Analytical solutions of Dirac-Bogoliubov-de Gennes equations for inhomogeneous quantum many-body systems

Per Moosavi

ETH Zurich

Integrability in Condensed Matter Physics and Quantum Field Theory
SwissMAP Research Station in Les Diablerets

February 10, 2023
Dirac-Bogoliubov-de Gennes (DBdG) equations

**Problem:** Given smooth functions $v(x)$ and $K(x)$, consider

$$
\begin{pmatrix}
  v(x) \partial_x + \partial_t & \Delta(x) \\
  \Delta(x) & v(x) \partial_x - \partial_t
\end{pmatrix}
\begin{pmatrix}
  u_+ \\
  u_-
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  0
\end{pmatrix}
$$

where

$$
\Delta(x) \equiv v(x) \partial_x \log \sqrt{K(x)}
$$

for $u_\pm = u_\pm(x, t)$ with given initial conditions. [P.M., arXiv:2208.14467]

**Questions:**

- What is the general solution?
- What is the effect of $\Delta(x) \neq 0$?
- What is the behavior as $t \to \infty$?
Applications of DBdG-type equations

- [Andreev, Sov. Phys. JETP (1964)]:
  Interfaces between normal metals and superconductors

- [Takayama, Lin-Liu, Maki, PRB (1980)]:
  Continuum description of Su-Schrieffer-Heeger model for polyacetylene

- [P.M., arXiv:2208.14467]:
  Dynamics in inhomogeneous Tomonaga-Luttinger liquids (TLLs)

\[ n(x) \]
a(x)
\[ a(x) \]
\[ J(x) \]
\[ J(x) \]

Inhomogeneous Tomonaga-Luttinger liquids

- a) Cold atoms
- b) Charges in a nanowire
- c) Superconducting circuits
- d) Spin chains

[Gluza, P.M., Sotiriadis, JPA (2022)]
Applications of DBdG-type equations

- [Andreev, Sov. Phys. JETP (1964)]:
  Interfaces between normal metals and superconductors

- [Takayama, Lin-Liu, Maki, PRB (1980)]:
  Continuum description of Su-Schrieffer-Heeger model for polyacetylene

- [P.M., arXiv:2208.14467]:
  \(\mu_L\) and \(\mu_R\) Quantum wires

  \(K_L\), \(1\), \(K_R\)

  \(K(x)\)

  \(\lambda(x)\), \(1\), \(K(x)\) Fractional quantum Hall (FQH) edges

  \(x\)
Some previous works on inhomogeneous TLLs


  Quantum wires


  Effective descriptions of trapped ultra-cold atoms in equilibrium


  Inhomogeneous TLLs out of equilibrium
Outline

- Tomonaga-Luttinger liquids / Compactified free bosons
- Examples of TLLs
- Why PDE approach?
- DBdG equations from TLL theory
- Solving the DBdG equations
- Application to quantum wires
Tomonaga-Luttinger liquids / Compactified free bosons
Tomonaga-Luttinger-liquid (TLL) theory

Given $v > 0$ and $K > 0$. Consider the action functional

$$ S = \frac{R^2}{8\pi} \int_{\mathbb{R} \times S^1_L} \text{d}^2 x \left( \partial^\mu \varphi \right) \left( \partial_\mu \varphi \right) $$

for fields $\varphi : S^1_L \to S^1_{2\pi}$ with compactification radius $R$ satisfying

$$ K = \frac{R^2}{4} $$

and metric $(h_{\mu\nu}) = \text{diag}(1, -1)$ in coordinates $(x^0, x^1) = (vt, x)$. 
Quantum field theory in Hamiltonian framework

Hamiltonian of free compactified bosons

\[ H_{v,K} = \frac{1}{2\pi} \int_{S^1_L} dx : \left( \frac{v}{K} [\Pi(x)]^2 + vK [\partial_x \varphi(x)]^2 \right) : \]

with bosonic field \( \varphi(x) \) and conjugate \( \Pi(x) \) for \( x \in S^1_L \) satisfying

\[ [\partial_x \varphi(x), \Pi(y)] = i\delta'(x - y). \]

How to understand this? Expanding \( \varphi(x) \) and \( \Pi(x) \) in plane waves:

\[ H_{v,K} = (\text{zero modes}) + \frac{\pi v}{2K} \sum_{n \neq 0} \frac{1}{L} : \left( \Pi_{-n} \Pi_n + [Kn]^2 \varphi_{-n} \varphi_n \right) : \]

with \( [\varphi_n, \Pi_m] = i\delta_{n,m} \).

Note: Mathematically, \( \varphi(x) \) and \( \Pi(x) \) are operator-valued distributions. Formal Fourier transforms: \( \varphi_n = L^{-1} \int dx \varphi(x)e^{-2\pi inx/L} \) and \( \Pi_n = \int dx \Pi(x)e^{2\pi inx/L} \).
Quantum field theory in Hamiltonian framework

Hamiltonian of free compactified bosons

\[ H_{v,K} = \frac{1}{2\pi} \int_{S^1_L} dx : \left( \frac{v}{K}[\pi \Pi(x)]^2 + vK[\partial_x \varphi(x)]^2 \right) : \]

with bosonic field \( \varphi(x) \) and conjugate \( \Pi(x) \) for \( x \in S^1_L \) satisfying

\[ [\partial_x \varphi(x), \Pi(y)] = i\delta'(x-y). \]

How to understand this? Expanding \( \varphi(x) \) and \( \Pi(x) \) in plane waves:

\[ H_{v,K} = \text{(zero modes)} + \frac{\pi v}{2K} \sum_{n \neq 0} \frac{1}{L} : (\Pi_{-n}\Pi_n + [Kn]^2 \varphi_{-n}\varphi_n) : \]

with \( [\varphi_n, \Pi_m] = i\delta_{n,m} \).

