Conformal Bootstrap from Gaudin Integrability

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Integrability in CMP and QFT, February 2023

Based on

- Kaviraj, A. *et al.* Multipoint Lightcone Bootstrap from Differential Equations. arXiv: 2212.10578 [hep-th] (Dec. 2022)
- Buric, I. *et al.* Gaudin Models and Multipoint Conformal Blocks III: Comb channel coordinates and OPE factorisation. arXiv: 2112.10827 [hep-th] (Dec. 2021)
- Buric, I. *et al.* Gaudin models and multipoint conformal blocks. Part II. Comb channel vertices in 3D and 4D. *JHEP* **11,** 182. arXiv: 2108.00023 [hep-th] (2021)
- Buric, I. *et al.* Gaudin models and multipoint conformal blocks: general theory. *JHEP* **10,** 139. arXiv: 2105.00021 [hep-th] (2021)
- Buric, I. *et al.* From Gaudin Integrable Models to *d*-dimensional Multipoint Conformal Blocks. *Phys. Rev. Lett.* **126,** 021602. arXiv: 2009.11882 [hep-th] (2021)

Plan

1. Motivation: conformal bootstrap
2. Review: Four-point lightcone bootstrap
3. Lightcone singularities of multipoint integrable system
4. Five-point lightcone bootstrap

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Ingredients of CFT in $d > 2$

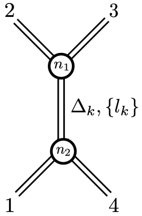
- Primaries fields $\mathcal{O}_{\Delta_i,\{l_{i\alpha}\}}(x_i)$ and descendants $\partial_{\mu_1}\ldots\partial_{\mu_n}\mathcal{O}_{\Delta_i,\{l_{i\alpha}\}}(x_i)$ labeled by a $\mathfrak{so}(1,1)$ weight Δ and $\mathfrak{so}(d)$ spin labels $\{l_{i\alpha}\}$.
- Unitarity bounds: $\Delta_i \geq \begin{cases} \frac{d-2}{2}, & l = 0 \\ d-2+l_{i1}, & l_{i1} > 0 \end{cases}$.
- Operator product expansion: Coefficients of the conformal block expansion are products OPE coefficients $C_{ijk}^{(n)}$.

The conformal bootstrap program

Goal

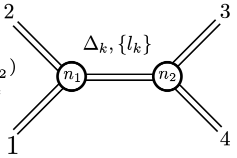
Constrain the scaling dimensions, spins and structure constants from the associativity of the OPE $\iff$ crossing symmetry of all four-point correlation functions.

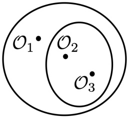
$$\sum_{\mathcal{O}_k, n_1, n_2} C^{(n_1)}_{23k} C^{(n_2)}_{k14}$$

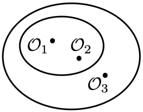


$$=$$

$$\sum_{\mathcal{O}_k, n_1, n_2} C^{(t_{n_1})}_{12k} C^{(t_{n_2})}_{k34}$$



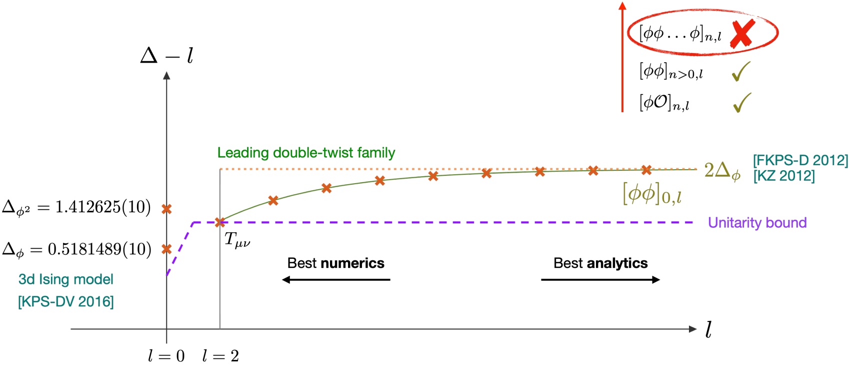




Current status

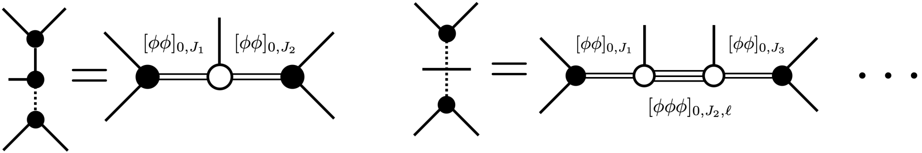
2 approaches give complementary information about CFT data :

- Numerical solutions near regular points (OPE converges).
- Analytic solutions near singularities, e.g. $(x_i - x_j)^2 \to 0$.



Higher point crossing equations and Gaudin Integrability

- So far : only finite subset of CSEs of low spin fields
- Exploring all crossing constraints requires infinite system of 4-point CSEs
- Alternative : higher points CSEs



This talk:

- Singularities of multipoint blocks from integrability
- Applications to large spin CFT data in five-point function.

