# Towards a mathematical theory of the ODE/IM correspondence

Davide Masoero
Grupo de Física Matemática da Universidade de Lisboa
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Integrability in Condensed Matter and Quantum Field Theories

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# References

The talk is based on three recent papers with R. Conti and A. Raimondo:

- R. Conti and D.M., *On solutions of the Bethe Ansatz for the Quantum KdV model*. arXiv 2022
- R. Conti and D.M., *Counting Monster Potentials*. JHEP 2021
- D.M. and Andrea Raimondo, *Opers for higher states of quantum KdV models*, Commm. Math. Phys, 2020.

And ongoing work with G. Degano, E. Mukhin, and A. Raimondo.

# Plan of the talk

- Introduction
- If the momentum is large enough, the Destri-De Vega equation for Quantum KdV is well-posed.
- The monster potentials are complete (proven up to a technical assumption).

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## Linear ODEs and Math Phys

It is a general fact that many interesting objects in mathematical physics can be expressed via the (generalised) monodromy data of a of linear ODE in the complex plane.

Quantum field theories & related geometrical objects often correspond to equations with irregular singularities (e.g. Frobenius manifolds and TQFT, ODE/IM correspondence).
This correspondence is somehow mediated by the Nonlinear Stokes Phenomenon (Wall-Crossing).

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# (About names)

‘ODE/IM correspondence’ and ‘ODE/IQFT correspondence’ are two different names for the same thing.

We should all agree to use the same name.

# Bethe Equations for Quantum KdV

Fix $\alpha > 0, p \geq 0$ (in most of this talk, $\alpha > 1$).
We look for a real entire function $Q(E)$ of order $\frac{1+\alpha}{2\alpha}$ such that

- All roots are simple, zero is not a root ($Q(0) = 1$).
- (Almost) all roots are real and positive.
- The counting function $n(E)$ satisfies $\lim_{E\to+\infty} \frac{n(E)}{E^{\frac{1+\alpha}{2\alpha}}} = 1$.
- If $E_*$ is a root, then

$$e^{-i4\pi p} \frac{Q\left(e^{-\frac{2\pi i}{\alpha+1}} E_*\right)}{Q\left(e^{\frac{2\pi i}{\alpha+1}} E_*\right)} = -1.$$

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# Where do they appear?

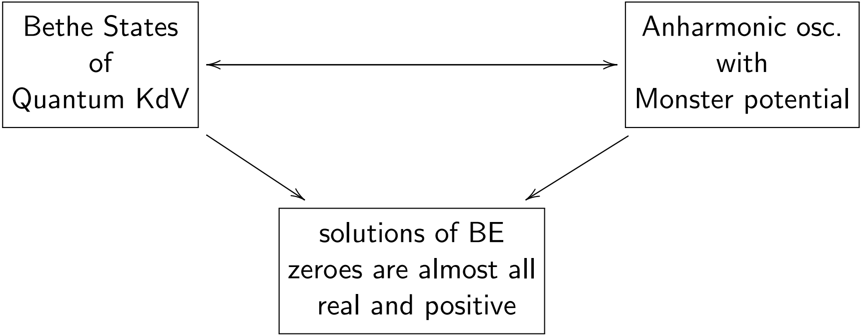
$$Q(E_*) = 0 \Longrightarrow e^{-i4\pi p} \frac{Q\left(e^{-\frac{2\pi i}{\alpha+1}} E_*\right)}{Q\left(e^{\frac{2\pi i}{\alpha+1}} E_*\right)} = -1.$$

- Edge asymptotics of Bethe roots for XXZ with $-1 < \Delta < 1$ (cf. H. Boos talk).
- Bethe Equations satisfied by the eigenvalues of the $\mathcal{Q}_+$ operator of Quantum KdV (Bazhanov-Lukyanov-Zamolodchikov '94-'96).

$$c = 1 - \frac{6\alpha^2}{\alpha+1}, \Delta = p^2(1+\alpha) - \frac{\alpha^2}{4\alpha+4}.$$

- The same equations are also satisfied by spectral determinants of some anharmonic oscillator (Dorey-Tateo '98 & BLZ '98).

# The ODE/IM Conjecture for Quantum KdV



# Topological classification of purely real solutions

- Problem: Classify solutions of the BE whose roots are **all** real and positive.
- Introducing the *'counting function'*,

$$z(E) = -2p + \frac{1}{2\pi i} \log \frac{Q\left(e^{-i\frac{2\pi}{\alpha+1}}E\right)}{Q\left(e^{i\frac{2\pi}{\alpha+1}}E\right)}, E \geq 0, z(0) = -2p.$$

- BE reads $Q(E_*) = 0 \Longrightarrow z(E_*) - \frac{1}{2} \in \mathbb{Z}$
- To classify solution we must add information on which quantum numbers are occupied.

