k,x} \times \mathsf{R}_{0,\varepsilon q}$

- Gluing of states in $\mathsf{W}_{k,x} \times \mathsf{R}_{0,\varepsilon q}$:



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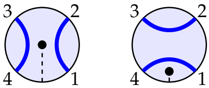
- Decomposition for $\varepsilon = -1$:

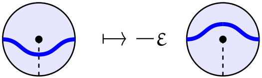
$$\mathsf{W}_{k,x} \times_z \mathsf{R}_{0,-q} \simeq \begin{cases} \dfrac{\mathsf{X}_{k,0,x,-q,z}}{\mathsf{W}_{k,x}} & \text{for } \begin{cases} k = 0, \; z = x^{\pm 1} \\ k > 0, \; z = x \end{cases} \\ \mathsf{X}_{k,0,x,-q,z} & \text{otherwise} \end{cases}$$

Case $m = 1$: The fusion $\mathsf{W}_{k,x} \times \mathsf{Q}_{0,\varepsilon q}$

- Basis for $\mathsf{Q}_{0,\varepsilon q}(4)$:



- The action of $\mathsf{aTL}_N(\beta)$ on $\mathsf{Q}_{0,\varepsilon q}(N)$ has an extra relation:



- The fusion $\mathsf{W}_{k,x} \times \mathsf{Q}_{0,\varepsilon q}$ is defined like $\mathsf{W}_{k,x} \times \mathsf{W}_{0,\varepsilon q}$ but with this extra relation for the second marked point.

- Decomposition of $\mathsf{W}_{k,x} \times \mathsf{Q}_{0,-q}$:

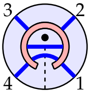
$$\mathsf{W}_{k,x} \times_z \mathsf{Q}_{0,-q} = \frac{\mathsf{W}_{k,x} \times_z \mathsf{W}_{0,-q}}{\mathsf{W}_{k,x} \times_z \mathsf{R}_{0,-q}} = \begin{cases} \mathsf{W}_{k,x} & \text{for} \begin{cases} k = 0,\ z = x^{\pm 1} \\ k > 0,\ z = x \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathsf{W}_{k,x} \times \mathsf{Q}_{0,-q} = \bigoplus_z \left[\mathsf{W}_{k,x} \times_z \mathsf{Q}_{0,-q}\right] = \mathsf{W}_{k,x}$$

Case $m = 2$: $\mathsf{W}_{k,x} \times \mathsf{Q}_{0,\varepsilon q^2}$

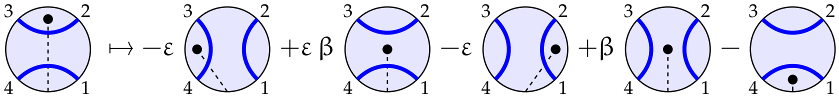
- Basis of $\mathsf{R}_{0,\varepsilon q^2}$ built using projectors P_4:

for $\mathsf{R}_{0,\varepsilon q^2}(4)$:



with $\alpha = \varepsilon(\beta^2 - 1)$

- Quotient relation for $\mathsf{Q}_{0,\varepsilon q^2}$:



- Decomposition of $\mathsf{W}_{k,x} \times \mathsf{Q}_{0,-q^2}$:

$$\mathsf{W}_{k,x} \times \mathsf{Q}_{0,-q^2} = \begin{cases} \mathsf{W}_{0,xq} \oplus \mathsf{W}_{1,x} & k = 0,\ x = \pm 1 \\ \mathsf{W}_{0,xq} \oplus \mathsf{W}_{0,xq^{-1}} & k = 0,\ x \neq 1 \\ \mathsf{W}_{\frac{1}{2},xq} \oplus \mathsf{W}_{\frac{1}{2},xq^{-1}} \oplus \mathsf{W}_{\frac{1}{2},x^{-1}} \oplus \mathsf{W}_{\frac{3}{2},x} & k = \frac{1}{2} \\ \mathsf{W}_{k,xq} \oplus \mathsf{W}_{k,xq^{-1}} \oplus \mathsf{W}_{k+1,x} & k \geqslant 1 \end{cases}$$

