Analytical solutions of Dirac-Bogoliubov-de Gennes equations for inhomogeneous quantum many-body systems

Per Moosavi

ETH Zurich

Integrability in Condensed Matter Physics and Quantum Field Theory
SwissMAP Research Station in Les Diablerets

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Dirac-Bogoliubov-de Gennes (DBdG) equations

Problem: Given smooth functions $v(x)$ and $K(x)$, consider

$$\begin{pmatrix} v(x)\partial_x + \partial_t & \Delta(x) \\ \Delta(x) & v(x)\partial_x - \partial_t \end{pmatrix} \begin{pmatrix} u_+ \\ u_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where

$$\Delta(x) \equiv v(x)\partial_x \log \sqrt{K(x)}$$

for $u_\pm = u_\pm(x,t)$ with given initial conditions. [P.M., arXiv:2208.14467]

Questions:

- What is the general solution?
- What is the effect of $\Delta(x) \neq 0$?
- What is the behavior as $t \to \infty$?

Applications of DBdG-type equations

◇ [Andreev, Sov. Phys. JETP (1964)]:

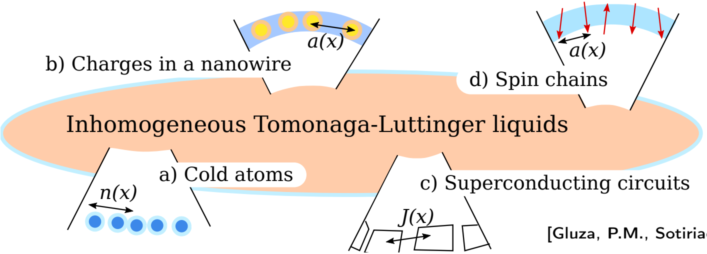
Interfaces between normal metals and superconductors

◇ [Takayama, Lin-Liu, Maki, PRB (1980)]:

Continuum description of Su-Schrieffer-Heeger model for polyacetylene

◇ [**P.M.**, arXiv:2208.14467]:

Dynamics in inhomogeneous Tomonaga-Luttinger liquids (TLLs)



[Gluza, P.M., Sotiriadis, JPA (2022)]

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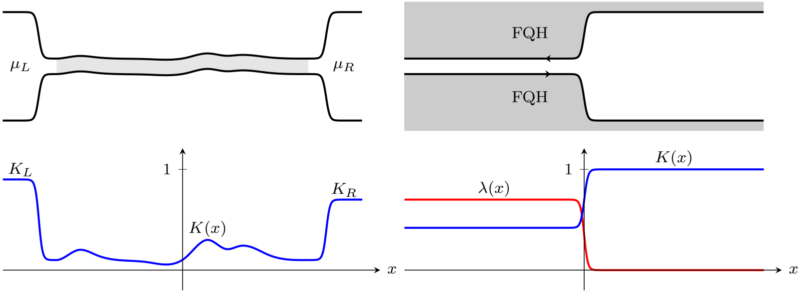
Quantum wires | Fractional quantum Hall (FQH) edges



Some previous works on inhomogeneous TLLs

◇ [Maslov, Stone, PRB (1995)], [Safi, Schulz, PRB (1995)], [Ponomarenko, PRB (1995)]:

Quantum wires

◇ [Stringari, PRL (1996)], . . . , [Citro et al., New J. Phys. (2008)]:

Effective descriptions of trapped ultra-cold atoms in equilibrium

◇ . . . , [Brun, Dubail, SciPost (2018)], [Bastianello, Dubail, Stéphan, JPA (2020)], [Gluza, **P.M.**, Sotiriadis, JPA (2022)], [Ruggiero, Calabrese, Giamarchi, Foini, SciPost (2022)]:

Inhomogeneous TLLs out of equilibrium

Outline

- Tomonaga-Luttinger liquids / Compactified free bosons
- Examples of TLLs
- Why PDE approach?
- DBdG equations from TLL theory
- Solving the DBdG equations
- Application to quantum wires

Tomonaga-Luttinger liquids / Compactified free bosons

Tomonaga-Luttinger-liquid (TLL) theory

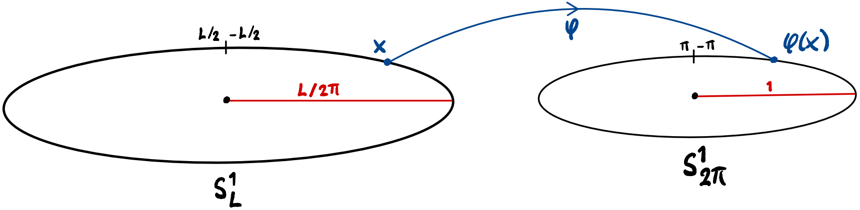
Given $v > 0$ and $K > 0$. Consider the action functional

$$S = \frac{R^2}{8\pi} \int_{\mathbb{R}\times S^1_L} \mathrm{d}^2x \, (\partial^\mu \varphi)(\partial_\mu \varphi)$$

for fields $\varphi : S^1_L \to S^1_{2\pi}$ with compactification radius R satisfying

$$K = \frac{R^2}{4}$$

and metric $(h_{\mu\nu}) = \mathrm{diag}(1, -1)$ in coordinates $(x^0, x^1) = (vt, x)$.



