

Fusion products and finite-dimensional quotients for periodic Temperley-Lieb algebras

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Joint works with Y. Ikhlef and A. Langlois-Rémillard



Integrability in Condensed Matter Physics and Quantum Field Theory

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Outline

- 1) Affine and periodic Temperley-Lieb algebras
- 2) Uncoiled affine/periodic Temperley-Lieb algebras
- 3) Fusion of standard representations
- 4) Fusion of standard and irreducible representations

1) Affine and periodic Temperley-Lieb algebras

Definition of the algebras

- Three algebras:

- Affine Temperley-Lieb algebra: $\mathfrak{aTL}_N(\beta) = \langle \Omega^{\pm 1}, e_0, \dots, e_{N-1} \rangle$
- Periodic Temperley-Lieb algebra: $\mathfrak{pTL}_N(\beta) = \langle \mathbf{1}, e_0, e_1, \dots, e_{N-1} \rangle$
- (Usual) Temperley-Lieb algebra: $\mathfrak{TL}_N(\beta) = \langle \mathbf{1}, e_1, \dots, e_{N-1} \rangle$

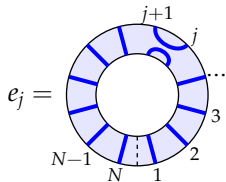
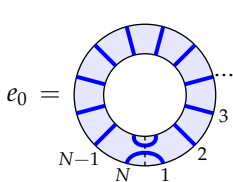
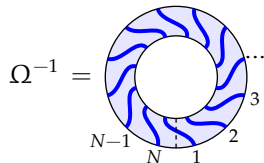
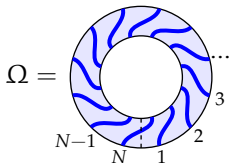
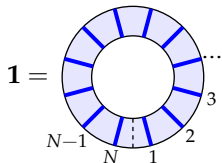
- Relations satisfied by the generators (with i, j taken mod N)

$$\begin{aligned} e_j^2 &= \beta e_j & e_j e_{j\pm 1} e_j &= e_j & e_i e_j &= e_j e_i & \text{for } |i-j| > 1 \\ \Omega e_j \Omega^{-1} &= e_{j-1} & \Omega \Omega^{-1} &= \Omega^{-1} \Omega = \mathbf{1} & \Omega^2 e_1 &= e_{N-1} e_{N-2} \cdots e_2 e_1 \end{aligned}$$

- Subalgebra structure: $\mathfrak{TL}_N(\beta) \subset \mathfrak{pTL}_N(\beta) \subset \mathfrak{aTL}_N(\beta)$
- $\mathfrak{TL}_N(\beta)$ is finite-dimensional
- $\mathfrak{pTL}_N(\beta)$ and $\mathfrak{aTL}_N(\beta)$ are infinite-dimensional

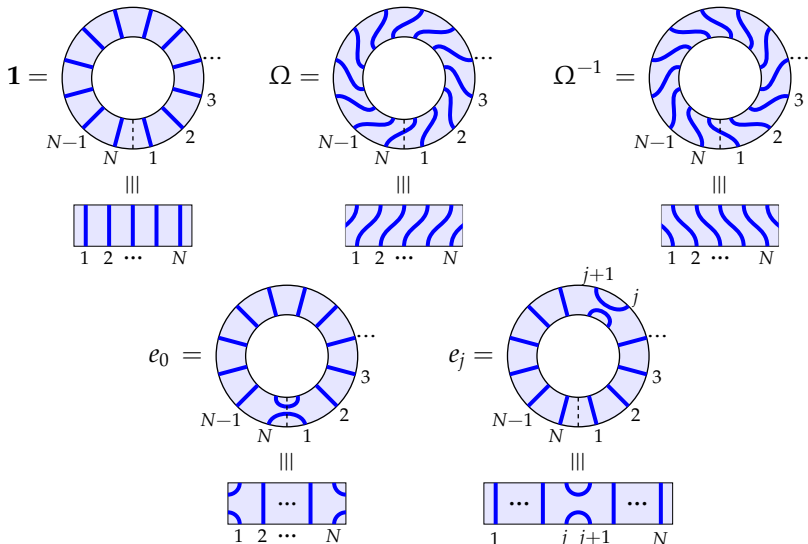
Generators and diagrams of $\mathfrak{aTL}_N(\beta)$

- **Connectivity diagrams** for the generators:



Generators and diagrams of $\mathfrak{aTL}_N(\beta)$

- **Connectivity diagrams** for the generators:



Product of diagrams

- Examples of products:

$$e_1 e_2 = \text{Diagram 1} = \text{Diagram 2}$$

$$(e_2)^2 = \text{Diagram 1} = \beta \text{Diagram 2} = \beta e_2$$

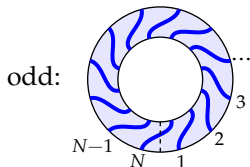
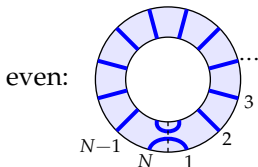
- Diagrams with arbitrary windings: $\Omega^k, k \in \mathbb{Z}$
- Non-contractible loops for N even:

$$(e_1 e_3 \cdots e_{N-1})(e_0 e_2 \cdots e_{N-2}) = \text{Diagram}$$

Diagrammatic definitions

algebra	set of connectivity diagrams
$\mathbf{aTL}_N(\beta)$	all possible diagrams, with arbitrary windings and non-contractible loops
$\mathbf{pTL}_N(\beta)$	subset of even diagrams of $\mathbf{aTL}_N(\beta)$ excluding $\{\Omega^{2k}, k \in \mathbb{Z}\}$
$\mathbf{TL}_N(\beta)$	subset of diagrams of $\mathbf{aTL}_N(\beta)$ without arcs crossing the dashed line