Note: Mathematically, \( \varphi(x) \) and \( \Pi(x) \) are operator-valued distributions. Formal Fourier transforms: \( \varphi_n = L^{-1} \int dx \varphi(x)e^{-2\pi inx/L} \) and \( \Pi_n = \int dx \Pi(x)e^{2\pi inx/L} \).
More precise definition of $H_{v,K}$

Infinitely many **uncoupled** quantum harmonic oscillators

$$H_{v,K} = \frac{\pi v}{L} (a_0^2 + \bar{a}_0^2) + \frac{\pi v}{L} \sum_{n \neq 0} : (a_{-n}a_n + \bar{a}_{-n}\bar{a}_n) :$$

with $a_n = a_{-n}^\dagger$ and $\bar{a}_n = \bar{a}_{-n}^\dagger$ ($n \in \mathbb{Z}$) for right/left movers satisfying

$$[a_n, a_m] = n\delta_{n+m,0} = [\bar{a}_n, \bar{a}_m], \quad [a_n, \bar{a}_m] = 0,$$

and $a_n|\Omega\rangle = \bar{a}_n|\Omega\rangle = 0$ for $n \geq 0$, defining the vacuum $|\Omega\rangle$, where

$$:a_n a_m: = a_n a_m - \langle \Omega | a_n a_m | \Omega \rangle.$$

**Note:** Relation to $\varphi_n$ and $\Pi_n$ for $n \neq 0$:

$$\varphi_n = \frac{1}{2\sqrt{K}} \frac{i}{n} (a_n - \bar{a}_{-n}), \quad \Pi_n = \sqrt{K} (a_{-n} + \bar{a}_n).$$
Inhomogeneous TLL

Hamiltonian

\[ H_{v(\cdot),K(\cdot)} = \frac{1}{2\pi} \int_{S^1_L} dx : \left( \frac{v(x)}{K(x)} \left[ \pi \Pi(x) \right]^2 + v(x)K(x)\left[ \partial_x \varphi(x) \right]^2 \right) : \]

with inhomogeneous periodic \( v(x) > 0 \) and \( K(x) > 0 \) on the circle \( S^1_L \).

For inhomogeneous periodic \( v(x) > 0 \) and \( K(x) = K > 0 \) constant:


Corresponding action functional

\[ S_{R(\cdot)} = \frac{1}{8\pi} \int_{\mathbb{R} \times S^1_L} d^2 x \sqrt{-h} R(x)^2 (\partial^\mu \varphi)(\partial_\mu \varphi) \]

with inhomogeneous compactification radius \( R(x) = 2\sqrt{K(x)} \) and metric \( (h_{\mu\nu}) = \text{diag}(v(x)^2/v^2, -1) \) in coordinates \( (x^0, x^1) = (vt, x) \).
Inhomogeneous TLL

Hamiltonian

\[ H_{v(\cdot), K(\cdot)} = \frac{1}{2\pi} \int_{S_1^L} d x : \left( \frac{v(x)}{K(x)} [\pi \Pi(x)]^2 + v(x) K(x) [\partial_x \varphi(x)]^2 \right) : \]

with inhomogeneous periodic \( v(x) > 0 \) and \( K(x) > 0 \) on the circle \( S_1^L \).

For inhomogeneous periodic \( v(x) > 0 \) and \( K(x) = K > 0 \) constant:


Corresponding action functional

\[ S_{R(\cdot)} = \frac{1}{8\pi} \int_{\mathbb{R} \times S_1^L} d^2 x \sqrt{-h} R(x)^2 (\partial^\mu \varphi)(\partial_\mu \varphi) \]

with inhomogeneous compactification radius \( R(x) = 2\sqrt{K(x)} \) and metric \( (h_{\mu\nu}) = \text{diag}(v(x)^2/v^2, -1) \) in coordinates \((x^0, x^1) = (vt, x)\).
Inhomogeneous marginal \((J\bar{J})\) deformation

Changing \(R(x)\) to \(R(x) + \delta R(x)\):

\[
\delta S \equiv S_{R(\cdot)+\delta R(\cdot)} - S_{R(\cdot)} = \int_{\mathbb{R} \times S^1_L} \text{d}^2x \sqrt{-h} \Phi
\]

with

\[
\Phi = \frac{1}{\pi} \frac{\delta R(x)}{R(x)} J\bar{J}
\]

where

\[
J \equiv -\frac{1}{\sqrt{K(x)}} \left[ \pi \Pi(x) - K(x) \partial_x \varphi(x) \right],
\]

\[
\bar{J} \equiv -\frac{1}{\sqrt{K(x)}} \left[ \pi \Pi(x) + K(x) \partial_x \varphi(x) \right].
\]

Similar to usual marginal deformation by an operator with conformal weights \((1, 1)\) in the case of constant compactification radius.
Related special case: Conformal interfaces

[Bachas, Brunner, JHEP (2008)]:

\[ \text{Conformal interface} \]
Examples of TLLs
Example: Quantum XXZ spin chain

Hamiltonian

\[ H = -J \sum_{j=1}^{N} \left( S_j^x S_{j+1}^x + S_j^y S_{j+1}^y - \Delta S_j^z S_{j+1}^z \right) \]

with \([S_j^\alpha, S_{j'}^\beta] = i \delta_{j,j'} \epsilon_{\alpha \beta \gamma} S_j^\gamma\) for \(\alpha, \beta, \gamma \in \{x, y, z\}\).