Quantum field theory in Hamiltonian framework

Hamiltonian of free compactified bosons

$$H_{v,K} = \frac{1}{2\pi} \int_{S^1_L} \mathrm{d}x \, :\!\Big(\frac{v}{K} [\pi \Pi(x)]^2 + vK [\partial_x \varphi(x)]^2 \Big)\!:$$

with bosonic field $\varphi(x)$ and conjugate $\Pi(x)$ for $x \in S^1_L$ satisfying

$$[\partial_x \varphi(x), \Pi(y)] = \mathrm{i}\delta'(x-y).$$

How to understand this? Expanding $\varphi(x)$ and $\Pi(x)$ in plane waves:

$$H_{v,K} = (\text{zero modes}) + \frac{\pi v}{2K} \sum_{n \neq 0} \frac{1}{L} :\!\big(\Pi_{-n}\Pi_n + [Kn]^2 \varphi_{-n}\varphi_n \big)\!:$$

with $[\varphi_n, \Pi_m] = \mathrm{i}\delta_{n,m}$.

Note: Mathematically, $\varphi(x)$ and $\Pi(x)$ are operator-valued distributions. Formal Fourier transforms: $\varphi_n = L^{-1} \int \mathrm{d}x \, \varphi(x) \mathrm{e}^{-2\pi \mathrm{i} n x/L}$ and $\Pi_n = \int \mathrm{d}x \, \Pi(x) \mathrm{e}^{2\pi \mathrm{i} n x/L}$.

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More precise definition of $H_{v,K}$

Infinitely many uncoupled quantum harmonic oscillators

$$H_{v,K} = \frac{\pi v}{L}\big(a_0^2 + \bar{a}_0^2\big) + \frac{\pi v}{L}\sum_{n\neq 0} :\!(a_{-n}a_n + \bar{a}_{-n}\bar{a}_n)\!:$$

with $a_n = a_{-n}^\dagger$ and $\bar{a}_n = \bar{a}_{-n}^\dagger$ ($n \in \mathbb{Z}$) for right/left movers satisfying

$$[a_n, a_m] = n\delta_{n+m,0} = [\bar{a}_n, \bar{a}_m], \qquad [a_n, \bar{a}_m] = 0,$$

and $a_n|\Omega\rangle = \bar{a}_n|\Omega\rangle = 0$ for $n \geq 0$, defining the vacuum $|\Omega\rangle$, where

$$:\!a_n a_m\!: = a_n a_m - \langle\Omega|a_n a_m|\Omega\rangle.$$

Note: Relation to φ_n and Π_n for $n \neq 0$:

$$\varphi_n = \frac{1}{2\sqrt{K}}\frac{\mathrm{i}}{n}\big(a_n - \bar{a}_{-n}\big), \qquad \Pi_n = \sqrt{K}\big(a_{-n} + \bar{a}_n\big).$$

Inhomogeneous TLL

Hamiltonian

$$H_{v(\cdot),K(\cdot)} = \frac{1}{2\pi}\int_{S^1_L} \mathrm{d}x : \left(\frac{v(x)}{K(x)}[\pi\Pi(x)]^2 + v(x)K(x)[\partial_x\varphi(x)]^2 \right):$$

with inhomogeneous periodic $v(x) > 0$ and $K(x) > 0$ on the circle S^1_L.

For inhomogeneous periodic $v(x) > 0$ and $K(x) = K > 0$ constant:

[Dubail, Stéphan, Viti, Calabrese, SciPost Phys. (2017)], [Dubail, Stéphan, Calabrese, SciPost Phys. (2017)] [Gawedzki, Langmann, P.M., JSP (2018)], [Langmann, P.M., PRL (2019)], [P.M., AHP (2021)]

Corresponding action functional

$$S_{R(\cdot)} = \frac{1}{8\pi}\int_{\mathbb{R}\times S^1_L} \mathrm{d}^2x\,\sqrt{-h}R(x)^2(\partial^\mu\varphi)(\partial_\mu\varphi)$$

with inhomogeneous compactification radius $R(x) = 2\sqrt{K(x)}$ and metric $(h_{\mu\nu}) = \mathrm{diag}(v(x)^2/v^2, -1)$ in coordinates $(x^0, x^1) = (vt, x)$.