- Parity of a diagram = number of arcs crossing the dashed line



Various representations of $\mathrm{TL}_N(\beta)$

- XXZ representation: (defined on $(\mathbb{C}^2)^{\otimes N}$ with $\beta = -q - q^{-1}$)

$$\chi(e_j) = \underbrace{\mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_2}_{j-1} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & 1 & -q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \underbrace{\mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_2}_{N-j-1}$$

$$H_{\mathrm{XXZ}} = - \sum_{j=1}^{N-1} \chi(e_j) \quad \text{Pasquier, Saleur (1990)}$$

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- Trivial representation: $\tau(e_j) = 0$ $\tau(\mathbf{1}) = 1$
- Adjoint representation: (example for $\mathrm{TL}_3(\beta)$)

$$v_1 = \begin{array}{|c|c|c|c|} \hline \text{blue} & \text{blue} & \text{blue} & \text{blue} \\ \hline \end{array} \quad v_2 = \begin{array}{|c|c|c|c|} \hline \text{blue} & \text{blue} & \text{blue} & \text{blue} \\ \hline \end{array} \quad v_3 = \begin{array}{|c|c|c|c|} \hline \text{blue} & \text{blue} & \text{blue} & \text{blue} \\ \hline \end{array} \quad v_4 = \begin{array}{|c|c|c|c|} \hline \text{blue} & \text{blue} & \text{blue} & \text{blue} \\ \hline \end{array} \quad v_5 = \begin{array}{|c|c|c|c|} \hline \text{blue} & \text{blue} & \text{blue} & \text{blue} \\ \hline \end{array}$$

$$\mu(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & \beta & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mu(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & \beta \end{pmatrix}$$


Various representations of $\mathrm{TL}_N(\beta)$


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$$\chi(e_j) = \underbrace{\mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_2}_{j-1} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & 1 & -q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \underbrace{\mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_2}_{N-j-1}$$


$$H_{\mathrm{XXZ}} = \sum_{j=1}^{N-1} \chi(e_j)$$


- Trivial representation: $\tau(e_j) = 0$ $\tau(\mathbf{1}) = 1$
- Adjoint representation: (example for $\mathrm{TL}_3(\beta)$)

$v_1 =$


 $v_2 =$


 $v_3 =$


 $v_4 =$


 $v_5 =$


$$\mu(e_1) = \begin{pmatrix} 0 & | & 0 & 0 & | & 0 & 0 \\ - & | & \beta & 1 & | & 0 & 0 \\ 0 & | & 0 & 0 & | & 0 & 0 \\ - & | & 0 & 0 & | & \beta & 1 \\ 0 & | & 0 & 0 & | & 0 & 0 \end{pmatrix}$$

$$\mu(e_2) = \begin{pmatrix} 0 & | & 0 & 0 & | & 0 & 0 \\ - & | & 0 & 0 & | & 0 & 0 \\ 0 & | & 1 & \beta & | & 0 & 0 \\ - & | & 0 & 0 & | & 0 & 0 \\ 1 & | & 0 & 0 & | & 1 & \beta \end{pmatrix}$$

Standard modules $W_{k,z}(N)$ over $\text{aTL}_N(\beta)$

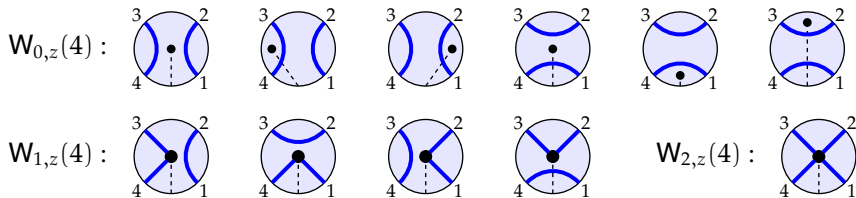
- Bases of link states of $W_{k,z}(N)$ with $2N$ nodes and $2k$ defects:



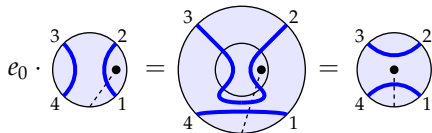
- Action of the algebra: $e_0 \cdot$ $=$ $=$

Standard modules $W_{k,z}(N)$ over $\mathfrak{aTL}_N(\beta)$

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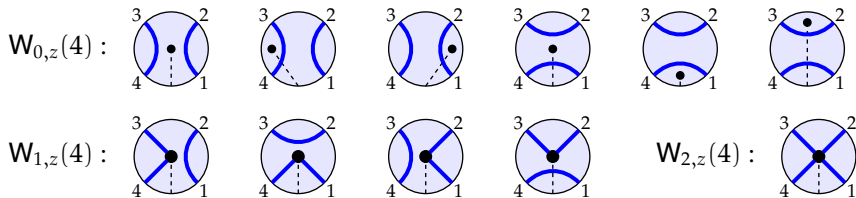


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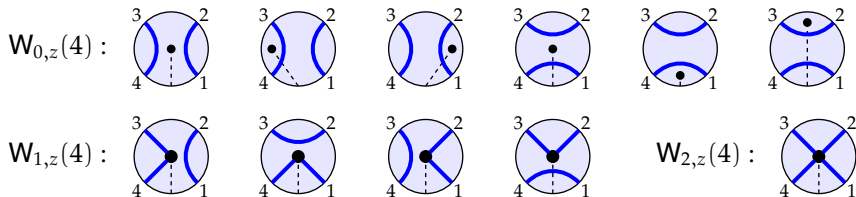
- Action of the algebra:

$$e_1 \cdot \text{diagram} = \text{diagram} = \alpha \cdot \text{diagram}$$

The equation shows the action of the algebra element e_1 on a link state. The first diagram is a circle with nodes 1, 2, 3, 4 and a black dot. Blue arcs connect 3-2 and 4-1. A dashed line connects node 4 to the dot. The second diagram is the same as the first but with an additional blue arc connecting nodes 3 and 4, and a blue loop around the dot. The third diagram is the original link state with a scalar factor α .