For \(|\Delta| < 1\), the low-energy description is a homogeneous TLL with

\[ v = Ja \frac{\pi}{2} \frac{\sqrt{1 - \Delta^2}}{\arccos(\Delta)}, \quad K = \frac{\pi}{2[\pi - \arccos(\Delta)]} \]

(from exact Bethe-ansatz solution) with \(a = L/N\) the lattice spacing.
Example: Inhomogeneous quantum XXZ spin chain

Hamiltonian

\[
H = - \sum_{j=1}^{N} J_j \left( S_j^x S_{j+1}^x + S_j^y S_{j+1}^y - \Delta S_j^z S_{j+1}^z \right)
\]

with \( J_j = \frac{[J(x_j) + J(x_{j+1})]}{2} \) given by a smooth function \( J(x) \).

For \( |\Delta| < 1 \), the low-energy description is an inhomogeneous TLL with

\[
v(x) = J(x) a \sqrt{1 + 4\Delta / \pi}, \quad K = 1 / \sqrt{1 + 4\Delta / \pi}
\]

(to lowest order in \( \Delta \)) with \( a = L/N \) the lattice spacing. [P.M., AHP (2021)]
Example: Lieb-Liniger model

Hamiltonian

\[
H = \int_{-L/2}^{L/2} dx \left( \frac{1}{2m} \partial_x \Psi(x) \dagger \partial_x \Psi(x) + \frac{g}{2} \Psi(x) \dagger \Psi(x) \Psi(x) \dagger \Psi(x) \right)
\]

with particle mass \( m \), repulsive coupling constant \( g > 0 \), and bosonic field \( \Psi(x) \) satisfying \([\Psi(x) \dagger, \Psi(y)] = \delta(x - y)\).

The low-energy description is a homogeneous TLL with

\[
v = \frac{v_F}{K}, \quad K \sim \begin{cases} 
1 + \frac{4}{\gamma} & \text{for } \gamma \gg 1, \\
\frac{\pi}{\sqrt{\gamma}} \left(1 - \frac{\sqrt{\gamma}}{2\pi}\right)^{-1/2} & \text{for } \gamma \ll 1,
\end{cases}
\]

in terms of the dimensionless coupling \( \gamma = mg/\rho_0 \).
Example: Trapped ultra-cold atoms

Hamiltonian

\[ H = \int_{-L/2}^{L/2} dx \left( \frac{1}{2m} \partial_x \Psi(x) \dagger \partial_x \Psi(x) + \frac{g}{2} \Psi(x) \dagger \Psi(x) \Psi(x) \dagger \Psi(x) \right. \]

\[ + \left. [V(x) - \mu] \Psi(x) \dagger \Psi(x) \right) \]

with quantities as before, trap \( V(x) \), and chemical potential \( \mu \).

The low-energy description is an inhomogeneous TLL with

\[ v(x) = \sqrt{\rho_0(x) g/m}, \quad K(x) = \pi \sqrt{\rho_0(x)/mg}, \]

where \( \rho_0(x) = [\mu - V(x)]/g \) in the Thomas-Fermi regime.
Why PDE approach?
Inhomogeneous TLL

Recall:

\[ H_{v(\cdot),K(\cdot)} = \frac{1}{2\pi} \int_{-L/2}^{L/2} dx : \left( \frac{v(x)}{K(x)}[\pi \Pi(x)]^2 + v(x)K(x)[\partial_x \varphi(x)]^2 \right) : \]

for periodic \( v(x) > 0 \) and \( K(x) > 0 \).
“Diagonalization” approaches

**Naively**: “Diagonalize” $H_{v(\cdot), K(\cdot)}$ by expressing in terms of $a_n$ and $\bar{a}_n$. If $K(x) = K$, achieved by a “simple” Bogoliubov transformation.

**Problem**: Does not work for $K(x)$ since $[\partial_x \varphi(x), \Pi(y)] = i \delta'(x - y)$ not satisfied by the transformed fields.

**Alternatively**: Expand $\partial_x \varphi(x)$ and $\Pi(x)$ not in plane waves but in other eigenfunctions obtained by solving a Sturm-Liouville problem.


For ultra-cold atoms in parabolic trap, then Legendre polynomials.

**Problem**: Again, not practical if eigenfunctions not known.
“Diagonalization” approaches

Naively: “Diagonalize” $H_{\nu(\cdot),K(\cdot)}$ by expressing in terms of $a_n$ and $\bar{a}_n$. If $K(x) = K$, achieved by a “simple” Bogoliubov transformation.