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Inhomogeneous marginal $(J\bar{J})$ deformation

Changing $R(x)$ to $R(x) + \delta R(x)$:

$$\delta S \equiv S_{R(\cdot)+\delta R(\cdot)} - S_{R(\cdot)} = \int_{\mathbb{R}\times S^1_L} \mathrm{d}^2 x \sqrt{-h}\Phi$$

with

$$\Phi = \frac{1}{\pi}\frac{\delta R(x)}{R(x)} J\bar{J}$$

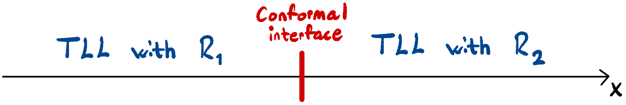
where

$$J \equiv -\frac{1}{\sqrt{K(x)}}\Big[\pi\Pi(x) - K(x)\partial_x\varphi(x)\Big],$$
$$\bar{J} \equiv -\frac{1}{\sqrt{K(x)}}\Big[\pi\Pi(x) + K(x)\partial_x\varphi(x)\Big].$$

Similar to usual marginal deformation by an operator with conformal weights $(1,1)$ in the case of constant compactification radius.

Related special case: Conformal interfaces

[Bachas, Brunner, JHEP (2008)]:



Examples of TLLs

Example: Quantum XXZ spin chain

Hamiltonian

$$H = -J \sum_{j=1}^{N} \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y - \Delta S_j^z S_{j+1}^z \right)$$

with $[S_j^\alpha, S_{j'}^\beta] = \mathrm{i}\delta_{j,j'}\epsilon_{\alpha\beta\gamma}S_j^\gamma$ for $\alpha, \beta, \gamma \in \{x, y, z\}$.

For $|\Delta| < 1$, the low-energy description is a homogeneous TLL with

$$v = Ja\frac{\pi}{2}\frac{\sqrt{1-\Delta^2}}{\arccos(\Delta)}, \qquad K = \frac{\pi}{2[\pi - \arccos(\Delta)]}$$

(from exact Bethe-ansatz solution) with $a = L/N$ the lattice spacing.

Example: Inhomogeneous quantum XXZ spin chain

Hamiltonian

$$H = -\sum_{j=1}^{N} J_j \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y - \Delta S_j^z S_{j+1}^z \right)$$

with $J_j = [J(x_j) + J(x_{j+1})]/2$ given by a smooth function $J(x)$.

For $|\Delta| < 1$, the low-energy description is an inhomogeneous TLL with

$$v(x) = J(x) a \sqrt{1 + 4\Delta/\pi}, \qquad K = 1/\sqrt{1 + 4\Delta/\pi}$$

(to lowest order in Δ) with $a = L/N$ the lattice spacing. [P.M., AHP (2021)]

Example: Lieb-Liniger model

Hamiltonian

$$H = \int_{-L/2}^{L/2} \mathrm{d}x \, \left(\frac{1}{2m} \partial_x \Psi(x)^\dagger \partial_x \Psi(x) + \frac{g}{2} \Psi(x)^\dagger \Psi(x) \Psi(x)^\dagger \Psi(x) \right)$$

with particle mass m, repulsive coupling constant $g > 0$, and bosonic field $\Psi(x)$ satisfying $[\Psi(x)^\dagger, \Psi(y)] = \delta(x-y)$.

The low-energy description is a homogeneous TLL with

$$v = \frac{v_F}{K}, \qquad K \sim \begin{cases} 1 + \frac{4}{\gamma} & \text{for } \gamma \gg 1, \\ \frac{\pi}{\sqrt{\gamma}} \left(1 - \frac{\sqrt{\gamma}}{2\pi}\right)^{-1/2} & \text{for } \gamma \ll 1, \end{cases}$$

in terms of the dimensionless coupling $\gamma = mg/\rho_0$.

Example: Trapped ultra-cold atoms

Hamiltonian

$$H = \int_{-L/2}^{L/2} \mathrm{d}x \left(\frac{1}{2m} \partial_x \Psi(x)^\dagger \partial_x \Psi(x) + \frac{g}{2} \Psi(x)^\dagger \Psi(x) \Psi(x)^\dagger \Psi(x) + [V(x) - \mu] \Psi(x)^\dagger \Psi(x) \right)$$

with quantities as before, trap $V(x)$, and chemical potential μ.

The low-energy description is an inhomogeneous TLL with

$$v(x) = \sqrt{\rho_0(x) g/m}, \qquad K(x) = \pi \sqrt{\rho_0(x)/mg},$$

where $\rho_0(x) = [\mu - V(x)]/g$ in the Thomas-Fermi regime.

Why PDE approach?

Inhomogeneous TLL

Recall:

$$H_{v(\cdot),K(\cdot)} = \frac{1}{2\pi}\int_{-L/2}^{L/2} \mathrm{d}x :\!\left(\frac{v(x)}{K(x)}[\pi\Pi(x)]^2 + v(x)K(x)[\partial_x\varphi(x)]^2\right)\!:$$

for periodic $v(x) > 0$ and $K(x) > 0$.