Standard modules $W_{k,z}(N)$ over $\mathfrak{aTL}_N(\beta)$

- Bases of link states of $W_{k,z}(N)$ with $2N$ nodes and $2k$ defects:



- Action of the algebra:

$$e_1 \cdot \text{diagram} = \text{diagram} = \alpha \cdot \text{diagram}$$

The diagram shows the action of the algebra element e_1 on a link state. The first diagram is a link state with a blue arc between nodes 1 and 2, and a central dot. The second diagram shows the result of the action, which is a link state with a blue arc between nodes 1 and 2, and a central dot, multiplied by the scalar α .

- Diagrammatic rules:

$$\begin{aligned}
 \text{diagram} &= \beta \cdot \text{diagram} & \text{diagram} &= \alpha \cdot \text{diagram} & & \text{(with } \alpha = z + z^{-1}\text{)} \\
 \text{diagram} &= z \cdot \text{diagram} & \text{diagram} &= z^{-1} \cdot \text{diagram} & & \text{diagram} = 0
 \end{aligned}$$

The diagrammatic rules show how various link configurations are related to scalars. The first rule shows a link with a blue loop on top of the central dot is equal to β times the link with just the central dot. The second rule shows a link with a blue loop on the bottom of the central dot is equal to α times the link with just the central dot. The third rule shows a link with a blue line from the central dot to node 1 is equal to z times the link with just the central dot. The fourth rule shows a link with a blue line from the central dot to node 2 is equal to z^{-1} times the link with just the central dot. The fifth rule shows a link with a blue loop on the central dot is equal to 0.

2) Uncoiled affine/periodic Temperley-Lieb algebras

Joint work with A. Langlois-Rémillard

Uncoiled algebras for N odd

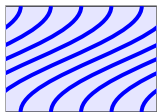
- Definition of the **uncoiled affine Temperley-Lieb algebra**:

$$\text{uaTL}_N(\beta, \gamma) = \frac{\text{aTL}_N(\beta)}{\{\Omega^N = \gamma \mathbf{1}\}}$$

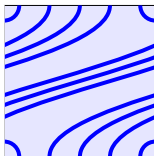
- Definition of the **uncoiled periodic Temperley-Lieb algebra**:

$$\text{upTL}_N(\beta, \gamma) = \frac{\text{pTL}_N(\beta)}{\{e_0(e_{N-1}e_{N-2} \cdots e_1e_0)^{N-2} = \gamma^2 e_0\}}$$

- Extra relations in diagrams:



$= \gamma$



$= \gamma^2$



- Subalgebra structure: $\text{upTL}_N(\beta, \gamma) \subset \text{uaTL}_N(\beta, \gamma)$

Sandwich diagrams for N odd

- Sets of sandwich diagrams

$$S_k(\mathbf{E}) = \left\{ \begin{array}{c} \text{---} \\ \boxed{w} \\ \text{---} \\ \dots \\ \text{---} \\ \boxed{c} \\ \text{---} \\ \dots \\ \text{---} \\ \boxed{v} \\ \text{---} \end{array} \right\} \left| v, w \in \mathbf{W}_{k,z}(N), c \in \mathbf{E} \subset \mathbf{aTL}_{2k}(\beta) \right\}$$

- Sandwich basis for $\mathbf{uaTL}_N(\beta, \gamma)$:

$$\mathbf{uaTL}_N(\beta, \gamma) : \bigcup_{k=\frac{1}{2}, \frac{3}{2}, \dots, \frac{N}{2}} S_k(\{\Omega^r \mid 0 \leq r < 2k\})$$

- Dimension of the algebra:

$$\dim \mathbf{uaTL}_N(\beta, \gamma) = \sum_{k=\frac{1}{2}, \frac{3}{2}, \dots, \frac{N}{2}} 2k \underbrace{[\dim \mathbf{W}_{k,z}(N)]}_{= \binom{N}{N/2-k}}^2 = N \binom{N-1}{\frac{N-1}{2}}^2$$

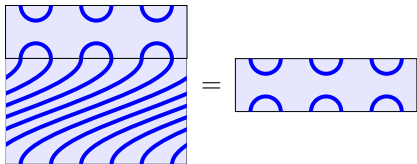
- $\mathbf{uaTL}_N(\beta, \gamma)$ is a **sandwich cellular algebra**. D. Tubbenhauer (2022)

Uncoiled algebras for N even

- Constraining relation in $\mathbf{aTL}_N(\beta)$ for N even:

$$\Omega^N E = E$$

$$(E = e_1 e_3 \cdots e_{N-1})$$



- Two families of uncoiled affine Temperley-Lieb algebras:

$$\mathbf{uaTL}_N^{(1)}(\beta, \alpha) = \frac{\mathbf{aTL}_N(\beta)}{\{\Omega^N = \mathbf{1}, E\Omega E = \alpha E\}}$$

$$\mathbf{uaTL}_N^{(2)}(\beta, \gamma) = \frac{\mathbf{aTL}_N(\beta)}{\{\Omega^N = \gamma \mathbf{1}, E = 0\}}$$

- Similarly, there are two families of uncoiled algebras associated to $\mathbf{pTL}_N(\beta)$ for N even: $\mathbf{upTL}_N^{(1)}(\beta, \alpha)$ and $\mathbf{upTL}_N^{(2)}(\beta, \gamma)$.
- We described these algebras with sandwich diagrams and computed their dimensions.