# Roots and Holes

- $E_k : z(E_k) = k + \frac{1}{2}$ with $k \in \mathbb{Z}, k \geq -2p + \frac{1}{2}$
- $k$ is a root number if $Q(E_k) = 0$, a hole-number otherwise.
- Root numbers form as increasing sequence $\{k_n\}_{n \in \mathbb{N}}$
- Now the BE reads

$$z(E_{k_n}) = k_n + \frac{1}{2}, \; n \in \mathbb{N}, \; \text{with } Q(E) = \prod_n \left(1 - \frac{E}{E_{k_n}}\right).$$

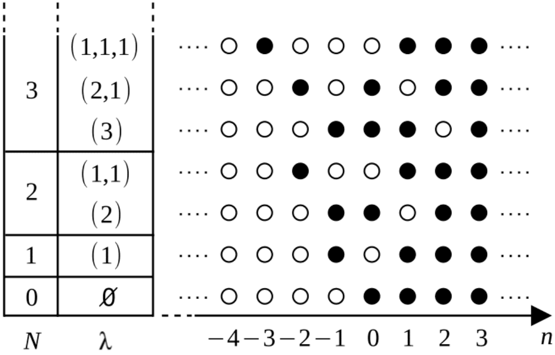
We want to study well-posedness of the above equation, when the sequence $\{k_n\}_{n \in \mathbb{N}}$ is given.

# Roots and integer partitions

Root-numbers are sequences that stabilizes: $k_n = n$, if $n \gg 0$.

$$\Downarrow$$

Root-numbers sequences are classified by integer partitions $\{k_n^\lambda\}_{n\in\mathbb{N}}$.



Bazhanov-Lukyanov-Zamolodchikov, Adv. Theor. Math. Phys., (2003) made the following conjecture:

1. Let $N \in \mathbb{N}$ and $2p \geq N + \frac{1}{2}$. For every $\lambda \vdash N$, the BE admit a unique (normalised) solution $Q^{\lambda}(E;p)$ whose sequence of root-numbers coincide with $\{k_n^{\lambda}\}_{n \in \mathbb{N}}$.
2. Any such solution of the BE coincides with the spectral determinant of a certain anharmonic oscillator.

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# Our results. 1. Well-posedness of BE

**(1) Theorem, M. - Conti 2022**

Fix $\alpha > 1$, $(N, \lambda \vdash N)$.
If $p$ is sufficiently large:

1. The BE admit a unique solution $Q^\lambda(E;p)$ whose sequence of root-numbers coincide with $\{k_n^\lambda\}_{n \in \mathbb{N}}$.

2. $\forall n \in \mathbb{N}$, $\exists C_n > 0$ such that

$$\left| \frac{E_{k_n}(p)}{p^{\frac{2\alpha}{1+\alpha}}} - \left[ A + B \left( k_n + \frac{1}{2} \right) \frac{1}{p} \right] \right| \leq \frac{C_n}{p^2}.$$

3. Uniform asymptotics of $z$ and of roots.

# Earlier results in the mathematical literature

- Well-posedness for $\alpha > 1$, $p = \frac{1}{2\alpha+2}$ and $\lambda = \emptyset$ by A. Avila in Comm. Math. Phys. (2004) - after Voros.
- Well-posedness for $2\alpha$ integer and $\lambda = \emptyset$ by Hilfiker and Runke, Ann. Henri Poincaré (2020), using TBA.

Remark. A variational approach (à la Yang & Yang) should yield sharp bound on the range of $p$ for which BE with real roots only is well-posed.

The range $0 < \alpha < 1$ seems more difficult to study.

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# Strategy of the proof

- Transform the logarithmic BE into a Free-Boundary Nonlinear Integral Equation (known as Destri-De Vega).

- Linearise in the large $p$ limit and do perturbation analysis.

The strategy is standard, the analysis is completely new.

Given $\lambda \vdash N$, let $H = \#\{\text{holes greater than the lowest root}\}$.
The unknown is a tuple $(\omega, h_1, \dots, h_H, z)$

- $\omega > 0$, the left endpoint of the integration interval $[\omega, +\infty[$;
- $h_1 < \dots < h_H$ are the holes greater than the lowest root;
- $z : C^1([\omega, \infty[)$, strictly monotone, $z(E) \sim E^{\frac{1+\alpha}{2\alpha}}, x \to +\infty$.