"Diagonalization" approaches

Naively: "Diagonalize" $H_{v(\cdot),K(\cdot)}$ by expressing in terms of a_n and $\bar{a}_n$. If $K(x) = K$, achieved by a "simple" Bogoliubov transformation.

Problem: Does not work for $K(x)$ since $[\partial_x \varphi(x), \Pi(y)] = \mathrm{i}\delta'(x-y)$ not satisfied by the transformed fields.

Alternatively: Expand $\partial_x \varphi(x)$ and $\Pi(x)$ not in plane waves but in other eigenfunctions obtained by solving a Sturm-Liouville problem.

[Stringari, PRL (1996)], [Ho, Ma, J. Low Temp. Phys. (1999)], [Menotti, Stringari, PRA (2002)] [Ghosh, arXiv:cond-mat/0402080], [Petrov, Gangardt, Shlyapnikov, J. Phys. IV (2004)] [Citro, De Palo, Orignac, Pedri, Chiofalo, New J. Phys. (2008)], [Gluza, P.M., Sotiriadis, JPA (2022)]

For ultra-cold atoms in parabolic trap, then Legendre polynomials.

Problem: Again, not practical if eigenfunctions not known.

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DBdG equations from TLL theory

PDE approach

Instead of diagonalizing $H_{v(\cdot),K(\cdot)}$ rewrite it as [P.M., arXiv:2208.14467]

$$H_{v(\cdot),K(\cdot)} = \int_{-L/2}^{L/2} \mathrm{d}x\, \pi v(x) : \Big(\widetilde{\rho}_+(x)^2 + \widetilde{\rho}_-(x)^2 \Big) :$$

with right/left-moving densities

$$\widetilde{\rho}_\pm(x) \equiv \frac{1}{2\pi\sqrt{K(x)}} \Big[\pi\Pi(x) \mp K(x)\partial_x\varphi(x) \Big].$$

Result: $\widetilde{\rho}_\pm(x)$ satisfy

$$[\widetilde{\rho}_\pm(x), \widetilde{\rho}_\pm(y)] = \mp\frac{\mathrm{i}}{2\pi}\delta'(x-y),$$
$$[\widetilde{\rho}_+(x), \widetilde{\rho}_-(y)] = \frac{\mathrm{i}}{2\pi}\Lambda(x)\delta(x-y)$$

with $\Lambda(x) \equiv \partial_x \log\sqrt{K(x)}$ coupling right/left movers.

Dirac-Bogoliubov-de Gennes (DBdG) equations

Heisenberg equation and commutation relations imply that $\widetilde{\rho}_\pm(x)$ and $\widetilde{j}_\pm(x) \equiv \pm v(x)\widetilde{\rho}_\pm(x)$ satisfy coupled continuity equations

$$\partial_t \widetilde{\rho}_\pm + \partial_x \widetilde{j}_\pm = \pm \Delta(x) \widetilde{\rho}_\mp$$

with $\Delta(x) \equiv v(x)\Lambda(x)$.

Result: $\widetilde{j}_\pm(x,t)$ satisfy the inhomogeneous DBdG equations

$$\begin{pmatrix} v(x)\partial_x + \partial_t & \Delta(x) \\ \Delta(x) & v(x)\partial_x - \partial_t \end{pmatrix} \begin{pmatrix} \widetilde{j}_+(x,t) \\ \widetilde{j}_-(x,t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with a local gap $\Delta(x) = v(x)\partial_x \log \sqrt{K(x)}$. [P.M., arXiv:2208.14467]

Remark 1: Vector and axial currents

The PDEs are equivalent to existence of vector and axial current with

$$\rho(x) = \Pi(x), \qquad \jmath(x) = v(x)K(x)\rho_5(x),$$
$$\rho_5(x) = -\partial_x\varphi(x)/\pi, \qquad \jmath_5(x) = \frac{v(x)}{K(x)}\rho(x),$$

satisfying

$$\partial_t\rho + \partial_x\jmath = 0, \qquad \partial_t\jmath + v(x)K(x)\partial_x\big[v(x)K(x)^{-1}\rho\big] = 0,$$
$$\partial_t\rho_5 + \partial_x\jmath_5 = 0, \qquad \partial_t\jmath_5 + v(x)K(x)^{-1}\partial_x\big[v(x)K(x)\rho_5\big] = 0,$$

In terms of quantities for right/left movers:

$$\rho = \sqrt{K(x)}\big(\widetilde{\rho}_+ + \widetilde{\rho}_-\big), \qquad \jmath = \sqrt{K(x)}\big(\widetilde{j}_+ + \widetilde{j}_-\big).$$