Problem: Does not work for $K(x)$ since $[\partial_x \varphi(x), \Pi(y)] = i\delta'(x - y)$ not satisfied by the transformed fields.

Alternatively: Expand $\partial_x \varphi(x)$ and $\Pi(x)$ not in plane waves but in other eigenfunctions obtained by solving a Sturm-Liouville problem.


For ultra-cold atoms in parabolic trap, then Legendre polynomials.

Problem: Again, not practical if eigenfunctions not known.
DBdG equations from TLL theory
Instead of diagonalizing $H_{v(\cdot)}, K(\cdot)$ rewrite it as [P.M., arXiv:2208.14467]

$$H_{v(\cdot), K(\cdot)} = \int_{-L/2}^{L/2} dx \pi v(x) \left( (\tilde{\rho}_+(x))^2 + (\tilde{\rho}_-(x))^2 \right)$$

with right/left-moving densities

$$\tilde{\rho}_\pm(x) \equiv \frac{1}{2\pi \sqrt{K(x)}} \left[ \pi \Pi(x) \mp K(x) \partial_x \varphi(x) \right].$$

**Result:** $\tilde{\rho}_\pm(x)$ satisfy

$$[\tilde{\rho}_\pm(x), \tilde{\rho}_\pm(y)] = \mp \frac{i}{2\pi} \delta'(x-y),$$

$$[\tilde{\rho}_+(x), \tilde{\rho}_-(y)] = \frac{i}{2\pi} \Lambda(x) \delta(x-y)$$

with $\Lambda(x) \equiv \partial_x \log \sqrt{K(x)}$ coupling right/left movers.
Heisenberg equation and commutation relations imply that $\tilde{\rho}_\pm(x)$ and $\tilde{j}_\pm(x) \equiv \pm v(x)\tilde{\rho}_\pm(x)$ satisfy coupled continuity equations

$$\partial_t\tilde{\rho}_\pm + \partial_x\tilde{j}_\pm = \pm \Delta(x)\tilde{\rho}_\mp$$

with $\Delta(x) \equiv v(x)\Lambda(x)$.

**Result:** $\tilde{j}_\pm(x,t)$ satisfy the inhomogeneous DBdG equations

$$\begin{pmatrix} v(x)\partial_x + \partial_t & \Delta(x) \\ \Delta(x) & v(x)\partial_x - \partial_t \end{pmatrix} \begin{pmatrix} \tilde{j}_+(x,t) \\ \tilde{j}_-(x,t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with a local gap $\Delta(x) = v(x)\partial_x \log \sqrt{K(x)}$. [P.M., arXiv:2208.14467]
Remark 1: Vector and axial currents

The PDEs are equivalent to existence of vector and axial current with

\[
\rho(x) = \Pi(x), \quad j(x) = v(x)K(x)\rho_5(x),
\]

\[
\rho_5(x) = -\partial_x \varphi(x)/\pi, \quad j_5(x) = \frac{v(x)}{K(x)}\rho(x),
\]

satisfying

\[
\partial_t \rho + \partial_x j = 0, \quad \partial_t j + v(x)K(x)\partial_x \left[ v(x)K(x)^{-1}\rho \right] = 0,
\]

\[
\partial_t \rho_5 + \partial_x j_5 = 0, \quad \partial_t j_5 + v(x)K(x)^{-1}\partial_x \left[ v(x)K(x)\rho_5 \right] = 0,
\]

In terms of quantities for right/left movers:

\[
\rho = \sqrt{K(x)}(\tilde{\rho}_+ + \tilde{\rho}_-), \quad j = \sqrt{K(x)}(\tilde{j}_+ + \tilde{j}_-).
\]
Remark 2: Coupled $\mathbb{U}(1)$ current algebras

Define

\[ a_n \equiv \int_{S^1_L} dx \tilde{\rho}_+(x)e^{-2\pi inx/L}, \quad \bar{a}_n \equiv \int_{S^1_L} dx \tilde{\rho}_-(x)e^{2\pi inx/L}. \]

Obtain coupled $\mathbb{U}(1)$ current algebras:

\[
[a_n, a_m] = n\delta_{n+m,0} = [\bar{a}_n, \bar{a}_m], \quad [a_n, \bar{a}_m] = \frac{i}{2\pi} \Lambda_{n-m},
\]

where $\Lambda_n \equiv \int_{S^1_L} dx \Lambda(x)e^{-2\pi inx/L}$.

\[\implies\] Infinitely many coupled quantum harmonic oscillators.

Special case: If $K(x) = K$, then $\Lambda_n = 0$ and the algebras decouple.
Solving the DBdG equations
Recall: \( \tilde{j}_\pm(x, t) \) satisfy

\[
\begin{pmatrix}
v(x) \partial_x + \partial_t & \Delta(x) \\
\Delta(x) & v(x) \partial_x - \partial_t
\end{pmatrix}
\begin{pmatrix}
\tilde{j}_+ \\
\tilde{j}_-
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

with \( \Delta(x) = v(x) \partial_x \log \sqrt{K(x)} \).
Inhomogeneous DBdG equations