Wenzl-Jones projectors for $TL_N(\beta)$

- The first three projectors P_1, P_2 and P_3 are

$$\boxed{1} = \begin{array}{|c|} \hline | \\ \hline \end{array} \quad \boxed{2} = \begin{array}{|c|} \hline || \\ \hline \end{array} - \frac{1}{\beta} \begin{array}{|c|} \hline \cup \\ \hline \cap \\ \hline \end{array}$$

$$\boxed{3} = \begin{array}{|c|} \hline ||| \\ \hline \end{array} - \frac{1}{\beta^2 - 1} \left(\begin{array}{|c|} \hline \cup || \\ \hline \cap \\ \hline \end{array} + \begin{array}{|c|} \hline || \cup \\ \hline \cap \\ \hline \end{array} \right) + \frac{\beta}{\beta^2 - 1} \left(\begin{array}{|c|} \hline \cup \cap \\ \hline \cup \cap \\ \hline \end{array} + \begin{array}{|c|} \hline \cap \cup \\ \hline \cap \cup \\ \hline \end{array} \right)$$

- Recursive definition:

$$P_m = \boxed{m} = \boxed{m-1} | - \frac{U_{m-1}(\frac{\beta}{2})}{U_{m-2}(\frac{\beta}{2})} \begin{array}{|c|} \hline \boxed{m-1} \\ \hline \cup \\ \hline \dots \\ \hline \cap \\ \hline \boxed{m-1} \\ \hline \end{array}$$

- Properties:

- satisfies the relations

$$P_m^2 = P_m \quad e_i P_m = P_m e_i = 0 \quad i = 1, \dots, m-1$$

- $P_N \in TL_N(\beta)$ projects on the one-dimensional trivial representation τ

Wenzl-Jones projectors for $uaTL_N(\beta)$

- Standard modules for $uaTL_N(\beta)$ for N odd:

$$\{W_{k,z}(N) \mid z = \gamma^{1/2k} e^{\pi i r/k}, k = \frac{1}{2}, \frac{3}{2}, \dots, \frac{N}{2}, r = 0, 1, \dots, 2k-1\}$$

- We construct a projector $Q_{N,r}$ for each of the N one-dimensional modules.
- Example for $N = 3$: (with $\omega = \gamma^{1/3} e^{2\pi i r/3}, r = 0, 1, 2$)

$$Q_{3,r} = \frac{1}{3} \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right]$$

The diagram shows four terms in brackets, each with a top and bottom pink box labeled '3'.
 1. Three vertical blue lines.
 2. Three blue wavy lines.
 3. Three blue wavy lines with a different phase.
 4. Three blue lines with semi-circular caps at the ends.

- General construction for N odd:

$$Q_{N,r} = \sum_{k=0}^{\frac{N-1}{2}} \sum_{\ell=0}^{N-2k-1} \Gamma_{k,\ell} \begin{array}{c} \text{Diagram} \end{array} \quad \text{where} \quad k = \underbrace{\text{Diagram}}_k$$

The diagram for $Q_{N,r}$ consists of a central blue box labeled Ω^ℓ flanked by two blue boxes labeled $N-2k$. These are sandwiched between two pink boxes labeled N . Blue semi-circular caps are on the sides of the $N-2k$ boxes. The parameter k is defined as a set of k vertical blue lines.

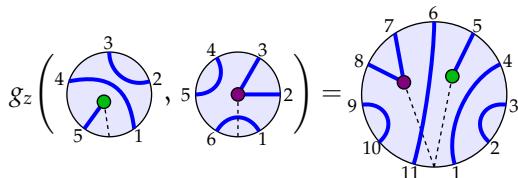
- Similar constructions for the other uncoiled algebras

3) Fusion of standard representations

Joint work with Y. Ikhlef – SciPost Phys. **12** (2022) 030

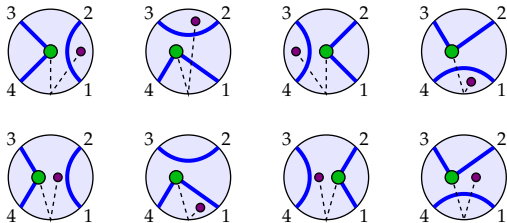
The representations $X_{k,\ell,x,y,z}(N)$

- Gluing operator over $W_{k,x}(N_a) \otimes W_{\ell,y}(N_b)$:



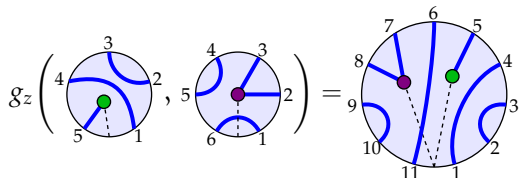
- Bases of link states with two marked points

for $X_{1,0,x,y,z}(4)$:



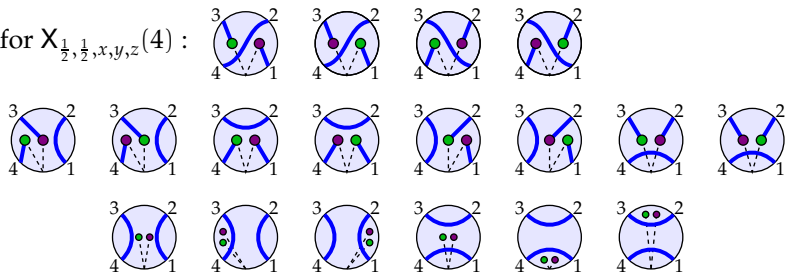
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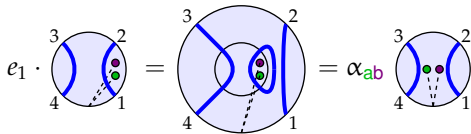
- Bases of link states with two marked points

for $X_{\frac{1}{2},\frac{1}{2},x,y,z}(4)$:



The representations $X_{k,\ell,x,y,z}(N)$

- Action of $\mathbf{aTL}_N(\beta)$:



The representations $X_{k,\ell,x,y,z}(N)$

- Action of $\mathfrak{aTL}_N(\beta)$:

$$e_1 \cdot \begin{array}{c} \text{3} \\ \text{2} \\ \text{1} \\ \text{4} \end{array} = \begin{array}{c} \text{3} \\ \text{2} \\ \text{1} \\ \text{4} \end{array} = \alpha_{ab} \begin{array}{c} \text{3} \\ \text{2} \\ \text{1} \\ \text{4} \end{array}$$

- Diagrammatic rules for the weights

	$= \beta$		$= \alpha_a$		$= \alpha_b$		$= \alpha_{ab}$	
	$= x$		$= \frac{1}{x}$		$= \frac{z}{x}$		$= \frac{x}{z}$	
	$= \frac{1}{y}$		$= y$		$= yz$		$= \frac{1}{zy}$	
	$= 1$		$= \mu$		$= \text{blue loop on top of blue arc from 2 to 4} = 0$		$= \text{blue loop on top of blue arc from 4 to 1} = 0$	

with

$$\alpha_a = x + x^{-1}$$

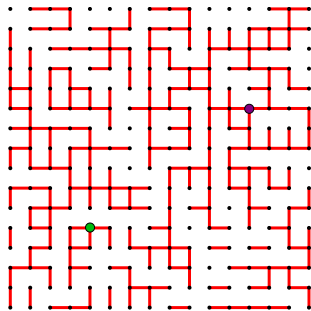
$$\alpha_b = y + y^{-1}$$

$$\alpha_{ab} = z + z^{-1}$$

$$\mu = \frac{zy}{x}$$

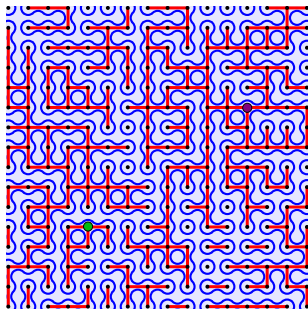
Geometric correlation functions

- Correlations of connectivity operators $\mathcal{O}(z)$ in critical percolation



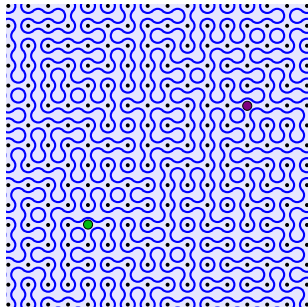
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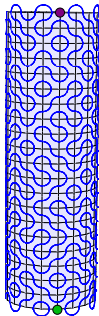
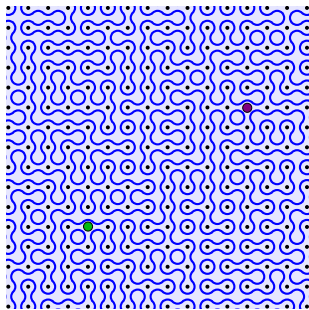
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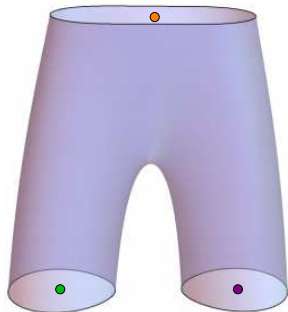
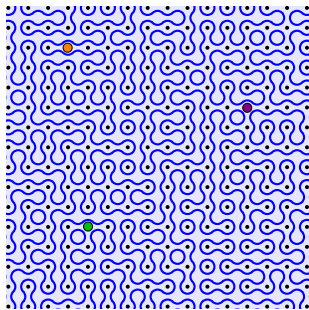
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Geometric correlation functions

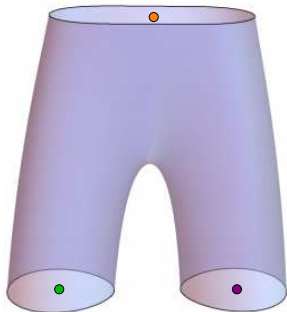
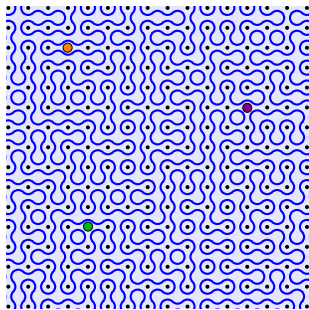
- Correlations of connectivity operators $\mathcal{O}(z)$ in critical percolation



$$\mathcal{O}(z_a) \times \mathcal{O}(z_b)$$

Geometric correlation functions

- Correlations of connectivity operators $\mathcal{O}(z)$ in critical percolation



$$\mathcal{O}(z_a) \times \mathcal{O}(z_b)$$

- Four-point correlation functions:

$$\langle \mathcal{O}(z_a) \mathcal{O}(z_b) \mathcal{O}(z_c) \mathcal{O}(z_d) \rangle = \sum_{\mu} \begin{array}{c} \mathcal{O}(z_b) \quad \mathcal{O}(z_c) \\ \quad \quad \mu \\ \mathcal{O}(z_a) \quad \mathcal{O}(z_d) \end{array}$$