Remark 2: Coupled $\mathrm{U}(1)$ current algebras

Define

$$a_n \equiv \int_{S^1_L} \mathrm{d}x\, \widetilde{\rho}_+(x) \mathrm{e}^{-2\pi \mathrm{i} n x/L}, \qquad \bar{a}_n \equiv \int_{S^1_L} \mathrm{d}x\, \widetilde{\rho}_-(x) \mathrm{e}^{2\pi \mathrm{i} n x/L}.$$

Obtain coupled $\mathrm{U}(1)$ current algebras:

$$[a_n, a_m] = n\delta_{n+m,0} = [\bar{a}_n, \bar{a}_m], \qquad [a_n, \bar{a}_m] = \frac{\mathrm{i}}{2\pi}\Lambda_{n-m},$$

where $\Lambda_n \equiv \int_{S^1_L} \mathrm{d}x\, \Lambda(x) \mathrm{e}^{-2\pi \mathrm{i} n x/L}$.

$\Longrightarrow$ Infinitely many coupled quantum harmonic oscillators.

Special case: If $K(x) = K$, then $\Lambda_n = 0$ and the algebras decouple.

Solving the DBdG equations

Inhomogeneous DBdG equations

Recall: $\widetilde{j}_{\pm}(x,t)$ satisfy

$$\begin{pmatrix} v(x)\partial_x + \partial_t & \Delta(x) \\ \Delta(x) & v(x)\partial_x - \partial_t \end{pmatrix} \begin{pmatrix} \widetilde{j}_+ \\ \widetilde{j}_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with $\Delta(x) = v(x)\partial_x \log \sqrt{K(x)}$.

Inhomogeneous DBdG equations

Recall: $\widetilde{j}_{\pm}(x,t)$ satisfy

$$\partial_x \begin{pmatrix} \widetilde{j}_+ \\ \widetilde{j}_- \end{pmatrix} + \begin{pmatrix} v(x)^{-1}\partial_t & \Lambda(x) \\ \Lambda(x) & -v(x)^{-1}\partial_t \end{pmatrix} \begin{pmatrix} \widetilde{j}_+ \\ \widetilde{j}_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with $\Lambda(x) = \partial_x \log \sqrt{K(x)}$.

Analogy with non-Hermitian (PT-symmetric) 2-level system

DBdG eqs. in frequency space ω for expectations in the infinite volume:

$$\partial_x \begin{pmatrix} \langle \hat{j}_+(x,\omega) \rangle \\ \langle \hat{j}_-(x,\omega) \rangle \end{pmatrix} = \mathrm{i}\mathsf{P}_\omega(x) \begin{pmatrix} \langle \hat{j}_+(x,\omega) \rangle \\ \langle \hat{j}_-(x,\omega) \rangle \end{pmatrix} + \frac{1}{v(x)} \sigma_3 \begin{pmatrix} \langle \widetilde{j}_+(x,0) \rangle \\ \langle \widetilde{j}_-(x,0) \rangle \end{pmatrix}$$

for $x \in \mathbb{R}$ with the $\mathfrak{sl}(2,\mathbb{C})$ matrix

$$\mathsf{P}_\omega(x) \equiv \frac{\omega}{v(x)} \sigma_3 + \mathrm{i}\Lambda(x)\sigma_1.$$

In general, $\mathsf{P}_\omega(x)\mathsf{P}_\omega(y) \neq \mathsf{P}_\omega(y)\mathsf{P}_\omega(x)$, so need spatial ordering $\overleftarrow{\mathcal{X}}$ ($\overrightarrow{\mathcal{X}}$) where positions decrease (increase) from left to right.

Note: Expectations $\langle \cdot \rangle$ w.r.t. arbitrary state in the infinite-volume limit $L \to \infty$. Assumed system prepared in a steady state for $t < 0$ and evolving for $t > 0$ with initial data $\langle \widetilde{j}_\pm(x, t = 0) \rangle$. Fourier transforms: $\hat{j}_\pm(x,\omega) = \int_0^\infty \mathrm{d}t \, \widetilde{j}_\pm(x,t) \mathrm{e}^{\mathrm{i}\omega t}$.

Problem studied by Magnus

$$\frac{\mathrm{d}}{\mathrm{d}s}Y(s) = A(s)Y(s), \qquad Y(s_0) = Y_0.$$

[Magnus, Comm. Pure Appl. Math. (1954)]