# Destri-De Vega Integral Equation II

The Destri-De Vega (DDV) is a free-boundary nonlinear integral equation:

1. $$z(E) = -2p + \int_{\omega}^{\infty} K_\alpha(E/y) \left[ z(y) - \frac{1}{2} \right] \frac{dy}{y} + H\, F_\alpha\left(\frac{E}{\omega}\right) - \sum_{j=1}^{H} F_\alpha\left(\frac{E}{h_k}\right), \; K_\alpha(x) = x\, F'_\alpha(x) = \frac{\sin\left(\frac{2\pi}{1+\alpha}\right)}{\pi} \frac{x}{1+x^2-2x\cos\left(\frac{2\pi}{1+\alpha}\right)}$$
2. $$\left[ z(\omega) - \frac{1}{2} \right] = -H$$
3. $z(h_j) = \sigma(j) + \frac{1}{2}$, $j = 1 \dots H$, $\sigma(j) =$ quantum number of $h_j$

Remark. If $z$ is a strictly increasing real analytic function

$$\lim_{\varepsilon \to 0^+} \frac{1}{\pi} \operatorname{Im} \log\left(1 + e^{2\pi i\, z(x+i\varepsilon)}\right) = z - \left[ z - \frac{1}{2} \right]$$

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$$z_{\omega,p}(E) = -2p + \int_{\omega}^{\infty} K_{\alpha}(E/y) z_{\omega,p}(y) \frac{dy}{y}, \; z_{\omega,p}(E) \sim E^{\frac{\alpha+1}{2\alpha}}, x \to \infty.$$

It is a Wiener-Hopf equation, solutions can be expressed via

$$\tau(\xi) = \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \frac{\alpha^{\frac{\alpha s}{1+\alpha}}}{2\sqrt{\pi}(1+\alpha)^{s-1}} \frac{\Gamma\left(-\frac{1}{2} - \frac{\alpha s}{1+\alpha}\right)\Gamma\left(1 - \frac{s}{1+\alpha}\right)}{s^2\,\Gamma(-s)} \xi^{-s} ds, \quad \xi = x/\omega.$$

We discovered a (much more useful) formula in terms of a WKB integral

$$\tau(\xi) = \frac{1}{\pi} \int_{u_-}^{u_+} \sqrt{u^2\xi - u^{2\alpha+2} - 1} \frac{du}{u}, \; \sqrt{\cdots}_{|u=u_\pm} = 0.$$

This is a first hint of the ODE/IM correspondence.

# Large $p$ Linearisation = WKB

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# Perturbation/Analytical challenges

We need to analyse integrals like

$$A_p[f,\varepsilon] = \int_1^\infty K_\alpha\left(\frac{x}{y}\right)\langle pf(y)+\varepsilon(y)\rangle \frac{dy}{y},\ \langle z\rangle = z - \left\lceil z-\frac{1}{2}\right\rceil$$
$$B_p[f,\varepsilon] = \int_1^\infty K_\alpha\left(\frac{x}{y}\right)\left\lceil pf(y)+\varepsilon(y)-\frac{1}{2}\right\rceil \frac{dy}{y}$$

As an example, we showed that if $f \sim x^{\frac{\alpha+1}{2\alpha}}$ and $\varepsilon, \tilde{\varepsilon}$ are bounded ( + some further hypotheses), then

$$\left| \|B_p[f,\varepsilon] - B_p[f,\tilde{\varepsilon}]\|_\infty - \frac{\alpha+1}{2\alpha}\|\varepsilon-\tilde{\varepsilon}\|_\infty \right| \lesssim_f \frac{\|\varepsilon-\tilde{\varepsilon}\|_\infty}{p}$$

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$\Longrightarrow$ contractiveness of the perturbation operator $B_p[I,\cdot]$ when $p$ is large.

$$-\Psi''(x) + \left(x^{2\alpha} + \frac{\ell(\ell+1)}{x^2} - E\right)\Psi(x) = 0, \alpha > 1, \ell \geq 0, E \in \mathbb{C}.$$

$E$ is said an eigenvalue if $\exists \Psi \neq 0$ such that

$$\lim_{x \to 0^+} \Psi(x) = \lim_{x \to +\infty} \Psi(x) = 0.$$

The **spectrum is discrete, simple and positive**, $E_n(\ell), n \in \mathbb{N}$:

$$E_n(\ell) = \left(\frac{2\Gamma\left(\frac{2\alpha+1}{2\alpha}\right)}{\sqrt{\pi}\Gamma\left(\frac{3\alpha+1}{2\alpha}\right)}\right)^{-\frac{2\alpha}{\alpha+1}} (4n + 2\ell + 3)^{\frac{2\alpha}{\alpha+1}} \left(1 + O(n^{-1})\right)$$

Spectral determinant $D_\ell(E)$ is an entire function of order $\frac{1+\alpha}{2\alpha}$.