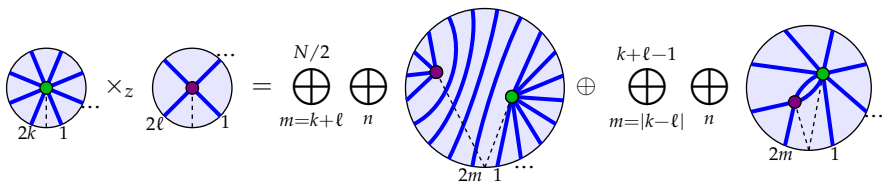
Properties of the representations $X_{k,\ell,x,y,z}(N)$

- Dimensions: $\dim X_{k,\ell,x,y,z}(N) = \left(\frac{N}{2} - |k - \ell| + 1\right) \binom{N}{\frac{N}{2} - |k - \ell|}$
- Decomposition over irreducible modules for β, z generic:

$$X_{k,\ell,x,y,z}(N) \simeq W_{|k-\ell|,z^\sigma}(N) \oplus \bigoplus_{m=|k-\ell|+1}^{N/2} \bigoplus_{n=0}^{2m-1} W_{m,\omega_{m,n}}(N)$$

$$\omega_{m,n} = z^{(k-\ell)/m} e^{in\pi/m} \quad \sigma = \begin{cases} 1 & k \geq \ell \\ -1 & k < \ell \end{cases}$$

- Diagrammatic interpretation:



Fusion of representations

- Fusion channels and full fusion products:

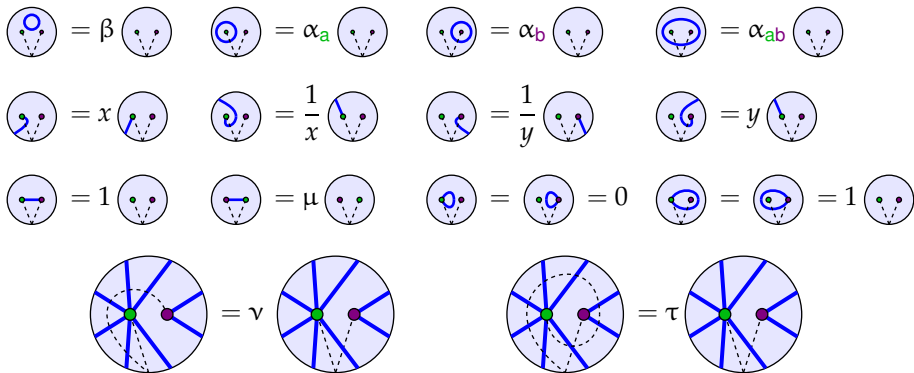
$$\mathbf{W}_{k,x}(N_a) \times_z \mathbf{W}_{\ell,y}(N_b) = \mathbf{X}_{k,\ell,x,y,z}(N)$$

$$\mathbf{W}_{k,x}(N_a) \times \mathbf{W}_{\ell,y}(N_b) = \bigoplus_{z \in \mathbb{C}^\times} [\mathbf{W}_{k,x}(N_a) \times_z \mathbf{W}_{\ell,y}(N_b)]$$

- Commutativity: $\mathbf{W}_{k,x}(N_a) \times \mathbf{W}_{\ell,y}(N_b) = \mathbf{W}_{\ell,y}(N_b) \times \mathbf{W}_{k,x}(N_a)$
- Remaining questions:
 - Is this the only way of defining representations with two marked points?
 - Can we give an algebraic definition of fusion for $\mathbf{aTL}_N(\beta)$?
 - Can we define fusion for other representations?
 - Are these fusion products associative?

The representations $Y_{k,\ell,x,y,z,w}(N)$

- Also defined with link states with two marked points
- Different diagrammatic rules for the weights:



with $\alpha_a = x + x^{-1}$ $\alpha_b = y + y^{-1}$ $\alpha_{ab} = w + w^{-1}$ $\mu = \frac{zy}{x}$

- Dimensions: $\dim Y_{k,\ell,x,y,z,w}(N) = \lfloor \frac{N+1}{2} \rfloor \binom{N}{\lfloor \frac{N+1}{2} \rfloor}$

Algebraic definitions of fusion

- Definition of fusion for the Temperley-Lieb algebra:

Read, Saleur (2007) Gainutdinov, Vasseur (2016)

$$\mathbf{V}(N_a) \times \mathbf{W}(N_b) = \mathbf{TL}_{N_a+N_b}(\beta) \otimes_{\mathbf{TL}_{N_a}(\beta) \otimes \mathbf{TL}_{N_b}(\beta)} (\mathbf{V}(N_a) \otimes \mathbf{W}(N_b))$$

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- Two previous attempts at defining fusion for $\mathbf{aTL}_N(\beta)$

Gainutdinov, Saleur (2016) Gainutdinov, Jacobsen, Saleur (2018)

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- New ideas for a general definition of fusion for $\mathbf{aTL}_N(\beta)$:

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$$\mathbf{V}(N_a) \times \mathbf{W}(N_b) \stackrel{?}{=} \mathbf{uaTL}_{N_a+N_b}(\beta, \gamma) \otimes_{\mathbf{TL}_{N_a}(\beta) \otimes \mathbf{TL}_{N_b}(\beta)} (\mathbf{V}(N_a) \otimes \mathbf{W}(N_b))$$