Green's functions

Result: Let $\langle \widetilde{j}_\pm(x,0)\rangle$ have compact support and $\lim_{|x|\to\infty} \langle \widetilde{j}_\pm(x,t)\rangle = 0$. Then,

$$\begin{pmatrix} \langle \widetilde{j}_+(x,t)\rangle \\ \langle \widetilde{j}_-(x,t)\rangle \end{pmatrix} = \int_{\mathbb{R}} \mathrm{d}y\, G(x,y;t) \frac{1}{v(y)} \begin{pmatrix} \langle \widetilde{j}_+(y,0)\rangle \\ \langle \widetilde{j}_-(y,0)\rangle \end{pmatrix}$$

using $G(x,y;t) = \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{2\pi} \hat{G}(x,y;\omega) \mathrm{e}^{-\mathrm{i}\omega t}$ with

$$\hat{G}(x,y;\omega) = \hat{G}_+(x,y;\omega)\frac{\sigma_0+\sigma_3}{2} + \hat{G}_-(x,y;\omega)\frac{\sigma_0-\sigma_3}{2},$$

$$\hat{G}_\pm(x,y;\omega) = \pm\theta(\pm[x-y])\overleftrightarrow{\mathcal{X}} \mathrm{e}^{\mathrm{i}\int_y^x \mathrm{d}s\, \mathsf{P}_\omega(s)} \sigma_3.$$

Special case: If $K(x) = K$, then $\hat{G}_\pm(x,y;\omega)$ equal

$$\hat{G}^0_\pm(x,y;\omega) = \pm\theta(\pm[x-y])\mathrm{e}^{\mathrm{i}\omega\tau_{x,y}\sigma_3}\sigma_3, \qquad \tau_{x,y} = \int_y^x \mathrm{d}s\, \frac{1}{v(s)}.$$

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using $G(x,y;t) = \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{2\pi} \hat{G}(x,y;\omega) \mathrm{e}^{-\mathrm{i}\omega t}$ with

$$\hat{G}(x,y;\omega) = \hat{G}_+(x,y;\omega) \frac{\sigma_0 + \sigma_3}{2} + \hat{G}_-(x,y;\omega) \frac{\sigma_0 - \sigma_3}{2},$$

$$\hat{G}_\pm(x,y;\omega) = \pm\theta(\pm[x-y]) \overset{\leftrightarrows}{\mathcal{X}} \mathrm{e}^{\mathrm{i}\int_y^x \mathrm{d}s\, \mathsf{P}_\omega(s)} \sigma_3.$$

Special case: If $K(x) = K$, then $\hat{G}_\pm(x,y;\omega)$ equal

$$\hat{G}^0_\pm(x,y;\omega) = \pm\theta(\pm[x-y]) \mathrm{e}^{\mathrm{i}\omega\tau_{x,y}\sigma_3} \sigma_3, \qquad \tau_{x,y} = \int_y^x \mathrm{d}s\, \frac{1}{v(s)}.$$

How to express the spatially-ordered exponentials?

The exponentials $\overleftrightarrow{\mathcal{X}}\mathrm{e}^{\mathrm{i}\int_y^x \mathrm{d}s\,\mathsf{P}_\omega(s)}$ are non-trivial to evaluate.

At least three possibilities:

- Dyson expansion
- Magnus expansion
- Product of exponentials of the $\mathfrak{sl}(2,\mathbb{C})$ generators

See review [Blanes, Casas, Oteo, Ros, Phys. Rep. (2009)].

Products of exponentials of the $\mathfrak{sl}(2,\mathbb{C})$ generators

Result: Let

$$\mathsf{H} = -\sigma_3, \qquad \mathsf{E} = \frac{\mathrm{i}\sigma_1 - \sigma_2}{2}, \qquad \mathsf{F} = \frac{\mathrm{i}\sigma_1 + \sigma_2}{2}.$$

Then, for $x > y$,

$$\overleftarrow{\mathcal{X}}\mathrm{e}^{\mathrm{i}\int_y^x \mathrm{d}s\,\mathsf{P}_\omega(s)} = \mathrm{e}^{h(x)\mathsf{H}}\mathrm{e}^{g(x)\mathsf{E}}\mathrm{e}^{f(x)\mathsf{F}},$$

where

$$\begin{cases} h'(x) = -\mathrm{i}\left[\omega v(x)^{-1} + g(x)\mathrm{e}^{-2h(x)}\Lambda(x)\right] \\ g'(x) = \mathrm{i}\left[\mathrm{e}^{2h(x)} - g(x)^2\mathrm{e}^{-2h(x)}\right]\Lambda(x) \\ f'(x) = \mathrm{i}\mathrm{e}^{-2h(x)}\Lambda(x) \end{cases}$$

with $h(0) = g(0) = f(0) = 0$, and similar for $x < y$.

Follows from [Wei, Norman, JMP (1963); Proc. Amer. Math. Soc. (1964)].