Recall: $\tilde{j}_\pm(x, t)$ satisfy

$$\partial_x \begin{pmatrix} \tilde{j}_+ \\ \tilde{j}_- \end{pmatrix} + \begin{pmatrix} v(x)^{-1} \partial_t & \Lambda(x) \\ \Lambda(x) & -v(x)^{-1} \partial_t \end{pmatrix} \begin{pmatrix} \tilde{j}_+ \\ \tilde{j}_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with $\Lambda(x) = \partial_x \log \sqrt{K(x)}$. 
Analogy with non-Hermitian (PT-symmetric) 2-level system

DBdG eqs. in frequency space $\omega$ for expectations in the infinite volume:

$$\partial_x \left( \left( \langle \hat{j}_+ (x, \omega) \rangle \right) \langle \hat{j}_- (x, \omega) \rangle \right) = i P_\omega (x) \left( \left( \langle \hat{j}_+ (x, \omega) \rangle \right) \langle \hat{j}_- (x, \omega) \rangle \right) + \frac{1}{v(x)} \sigma_3 \left( \langle \tilde{j}_+ (x, 0) \rangle \right) \langle \tilde{j}_- (x, 0) \rangle$$

for $x \in \mathbb{R}$ with the $\mathfrak{sl}(2, \mathbb{C})$ matrix

$$P_\omega (x) \equiv \frac{\omega}{v(x)} \sigma_3 + i \Lambda(x) \sigma_1.$$ 

In general, $P_\omega (x) P_\omega (y) \neq P_\omega (y) P_\omega (x)$, so need spatial ordering $\leftarrow \rightarrow$, where positions decrease (increase) from left to right.

Note: Expectations $\langle \cdot \rangle$ w.r.t. arbitrary state in the infinite-volume limit $L \to \infty$. Assumed system prepared in a steady state for $t < 0$ and evolving for $t > 0$ with initial data $\langle \tilde{j}_\pm (x, t = 0) \rangle$. Fourier transforms: $\hat{j}_\pm (x, \omega) = \int_0^\infty dt \tilde{j}_\pm (x, t)e^{i\omega t}$. 

21/31
Problem studied by Magnus

\[
\frac{d}{ds} Y(s) = A(s)Y(s), \quad Y(s_0) = Y_0.
\]

Green’s functions

**Result:** Let \( \langle \tilde{j}_\pm(x, 0) \rangle \) have compact support and \( \lim_{|x| \to \infty} \langle \tilde{j}_\pm(x, t) \rangle = 0 \). Then,

\[
\begin{pmatrix}
\langle \tilde{j}_+(x, t) \rangle \\
\langle \tilde{j}_-(x, t) \rangle
\end{pmatrix} = \int_{\mathbb{R}} dy \, G(x, y; t) \frac{1}{v(y)} \begin{pmatrix}
\langle \tilde{j}_+(y, 0) \rangle \\
\langle \tilde{j}_-(y, 0) \rangle
\end{pmatrix}
\]

using \( G(x, y; t) = \int_{\mathbb{R}} \frac{d\omega}{2\pi} \hat{G}(x, y; \omega) e^{-i\omega t} \) with

\[
\hat{G}(x, y; \omega) = \hat{G}_+(x, y; \omega) \frac{\sigma_0 + \sigma_3}{2} + \hat{G}_-(x, y; \omega) \frac{\sigma_0 - \sigma_3}{2},
\]

\[
\hat{G}_\pm(x, y; \omega) = \pm \theta(\pm [x - y]) \chi e^{i \int_y^x ds P_\omega(s)} \sigma_3.
\]

**Special case:** If \( K(x) = K \), then \( \hat{G}_\pm(x, y; \omega) \) equal

\[
\hat{G}_\pm^0(x, y; \omega) = \pm \theta(\pm [x - y]) e^{i\omega \tau_{x,y} \sigma_3} \sigma_3, \quad \tau_{x,y} = \int_y^x \frac{1}{v(s)} ds.
\]
Green's functions

**Result:** Let \( \langle \tilde{j}_{\pm}(x, 0) \rangle \) have compact support and \( \lim_{|x| \to \infty} \langle \tilde{j}_{\pm}(x, t) \rangle = 0 \). Then,

\[
\begin{pmatrix}
\langle \tilde{j}^+_t(x, t) \rangle \\
\langle \tilde{j}^-_t(x, t) \rangle
\end{pmatrix} = \int_{\mathbb{R}} dy \ G(x, y; t) \frac{1}{v(y)} \begin{pmatrix}
\langle \tilde{j}^+_t(y, 0) \rangle \\
\langle \tilde{j}^-_t(y, 0) \rangle
\end{pmatrix}
\]

using \( G(x, y; t) = \int_{\mathbb{R}} \frac{d\omega}{2\pi} \hat{G}(x, y; \omega) e^{-i\omega t} \) with

\[
\hat{G}(x, y; \omega) = \hat{G}^+(x, y; \omega) \frac{\sigma_0 + \sigma_3}{2} + \hat{G}^-(x, y; \omega) \frac{\sigma_0 - \sigma_3}{2},
\]

\[
\hat{G}^\pm(x, y; \omega) = \pm \theta(\pm [x - y]) \hat{X} e^{i \int_x^y ds P_{\omega(s)} \sigma_3}.
\]

**Special case:** If \( K(x) = K \), then \( \hat{G}^\pm(x, y; \omega) \) equal

\[
\hat{G}_\pm^0(x, y; \omega) = \pm \theta(\pm [x - y]) e^{i\omega \tau_{x,y} \sigma_3} \sigma_3, \quad \tau_{x,y} = \int_y^x ds \frac{1}{v(s)}.
\]
How to express the spatially-ordered exponentials?