# A family of anharmonic oscillators. Dorey Tateo, BLZ '98

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## Monster potentials, BLZ (2003)

1. Let $R$ be a monic polynomial of degree $N$. The spectral determinant $D_\ell^R(E)$ for the potential

$$V^R = x^{2\alpha} + \frac{\ell(\ell+1)}{x^2} - 2\frac{d^2}{dx^2}\log R(x^{2\alpha+2})$$

satisfies the BE if the monodromy about the additional poles is trivial for every $E$.

2. Assuming that the roots of $R$ are distinct, the trivial monodromy is equivalent to the BLZ system

$$\sum_{j\neq k}\frac{z_k\left(z_k^2+(3+\alpha)(1+2\alpha)z_k z_j+\alpha(1+2\alpha)z_j^2\right)}{(z_k-z_j)^3} - \frac{\alpha z_k}{4(1+\alpha)} + \Delta(\ell,\alpha) = 0\,, \quad k=1,\dots,N.$$

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## Rational extensions of the harmonic oscillator

- A rational extension of degree $N$ is a potential

$$V^U(t) = t^2 - 2\frac{d^2}{dt^2} \ln U(t),$$

  where $U$ a polynomial of degree $N$ such that all monodromies of $\psi''(t) = (V^U(t) - E)\psi$ are trivial for every $E$.

- Oblomkov's theorem (1999)

$$U \propto U^\lambda := Wr[H_{\lambda_1+j-1}, \ldots, H_{\lambda_j}], \text{ for a } \lambda := (\lambda_1, \ldots, \lambda_j) \vdash N.$$

## (2) (Conditional) Theorem, M. - Conti 2021/2022

- We noticed that in the large momentum multi-scale limit, monster potentials converge to rational extensions of the harmonic oscillator:
  Assume there exists a sequence $R_\ell$ of monster potentials with $\ell \to \infty$, then – up to subsequences –

$$z_k = \frac{\ell^2}{\alpha} + \frac{(2\alpha+2)^{\frac{3}{4}}}{\alpha} v_k^\lambda \ell^{\frac{3}{2}} + O(\ell), \; k = 1, \dots, N$$

  where $v_k^\lambda$ are the roots of $U^\lambda$.
- (If a monster potential with a such an asymptotics exists and) $D_\ell^\lambda(E)$ is the corresponding spectral determinant, then

$$D^\lambda(E;\ell) = Q^\lambda(E/\eta; p), \; p = \frac{2\ell+1}{\alpha+1} \text{ and } \eta = \left( \frac{2\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+\frac{1}{2\alpha}\right)}{\Gamma\left(1+\frac{1}{2\alpha}\right)} \right)^{\frac{2\alpha}{1+\alpha}}.$$

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# An unproven identity

Let $\lambda \vdash N$, assume $U^{\lambda}$ has $N$ distinct zeroes (see conjecture by Felder-Hemery-Veselov 2010). Consider the Jacobian

$$J_{ij}^{\lambda} = \delta_{ij} \left( 1 + \sum_{l \neq j} \frac{6}{(v_i^{\lambda} - v_j^{\lambda})^4} \right) - (1 - \delta_{ij}) \frac{6}{(v_i^{\lambda} - v_j^{\lambda})^4}, \, {}_{i,j=1,\dots,N}.$$

We found an explicit (albeit unproven) formula for the eigenvalues of $J^{\lambda}$: these are square of the hook-lengths of the partition.

$\lambda=(N)$ stated/proven in Ahmed, Bruschi, Calogero, Olshanetsky, and Perelomov ('79).

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**The Big ODE/IM Conjecture**

If a QFT is Bethe Integrable then the corresponding solutions of the Bethe Equations are **spectral determinants of linear differential operators**.
$\rightarrow$ Bethe Roots are eigenvalues of a (possibly self-adjoint) differential operator (cf. Hilbert-Pólya Conjecture).

**M - Raimondo (- Valeri) ('16,'17, '20, ongoing) after Feigin-Frenkel (2011)**

$\widehat{\mathfrak{g}}$ an affine Kac-Moody Lie-algebra and ${}^L\widehat{\mathfrak{g}}$ the Langlands dual,

$$\left\{\text{Bethe states of } \widehat{\mathfrak{g}} - \text{quantum KdV}\right\} \longleftarrow\!\!-\!-\!-\!\!\longrightarrow \left\{{}^L\widehat{\mathfrak{g}} - \text{opers on } \mathbb{C}^*\right\}.$$

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# Open questions? A lot

The ODE/IM correspondence for Quantum KdV is just a tiny piece of an enormous field of research of which we know a lot but still very little.

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- Once, they are found, how do we prove them?
- Why the ODE/IM correspondence? Can we find a theory? Why is the nonlinear Stokes phenomenon that important?

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