Magnus expansion

Result: For $x > y$,

$$\overleftarrow{\mathcal{X}} \mathrm{e}^{\mathrm{i}\int_y^x \mathrm{d}s\, \mathsf{P}_\omega(s)} = \exp\left[\sum_{n=1}^{\infty} \Omega_\omega^n(x,y)\right] \mathrm{e}^{\mathrm{i}\omega\tau_{x,y}\sigma_3}$$

with

$$\Omega_\omega^1(x,y) = -\int_y^x \mathrm{d}x_1\, \Lambda(x_1)\mathsf{A}_\omega(x_1,x), \qquad \mathsf{A}_\omega(s,x) \equiv \begin{pmatrix} 0 & \mathrm{e}^{-2\mathrm{i}\omega\tau_{s,x}} \\ \mathrm{e}^{2\mathrm{i}\omega\tau_{s,x}} & 0 \end{pmatrix},$$

$$\Omega_\omega^2(x,y) = -\mathrm{i}\int_y^x \mathrm{d}x_1 \int_y^{x_1} \mathrm{d}x_2\, \Lambda(x_1)\Lambda(x_2)\sin(2\omega\tau_{x_1,x_2})\sigma_3,$$

and

$$\Omega_\omega^n(x,y) = -\sum_{k=1}^{n-1} \frac{B_k}{k!} \sum_{\substack{m_1\geq 1,\ldots,m_k\geq 1\\ m_1+\ldots+m_k=n-1}} \int_y^x \mathrm{d}s \prod_{j=1}^{k} \mathrm{ad}_{\Omega_\omega^{m_j}(s,y)}\, \Lambda(s)\mathsf{A}_\omega(s,x)$$

for $n \geq 3$ consist of similar nested spatial integrals of $\mathfrak{sl}(2,\mathbb{C})$-valued functions that vanish at $\omega = 0$, and similar for $x < y$.

Late-time asymptotics

If $\omega = 0$, then $\mathsf{P}_0(x) = \mathrm{i}\partial_x \log\left(\sqrt{K(x)}\right)\sigma_1$ for different x commute and the only non-zero contribution in the expansions is

$$\exp\left[-\int_y^x \mathrm{d}s\,\Lambda(s)\sigma_1\right] \equiv \mathsf{T}(x,y) = \begin{pmatrix} \frac{\sqrt{\frac{K(y)}{K(x)}}+\sqrt{\frac{K(x)}{K(y)}}}{2} & \frac{\sqrt{\frac{K(y)}{K(x)}}-\sqrt{\frac{K(x)}{K(y)}}}{2} \\ \frac{\sqrt{\frac{K(y)}{K(x)}}-\sqrt{\frac{K(x)}{K(y)}}}{2} & \frac{\sqrt{\frac{K(y)}{K(x)}}+\sqrt{\frac{K(x)}{K(y)}}}{2} \end{pmatrix}.$$

Result: Leading $t \gg 1$ contribution to $G(x,y;t)$ is $\mathsf{T}(x,y)G^0(x,y;t)$.

Example: For the current $\jmath = \sqrt{K(x)}\left(\widetilde{j}_+ + \widetilde{j}_-\right)$,

$$\langle \jmath(x,t)\rangle = \int_{\mathbb{R}} \mathrm{d}y\,\frac{\delta(\tau_{x,y}-t)-\delta(\tau_{x,y}+t)}{2}\langle\rho(y,0)\rangle + \int_{\mathbb{R}} \mathrm{d}y\,\frac{\delta(\tau_{x,y}-t)+\delta(\tau_{x,y}+t)}{2v(y)}\langle \jmath(y,0)\rangle + o(t^{-1})$$

when $t \gg 1$ for all $K(x)$.

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Transfer matrix

Consider a subsystem on a finite interval $[y, x]$ with $\langle \widetilde{j}_{\pm}(\cdot, 0)\rangle = 0$ inside and currents instead incident at y and x.

Result: The transfer matrix $\mathsf{T}(\omega)$ between $(\hat{j}_+(y,\omega), \hat{j}_-(y,\omega))^T$ and $(\hat{j}_+(x,\omega), \hat{j}_-(x,\omega))^T$ for $x > y$ is

$$\mathsf{T}(\omega) = \begin{pmatrix} \mathsf{T}_{++}(\omega) & \mathsf{T}_{+-}(\omega) \\ \mathsf{T}_{-+}(\omega) & \mathsf{T}_{--}(\omega) \end{pmatrix} = \overleftarrow{\mathcal{X}} \mathrm{e}^{\mathrm{i} \int_y^x \mathrm{d}s \, \mathsf{P}_\omega(s)}.$$

Simplifies for $\omega = 0$:

$$\mathsf{T}(\omega = 0) = \begin{pmatrix} \frac{\sqrt{\frac{K(y)}{K(x)}} + \sqrt{\frac{K(x)}{K(y)}}}{2} & \frac{\sqrt{\frac{K(y)}{K(x)}} - \sqrt{\frac{K(x)}{K(y)}}}{2} \\ \frac{\sqrt{\frac{K(y)}{K(x)}} - \sqrt{\frac{K(x)}{K(y)}}}{2} & \frac{\sqrt{\frac{K(y)}{K(x)}} + \sqrt{\frac{K(x)}{K(y)}}}{2} \end{pmatrix} = T(x, y).$$