The exponentials $\hat{X} e^{i \int_y^x ds \, P_\omega(s)}$ are non-trivial to evaluate.

At least three possibilities:

- Dyson expansion
- Magnus expansion
- Product of exponentials of the $\mathfrak{sl}(2, \mathbb{C})$ generators

Products of exponentials of the $\mathfrak{sl}(2, \mathbb{C})$ generators

**Result:** Let

$$H = -\sigma_3, \quad E = \frac{i\sigma_1 - \sigma_2}{2}, \quad F = \frac{i\sigma_1 + \sigma_2}{2}.$$  

Then, for $x > y$,

$$\chi e^{i \int_y^x ds \, P \omega(s)} = e^{h(x)H} e^{g(x)E} e^{f(x)F},$$

where

$$\begin{align*}
h'(x) &= -i \left[ \omega v(x)^{-1} + g(x) e^{-2h(x)} \Lambda(x) \right], \\
g'(x) &= i \left[ e^{2h(x)} - g(x)^2 e^{-2h(x)} \right] \Lambda(x), \\
f'(x) &= i e^{-2h(x)} \Lambda(x),
\end{align*}$$

with $h(0) = g(0) = f(0) = 0$, and similar for $x < y$.

Follows from [Wei, Norman, JMP (1963); Proc. Amer. Math. Soc. (1964)].
Magnus expansion

**Result:** For \( x > y \),

\[
\widehat{X} e^{i \int_{y}^{x} ds P_{\omega}(s)} = \exp \left[ \sum_{n=1}^{\infty} \Omega_{\omega}^{n}(x, y) \right] e^{i \omega \tau_{x,y} \sigma_{3}}
\]

with

\[
\Omega_{\omega}^{1}(x, y) = - \int_{y}^{x} dx_{1} \Lambda(x_{1}) A_{\omega}(x_{1}, x), \quad A_{\omega}(s, x) \equiv \begin{pmatrix} 0 & e^{-2i \omega \tau_{s, x}} \\ e^{2i \omega \tau_{s, x}} & 0 \end{pmatrix},
\]

\[
\Omega_{\omega}^{2}(x, y) = -i \int_{y}^{x} dx_{1} \int_{y}^{x_{1}} dx_{2} \Lambda(x_{1}) \Lambda(x_{2}) \sin(2 \omega \tau_{x_{1}, x_{2}}) \sigma_{3},
\]

and

\[
\Omega_{\omega}^{n}(x, y) = - \sum_{k=1}^{n-1} \frac{B_{k}}{k!} \sum_{m_{1} \geq 1, \ldots, m_{k} \geq 1}^{m_{1}+\ldots+m_{k}=n-1} \int_{y}^{x} ds \prod_{j=1}^{k} \text{ad}_{\Omega_{\omega}^{m_{j}}(s, y)} \Lambda(s) A_{\omega}(s, x)
\]

for \( n \geq 3 \) consist of similar nested spatial integrals of \( \mathfrak{sl}(2, \mathbb{C}) \)-valued functions that vanish at \( \omega = 0 \), and similar for \( x < y \).
Late-time asymptotics

If $\omega = 0$, then $P_0(x) = i\partial_x \log(\sqrt{K(x)}) \sigma_1$ for different $x$ commute and the only non-zero contribution in the expansions is

$$\exp \left[ - \int_y^x ds \, \Lambda(s) \sigma_1 \right] \equiv T(x, y) = \begin{pmatrix} \sqrt{\frac{K(y)}{K(x)}} + \sqrt{\frac{K(x)}{K(y)}} & \sqrt{\frac{K(y)}{K(x)}} - \sqrt{\frac{K(x)}{K(y)}} \\ \sqrt{\frac{K(y)}{K(x)}} - \sqrt{\frac{K(x)}{K(y)}} & \sqrt{\frac{K(y)}{K(x)}} + \sqrt{\frac{K(x)}{K(y)}} \end{pmatrix}.$$ 

Result: Leading $t \gg 1$ contribution to $G(x, y; t)$ is $T(x, y)G^0(x, y; t)$.

Example: For the current $j = \sqrt{K(x)}(\tilde{j}_+ + \tilde{j}_-)$,

$$\langle j(x, t) \rangle = \int_\mathbb{R} dy \frac{\delta(\tau_{x,y} - t) - \delta(\tau_{x,y} + t)}{2} \langle \rho(y, 0) \rangle$$

$$+ \int_\mathbb{R} dy \frac{\delta(\tau_{x,y} - t) + \delta(\tau_{x,y} + t)}{2v(y)} \langle j(y, 0) \rangle + o(t^{-1})$$

when $t \gg 1$ for all $K(x)$. 
Late-time asymptotics

If \( \omega = 0 \), then \( P_0(x) = i \partial_x \log(\sqrt{K(x)}) \sigma_1 \) for different \( x \) commute and the only non-zero contribution in the expansions is

\[
\exp \left[ - \int_y^x ds \Lambda(s) \sigma_1 \right] \equiv T(x, y) = \begin{pmatrix}
\sqrt{\frac{K(y)}{K(x)}} + \sqrt{\frac{K(x)}{K(y)}} & \sqrt{\frac{K(y)}{K(x)}} - \sqrt{\frac{K(x)}{K(y)}} \\
\sqrt{\frac{K(y)}{K(x)}} - \sqrt{\frac{K(x)}{K(y)}} & \sqrt{\frac{K(y)}{K(x)}} + \sqrt{\frac{K(x)}{K(y)}}
\end{pmatrix}.
\]