Associativity

- We compute $\mathsf{W}_{k,x} \times \mathsf{Q}_{0,-q^2} \times \mathsf{W}_{k+1,y}$ in two ways:

$$(\mathsf{W}_{k,x} \times \mathsf{Q}_{0,-q^2}) \times \mathsf{W}_{k+1,y} = (\mathsf{W}_{k,xq} \oplus \mathsf{W}_{k,xq^{-1}} \oplus \mathsf{W}_{k+1,x}) \times \mathsf{W}_{k+1,y}$$
$$= \cdots = \bigoplus_{z\in\mathbb{C}^\times} \bigoplus_{m\geqslant 0} \mathsf{W}_{m,z}$$

$$\mathsf{W}_{k,x} \times (\mathsf{Q}_{0,-q^2} \times \mathsf{W}_{k+1,y}) = \mathsf{W}_{k,x} \times (\mathsf{W}_{k+1,yq} \oplus \mathsf{W}_{k+1,yq^{-1}} \oplus \mathsf{W}_{k+2,y})$$
$$= \cdots = \bigoplus_{z\in\mathbb{C}^\times} \bigoplus_{m\geqslant 1} \mathsf{W}_{m,z}$$

- This fusion product is **non-associative!**
- An associative fusion product would instead satisfy

$$\mathsf{W}_{k,x} \times \mathsf{Q}_{0,-q^2} = \mathsf{W}_{k-1,x} \oplus \mathsf{W}_{k,xq} \oplus \mathsf{W}_{k,xq^{-1}} \oplus \mathsf{W}_{k+1,x}$$

- This holds for fusion defined with the modules $\mathsf{Y}_{k,\ell,x,y,z,w}(N)$!

Expectations from CFT

- Scaling limits for generic q, z:

$$\mathsf{W}_{k,z} \mapsto \bigoplus_{p=-\infty}^{\infty} \mathcal{V}(h_{\mu+p,k}) \otimes \mathcal{V}(h_{\mu+p,-k}) \qquad z = \mathrm{e}^{\mathrm{i}\pi\mu}$$

$$\mathsf{Q}_{0,-q^m} \mapsto \bigoplus_{p=1}^{\infty} \mathcal{K}(h_{p,m}) \otimes \mathcal{K}(h_{p,m}) \qquad \mathcal{K}(h_{r,s}) = \frac{\mathcal{V}(h_{r,s})}{\mathcal{V}(h_{r,s}+rs)}$$

- Fusion of chiral primary fields for integer r, s:

$$\phi_{\mu,\nu} \times \phi_{1,1} = \phi_{\mu,\nu} \qquad \phi_{\mu,\nu} \times \phi_{1,2} = \phi_{\mu,\nu-1} + \phi_{\mu,\nu+1}$$

Expectations from CFT

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- Fusion of chiral primary fields for integer r, s:

$$\phi_{\mu,\nu} \times \phi_{r,s} = \sum_{m=-(r-1)/2}^{(r-1)/2} \sum_{n=-(s-1)/2}^{(s-1)/2} \phi_{\mu+2m,\nu+2n}$$

Expectations from CFT

- Scaling limits for generic q, z:

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$$\mathsf{Q}_{0,-q^m} \mapsto \bigoplus_{p=1}^{\infty} \mathcal{K}(h_{p,m}) \otimes \mathcal{K}(h_{p,m}) \qquad \mathcal{K}(h_{r,s}) = \frac{\mathcal{V}(h_{r,s})}{\mathcal{V}(h_{r,s}+rs)}$$

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- Fusion rules for $\mathsf{W} \times \mathsf{Q}$:

$$\mathsf{W}_{k,x} \times \mathsf{Q}_{0,-q^m} = \begin{cases} \mathsf{W}_{k,x} & m = 1 \\ \mathsf{W}_{k-1,x} \oplus \mathsf{W}_{k,xq} \oplus \mathsf{W}_{k,xq^{-1}} \oplus \mathsf{W}_{k+1,x} & m = 2 \\ \bigoplus_{\mu,\nu=(m-1)/2}^{(m+1)/2} \mathsf{W}_{k+\mu-\nu,x\,q^{\mu+\nu}} & \text{general } m \end{cases}$$

Overview

- We introduced uncoiled affine and periodic Temperley-Lieb algebras as finite dimensional quotients of $\mathsf{aTL}_N(\beta)$ and $\mathsf{pTL}_N(\beta)$.
- We constructed their Wenzl-Jones projectors.
- We defined associative fusion products $\mathsf{Y}_{k,\ell,x,y,z,w}(N)$ for $\mathsf{aTL}_N(\beta)$ that reproduce the expected CFT fusion rules.
- The algebraic definition of fusion is still an open problem.