4) Fusion of standard and irreducible representations

Joint work with Y. Ikhlef

Structure of $W_{k,z}(N)$ for q generic

- For z generic, $W_{k,z}(N)$ is **irreducible** (with $\beta = -q - q^{-1}$).
Graham, Lehrer (2008)
- For $z = \varepsilon q^{\pm m}$ non-generic, there are non-zero homomorphisms between standard modules:

$$W_{m, \varepsilon q^{\pm k}}(N) \rightarrow W_{k, \varepsilon q^{\pm m}}(N) \quad k < m \leq \frac{N}{2} \quad \varepsilon \in \{+1, -1\}$$

- The module is **indecomposable yet reducible** and its structure is read from its **Loewy diagram**:

$$\begin{array}{ccc}
 & Q_{k, \varepsilon q^{\pm m}}(N) & (\text{quotient module}) \\
 & \searrow & \\
 W_{k, \varepsilon q^{\pm m}}(N) \simeq & & R_{k, \varepsilon q^{\pm m}}(N) \quad (\text{radical submodule})
 \end{array}$$

- **Next goal:** study the fusion products $W_{k,x}(N_a) \times_z R_{0, \varepsilon q^{\pm m}}(N_b)$ and $W_{k,x}(N_a) \times_z Q_{0, \varepsilon q^{\pm m}}(N_b)$

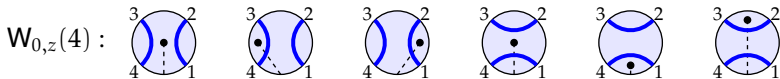
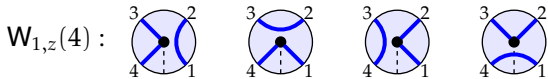
Case $m = 1$: Basis for $R_{0,\varepsilon q}$

- Homomorphism and Loewy diagram:

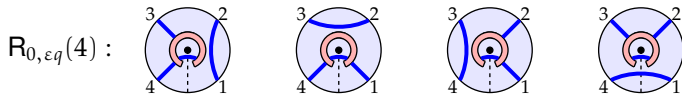
$$W_{1,1}(N) \rightarrow W_{0,\varepsilon q}(N) \quad W_{0,\varepsilon q}(N) \simeq$$




$$Q_{0,\varepsilon q}(N) \searrow R_{0,\varepsilon q}(N)$$

- Standard bases for $N = 4$:



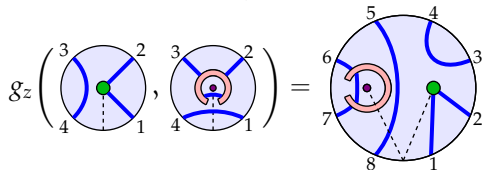
- A basis of $R_{0,\varepsilon q}(N)$ is obtained using the projectors P_2 :



where  =  - $\frac{1}{\beta}$  $\alpha = -\varepsilon\beta$

Case $m = 1$: The fusion $W_{k,x} \times R_{0,\varepsilon q}$

- Gluing of states in $W_{k,x} \times R_{0,\varepsilon q}$:

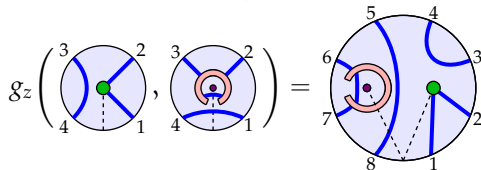


- The action of $\mathfrak{aTL}_N(\beta)$ is defined in the same way for $W_{k,x} \times R_{0,\varepsilon q}$ as for $W_{k,x} \times W_{0,\varepsilon q}$, with $\alpha_b = -\varepsilon\beta$.
- Submodule structure:

$$R_{0,\varepsilon q} \subset W_{0,\varepsilon q} \quad \longrightarrow \quad W_{k,x} \times R_{0,\varepsilon q} \subset W_{k,x} \times W_{0,\varepsilon q}$$

Case $m = 1$: The fusion $W_{k,x} \times R_{0,\varepsilon q}$

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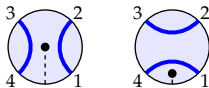
$$R_{0,\varepsilon q} \subset W_{0,\varepsilon q} \quad \longrightarrow \quad W_{k,x} \times R_{0,\varepsilon q} \subset W_{k,x} \times W_{0,\varepsilon q}$$

- Decomposition for $\varepsilon = -1$:

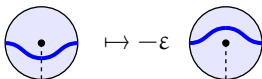
$$W_{k,x} \times_z R_{0,-q} \simeq \begin{cases} \frac{X_{k,0,x,-q,z}}{W_{k,x}} & \text{for } \begin{cases} k = 0, z = x^{\pm 1} \\ k > 0, z = x \end{cases} \\ X_{k,0,x,-q,z} & \text{otherwise} \end{cases}$$

Case $m = 1$: The fusion $W_{k,x} \times Q_{0,\varepsilon q}$

- Basis for $Q_{0,\varepsilon q}(4)$:



- The action of $\mathbf{aTL}_N(\beta)$ on $Q_{0,\varepsilon q}(N)$ has an extra relation:




- The fusion $W_{k,x} \times Q_{0,\varepsilon q}$ is defined like $W_{k,x} \times W_{0,\varepsilon q}$ but with this extra relation for the second marked point.
- Decomposition of $W_{k,x} \times Q_{0,-q}$:

$$W_{k,x} \times_z Q_{0,-q} = \frac{W_{k,x} \times_z W_{0,-q}}{W_{k,x} \times_z R_{0,-q}} = \begin{cases} W_{k,x} & \text{for } \begin{cases} k = 0, z = x^{\pm 1} \\ k > 0, z = x \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