**Result:** Leading \( t \gg 1 \) contribution to \( G(x, y; t) \) is \( T(x, y)G^0(x, y; t) \).

**Example:** For the current \( j = \sqrt{K(x)}(\tilde{j}_+ + \tilde{j}_-) \),

\[
\langle j(x, t) \rangle = \int_\mathbb{R} dy \frac{\delta(\tau_{x,y} - t) - \delta(\tau_{x,y} + t)}{2} \langle \rho(y, 0) \rangle
\]

\[
+ \int_\mathbb{R} dy \frac{\delta(\tau_{x,y} - t) + \delta(\tau_{x,y} + t)}{2v(y)} \langle j(y, 0) \rangle + o(t^{-1})
\]

when \( t \gg 1 \) for all \( K(x) \).
Consider a subsystem on a finite interval \([y, x]\) with \(\tilde{j}_\pm(\cdot, 0)\) = 0 inside and currents instead incident at \(y\) and \(x\).

**Result:** The transfer matrix \(T(\omega)\) between \((\hat{j}_+(y, \omega), \hat{j}_-(y, \omega))\)^T and \((\hat{j}_+(x, \omega), \hat{j}_-(x, \omega))\)^T for \(x > y\) is

\[
T(\omega) = \begin{pmatrix}
T_{++}(\omega) & T_{+-}(\omega) \\
T_{-+}(\omega) & T_{--}(\omega)
\end{pmatrix} = \mathcal{X} e^{i \int_y^x ds P_\omega(s)}.
\]

Simplifies for \(\omega = 0\):

\[
T(\omega = 0) = \begin{pmatrix}
\frac{\sqrt{K(y)}}{K(x)} + \frac{\sqrt{K(x)}}{K(y)} & \frac{\sqrt{K(y)}}{K(x)} - \frac{\sqrt{K(x)}}{K(y)} \\
\frac{\sqrt{K(y)}}{K(x)} - \frac{\sqrt{K(x)}}{K(y)} & \frac{\sqrt{K(y)}}{K(x)} + \frac{\sqrt{K(x)}}{K(y)}
\end{pmatrix} = T(x, y).
\]
Consider a subsystem on a finite interval \([y, x]\) with \(\left\langle \vec{j}_\pm(\cdot, 0) \right\rangle = 0\) inside and currents instead incident at \(y\) and \(x\).

**Result:** The transfer matrix \(T(\omega)\) between \((\hat{j}_+(y, \omega), \hat{j}_-(y, \omega))^T\) and \((\hat{j}_+(x, \omega), \hat{j}_-(x, \omega))^T\) for \(x > y\) is

\[
T(\omega) = \begin{pmatrix}
T_{++}(\omega) & T_{+-}(\omega) \\
T_{-+}(\omega) & T_{--}(\omega)
\end{pmatrix} = \mathcal{X} e^{i \int_y^x ds P_\omega(s)}.
\]

Simplifies for \(\omega = 0\):

\[
T(\omega = 0) = \begin{pmatrix}
\frac{\sqrt{K(y)} + \sqrt{K(x)}}{2} & \frac{\sqrt{K(y)} - \sqrt{K(x)}}{2} \\
\frac{\sqrt{K(y)} - \sqrt{K(x)}}{2} & \frac{\sqrt{K(y)} + \sqrt{K(x)}}{2}
\end{pmatrix} = T(x, y).
\]
Scattering matrix

**Result:** The scattering matrix is

\[
S(\omega) = \begin{pmatrix} T(\omega) & R(\omega) \\ \tilde{R}(\omega) & T(\omega) \end{pmatrix}
\]

with the transmission and reflection amplitudes \(|T(\omega)|^2 + |R(\omega)|^2 = 1\):

\[
T(\omega) = \frac{1}{T_{--}(\omega)}, \quad R(\omega) = \frac{T_{+-}(\omega)}{T_{--}(\omega)}, \quad \tilde{R}(\omega) = -\overline{R(\omega)} \frac{T(\omega)}{\overline{T(\omega)}}.
\]

Again, simplifies for \(\omega = 0\):

\[
T(\omega = 0) = \frac{2\sqrt{K(y)K(x)}}{K(y) + K(x)}, \quad R(\omega = 0) = \frac{K(y) - K(x)}{K(y) + K(x)}.
\]

Generalizes results for conformal interfaces and yields simple proof of independence on intermediate values of \(K(\cdot)\) for quantum wires.