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Scattering matrix

Result: The scattering matrix is

$$\mathsf{S}(\omega) = \begin{pmatrix} T(\omega) & R(\omega) \\ \widetilde{R}(\omega) & T(\omega) \end{pmatrix}$$

with the transmission and reflection amplitudes $\big(|T(\omega)|^2 + |R(\omega)|^2 = 1\big)$

$$T(\omega) = \frac{1}{\mathsf{T}_{--}(\omega)}, \qquad R(\omega) = \frac{\mathsf{T}_{+-}(\omega)}{\mathsf{T}_{--}(\omega)}, \qquad \widetilde{R}(\omega) = -\overline{R(\omega)}\,\frac{T(\omega)}{\overline{T(\omega)}}.$$

Again, simplifies for $\omega = 0$:

$$T(\omega = 0) = \frac{2\sqrt{K(y)K(x)}}{K(y) + K(x)}, \qquad R(\omega = 0) = \frac{K(y) - K(x)}{K(y) + K(x)}.$$

Generalizes results for conformal interfaces and yields simple proof of independence on intermediate values of $K(\cdot)$ for quantum wires.

[Bachas, Brunner, JHEP (2008)], [Maslov, Stone, PRB (1995)], [Safi, Schulz, PRB (1995)]

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Transfer matrix for density and current

Let $\rho(x,t) = \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{2\pi} \hat{\rho}(x,\omega) \mathrm{e}^{-\mathrm{i}\omega t}$ and $\jmath(x,t) = \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{2\pi} \hat{\jmath}(x,\omega) \mathrm{e}^{-\mathrm{i}\omega t}$.

Result: The corresponding zero-frequency transfer matrix is given by

$$\begin{pmatrix} \langle \hat{\rho}(x, \omega = 0) \rangle \\ \langle \hat{\jmath}(x, \omega = 0) \rangle \end{pmatrix} = \begin{pmatrix} \frac{K(x)/v(x)}{K(y)/v(y)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \langle \hat{\rho}(y, \omega = 0) \rangle \\ \langle \hat{\jmath}(y, \omega = 0) \rangle \end{pmatrix}.$$

Implies that $\langle \jmath_5 \rangle = \frac{v(x)}{K(x)} \langle \rho \rangle$ and $\langle \jmath \rangle$ are universal.

Application to quantum wires

Transport in quantum wire

Consider a quantum quench turning off a smooth chemical-potential profile $\mu(x)$ at $t = 0$. Suppose there is some finite $\ell > 0$ so that

$$\mu(x), K(x), v(x) = \begin{cases} \mu_L, K_L, v_L & \text{for } x < -\ell, \\ \mu_R, K_R, v_R & \text{for } x > +\ell. \end{cases}$$

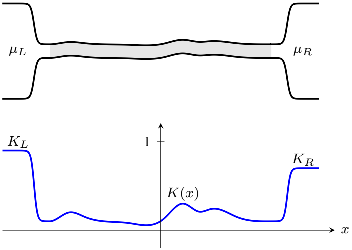
Due to universality of $\frac{v(x)}{K(x)}\langle\rho\rangle$ and equilibrium before the quench:

$$\langle\rho(y,0)\rangle = \frac{K(y)}{\pi v(y)}\mu(y), \quad \langle \jmath(y,0)\rangle = 0.$$

Inserted into the $t \gg 1$ expression for $\jmath$:

$$\lim_{t\to\infty}\langle \jmath(x,t)\rangle = \frac{\mu_+ - \mu_-}{2\pi}$$

with $\mu_+ = K_L\mu_L$ and $\mu_- = K_R\mu_R$.



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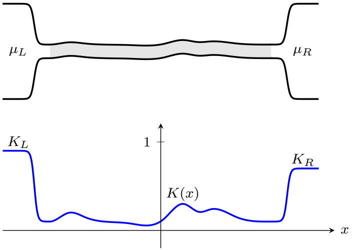

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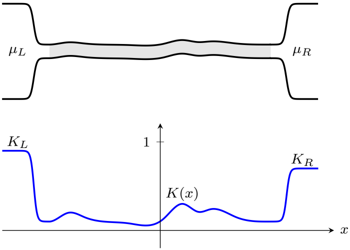

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Summary

- Showed that the dynamics of inhomogeneous TLLs are described by inhomogeneous DBdG equations.
- Obtained general solution of the DBdG equations.
- Derived explicit results at late time or at stationarity that generalize known results in the literature.
- Used results to study coupled FQH edges, quantum wires, and quantum quenches.
- Results applicable whenever DBdG-type equations appear and approach directly generalizable to other algebras than $\mathfrak{sl}(2, \mathbb{C})$.
- Interesting to extend to heat transport and correlation functions.

Thank you for your attention!