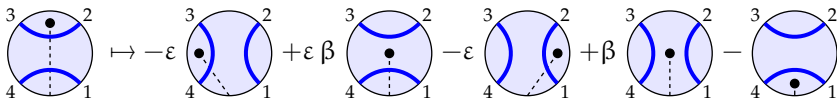
$$W_{k,x} \times Q_{0,-q} = \bigoplus_z [W_{k,x} \times_z Q_{0,-q}] = W_{k,x}$$

Case $m = 2$: $W_{k,x} \times Q_{0,\varepsilon q^2}$

- Basis of $R_{0,\varepsilon q^2}$ built using projectors P_4 :

for $R_{0,\varepsilon q^2}(4)$:  with $\alpha = \varepsilon(\beta^2 - 1)$

- Quotient relation for $Q_{0,\varepsilon q^2}$:



$$\begin{aligned} & \text{Diagram 1} \mapsto -\varepsilon \text{Diagram 2} + \varepsilon\beta \text{Diagram 3} - \varepsilon \text{Diagram 4} + \beta \text{Diagram 5} - \text{Diagram 6} \end{aligned}$$

- Decomposition of $W_{k,x} \times Q_{0,-q^2}$:

$$W_{k,x} \times Q_{0,-q^2} = \begin{cases} W_{0,xq} \oplus W_{1,x} & k = 0, x = \pm 1 \\ W_{0,xq} \oplus W_{0,xq^{-1}} & k = 0, x \neq 1 \\ W_{\frac{1}{2},xq} \oplus W_{\frac{1}{2},xq^{-1}} \oplus W_{\frac{1}{2},x^{-1}} \oplus W_{\frac{3}{2},x} & k = \frac{1}{2} \\ W_{k,xq} \oplus W_{k,xq^{-1}} \oplus W_{k+1,x} & k \geq 1 \end{cases}$$

Associativity

- We compute $W_{k,x} \times Q_{0,-q^2} \times W_{k+1,y}$ in two ways:

$$\begin{aligned} (W_{k,x} \times Q_{0,-q^2}) \times W_{k+1,y} &= (W_{k,xq} \oplus W_{k,xq^{-1}} \oplus W_{k+1,x}) \times W_{k+1,y} \\ &= \dots = \bigoplus_{z \in \mathbb{C}^\times} \bigoplus_{m \geq 0} W_{m,z} \end{aligned}$$

$$\begin{aligned} W_{k,x} \times (Q_{0,-q^2} \times W_{k+1,y}) &= W_{k,x} \times (W_{k+1,yq} \oplus W_{k+1,yq^{-1}} \oplus W_{k+2,y}) \\ &= \dots = \bigoplus_{z \in \mathbb{C}^\times} \bigoplus_{m \geq 1} W_{m,z} \end{aligned}$$

- This fusion product is **non-associative!**
- An associative fusion product would instead satisfy

$$W_{k,x} \times Q_{0,-q^2} = W_{k-1,x} \oplus W_{k,xq} \oplus W_{k,xq^{-1}} \oplus W_{k+1,x}$$

- This holds for fusion defined with the modules $Y_{k,\ell,x,y,z,w}(N)$!

Expectations from CFT

- Scaling limits for generic q, z :

$$W_{k,z} \mapsto \bigoplus_{p=-\infty}^{\infty} \mathcal{V}(h_{\mu+p,k}) \otimes \mathcal{V}(h_{\mu+p,-k}) \quad z = e^{i\pi\mu}$$

$$Q_{0,-q^m} \mapsto \bigoplus_{p=1}^{\infty} \mathcal{K}(h_{p,m}) \otimes \mathcal{K}(h_{p,m}) \quad \mathcal{K}(h_{r,s}) = \frac{\mathcal{V}(h_{r,s})}{\mathcal{V}(h_{r,s} + rs)}$$

- Fusion of chiral primary fields for integer r, s :

$$\phi_{\mu,\nu} \times \phi_{1,1} = \phi_{\mu,\nu} \quad \phi_{\mu,\nu} \times \phi_{1,2} = \phi_{\mu,\nu-1} + \phi_{\mu,\nu+1}$$

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$$\Phi_{\mu,\nu} \times \Phi_{r,s} = \sum_{m=-(r-1)/2}^{(r-1)/2} \sum_{n=-(s-1)/2}^{(s-1)/2} \Phi_{\mu+2m,\nu+2n}$$

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- Fusion rules for $W \times Q$:

$$W_{k,x} \times Q_{0,-q^m} = \begin{cases} W_{k,x} & m = 1 \\ W_{k-1,x} \oplus W_{k,xq} \oplus W_{k,xq^{-1}} \oplus W_{k+1,x} & m = 2 \\ \bigoplus_{\mu,\nu=(m-1)/2}^{(m+1)/2} W_{k+\mu-\nu,xq^{\mu+\nu}} & \text{general } m \end{cases}$$

Overview

- We introduced uncoiled affine and periodic Temperley-Lieb algebras as finite dimensional quotients of $\mathfrak{aTL}_N(\beta)$ and $\mathfrak{pTL}_N(\beta)$.
- We constructed their Wenzl-Jones projectors.
- We defined associative fusion products $Y_{k,\ell,x,y,z,w}(N)$ for $\mathfrak{aTL}_N(\beta)$ that reproduce the expected CFT fusion rules.
- The algebraic definition of fusion is still an open problem.