Scattering matrix

**Result:** The scattering matrix is

\[
S(\omega) = \begin{pmatrix} T(\omega) & R(\omega) \\ \tilde{R}(\omega) & T(\omega) \end{pmatrix}
\]

with the transmission and reflection amplitudes

\[
T(\omega) = \frac{1}{T_{--}(\omega)}, \quad R(\omega) = \frac{T_{+-}(\omega)}{T_{--}(\omega)}, \quad \tilde{R}(\omega) = -\overline{R(\omega)} \frac{T(\omega)}{T(\omega)}.
\]

Again, simplifies for \( \omega = 0 \):

\[
T(\omega = 0) = \frac{2 \sqrt{K(y)K(x)}}{K(y) + K(x)}, \quad R(\omega = 0) = \frac{K(y) - K(x)}{K(y) + K(x)}.
\]

Generalizes results for *conformal interfaces* and yields simple proof of independence on intermediate values of \( K(\cdot) \) for quantum wires.

Transfer matrix for density and current

Let \( \rho(x, t) = \int_{\mathbb{R}} \frac{d\omega}{2\pi} \hat{\rho}(x, \omega)e^{-i\omega t} \) and \( j(x, t) = \int_{\mathbb{R}} \frac{d\omega}{2\pi} \hat{j}(x, \omega)e^{-i\omega t} \).

**Result:** The corresponding zero-frequency transfer matrix is given by

\[
\left( \begin{array}{c}
\langle \hat{\rho}(x, \omega = 0) \rangle \\
\langle \hat{j}(x, \omega = 0) \rangle 
\end{array} \right) = \left( \begin{array}{cc}
\frac{K(x)/v(x)}{K(y)/v(y)} & 0 \\
0 & 1 
\end{array} \right) \left( \begin{array}{c}
\langle \hat{\rho}(y, \omega = 0) \rangle \\
\langle \hat{j}(y, \omega = 0) \rangle 
\end{array} \right).
\]

Implies that \( \langle j_5 \rangle = \frac{v(x)}{K(x)} \langle \rho \rangle \) and \( \langle j \rangle \) are universal.
Application to quantum wires
Transport in quantum wire

Consider a quantum quench turning off a smooth chemical-potential profile $\mu(x)$ at $t = 0$. Suppose there is some finite $\ell > 0$ so that

$$\mu(x), K(x), v(x) = \begin{cases} 
\mu_L, K_L, v_L & \text{for } x < -\ell, \\
\mu_R, K_R, v_R & \text{for } x > +\ell.
\end{cases}$$

Due to universality of $\frac{v(x)}{K(x)}\langle \rho \rangle$ and equilibrium before the quench:

$$\langle \rho(y, 0) \rangle = \frac{K(y)}{\pi v(y)} \mu(y), \quad \langle j(y, 0) \rangle = 0.$$

Inserted into the $t \gg 1$ expression for $j$:

$$\lim_{t \to \infty} \langle j(x, t) \rangle = \frac{\mu_+ - \mu_-}{2\pi}$$

with $\mu_+ = K_L \mu_L$ and $\mu_- = K_R \mu_R$. 

\[\text{Graph showing the transport in quantum wire.}\]
Transport in quantum wire

Consider a quantum quench turning off a smooth chemical-potential profile $\mu(x)$ at $t = 0$. Suppose there is some finite $\ell > 0$ so that

$$\mu(x), K(x), v(x) = \begin{cases} \mu_L, K_L, v_L & \text{for } x < -\ell, \\ \mu_R, K_R, v_R & \text{for } x > +\ell. \end{cases}$$

Due to universality of $\frac{v(x)}{K(x)} \langle \rho \rangle$ and equilibrium before the quench:

$$\langle \rho(y, 0) \rangle = \frac{K(y)}{\pi v(y)} \mu(y), \quad \langle j(y, 0) \rangle = 0.$$

Inserted into the $t \gg 1$ expression for $j$:

$$\lim_{t \to \infty} \langle j(x, t) \rangle = \frac{\mu_+ - \mu_-}{2\pi}$$

with $\mu_+ = K_L \mu_L$ and $\mu_- = K_R \mu_R$. 

---

**Figure 1. Illustrations of (a) a quantum wire coupled to leads and (b) a quantum wire with a smooth chemical-potential profile.**
Consider a quantum quench turning off a smooth chemical-potential profile $\mu(x)$ at $t = 0$. Suppose there is some finite $\ell > 0$ so that

$$
\mu(x), K(x), v(x) = \begin{cases} 
\mu_L, K_L, v_L & \text{for } x < -\ell, \\
\mu_R, K_R, v_R & \text{for } x > +\ell.
\end{cases}
$$

Due to universality of $\frac{v(x)}{K(x)}\langle \rho \rangle$ and equilibrium before the quench:

$$
\langle \rho(y, 0) \rangle = \frac{K(y)}{\pi v(y)} \mu(y), \quad \langle j(y, 0) \rangle = 0.
$$

Inserted into the $t \gg 1$ expression for $j$:

$$
\lim_{t \to \infty} \langle j(x, t) \rangle = \frac{\mu_+ - \mu_-}{2\pi}
$$

with $\mu_+ = K_L\mu_L$ and $\mu_- = K_R\mu_R$. 
Summary
Showed that the dynamics of inhomogeneous TLLs are described by inhomogeneous DBdG equations.

Obtained general solution of the DBdG equations.

Derived explicit results at late time or at stationarity that generalize known results in the literature.

Used results to study coupled FQH edges, quantum wires, and quantum quenches.

Results applicable whenever DBdG-type equations appear and approach directly generalizable to other algebras than $\mathfrak{sl}(2, \mathbb{C})$.

Interesting to extend to heat transport and correlation functions.

Thank you for your attention!