Plan

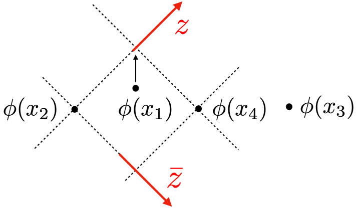
Limits of the BC_2 Calogero-Sutherland model

Crossing equation for $\langle\phi\phi\phi\phi\rangle$

$$u^{\Delta_\phi}\sum_{\Delta,J} C^2_{\Delta,J}\psi^{(14)}_{\Delta,J}(v,u) = v^{\Delta_\phi}\sum_{\Delta,J} C^2_{\Delta,J}\psi^{(12)}_{\Delta,J}(u,v).$$

Lightcone limit

$x^2_{14} \ll x^2_{12} \ll 1 \Rightarrow v \ll u \ll 1.$



Physical expectation

- LHS: isolated sum of leading-twist operators $\mathcal{O}_\star = 1, T_{\mu\nu}, \phi^2, \ldots$ [FGG71]
- RHS: large spin integral over leading double-twist operators $[\phi\phi]_{0,J}$ [Fit+13]; [KZ13]

Limits of the BC_2 Calogero-Sutherland model

Casimir operator/CS Hamiltonian:

$$\mathcal{D}_{12}^2 = (1-v-u)\partial_u \overbrace{\vartheta_u}^{:=u\partial_u} + \vartheta_v(2\vartheta_v - d) - (1+v-u)(\vartheta_v+\vartheta_u)^2 =$$

$$\Theta_4^{-1}\left\{\sum_{\alpha=L,R}\left(-\partial_{\tau_\alpha}^2 \underbrace{-\frac{1}{4\sinh^2\tau_\alpha}}_{V_\alpha(\tau_\alpha)}\right) + \underbrace{\frac{(d-2)(d-4)}{32}\left(\frac{1}{\sinh^2\frac{\tau_L-\tau_R}{2}} + \frac{1}{\sinh^2\frac{\tau_L+\tau_R}{2}}\right)}_{V_{LR}(\tau_L,\tau_R)}\right\}\Theta_4$$

Spectrum known

$$\mathcal{D}_{12}^2\psi_{\Delta,J} = (\Delta(\Delta-d) + J(J+d-2))\,\psi_{\Delta,J} = C^2(\Delta,J)\psi_{\Delta,J}.$$

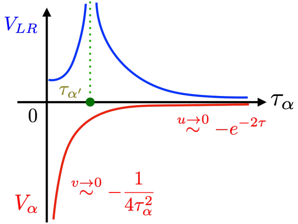
Boundary condition at $(u,v)\to(0,1) \iff (\tau_L,\tau_R)\to(+\infty,+\infty)$

$$\psi_{\Delta,J}(u,v)\sim u^h(1-v)^J \sim 4^\Delta e^{-h\tau_L-\bar{h}\tau_R}, \quad (\Delta,J) = (h+\bar{h},\bar{h}-h).$$

Limits of the BC_2 Calogero-Sutherland model

Coordinates

$$u = z(\tau_L)z(\tau_R), \quad v = (1 - z(\tau_L))(1 - z(\tau_R)), \quad z(\tau) = \frac{4}{e^\tau + e^{-\tau} + 2}.$$



$$\mathcal{D}_{12}^2 = (1-u)\partial_v \vartheta_v + \mathrm{O}(v^0).$$

$$\psi_{\bar{h}}^{rat}(v) \overset{v\to 0}{\sim} \begin{cases} \mathrm{B}_{\bar{h}}^{-1} \log v, & \bar{h}^2 = \mathrm{O}(1), \\ \mathcal{N}_{\bar{h}} K_0(2\bar{h}\sqrt{v}), & \bar{h}^2 = \mathrm{O}(v^{-1}). \end{cases}$$

$$\begin{array}{ccc} \psi_{h+\bar{h},\bar{h}-h}(u,v) & \xrightarrow{v\to 0} & F_h^{(d)}(u)\psi_{\bar{h}}^{rat}\left(\frac{v}{1-u}\right) \\ \Big\downarrow{\scriptstyle u\to 0} & & \Big\downarrow \end{array}$$

$$u^h(1-v)^{\bar{h}-h} \underset{v\to 1}{\longleftarrow} u^h(1-v)^{\bar{h}}\,{}_2F_1\left(\bar{h},\bar{h};2\bar{h};1-v\right) \longrightarrow u^h\psi_{\bar{h}}^{rat}(v)$$

Solving the CE for large spin CFT data

1. $X_{14} \to 0$: Block on LHS suppressed by v^h:

$$(u/v)^{\Delta_\phi}(1 + C^2_{\Delta_\star, J_\star}(1-u)^{\bar{h}_\star}{}_2F_1(1-u) + \dots) = \sum_{\Delta, J} C^2_{\Delta, J}\psi_{\Delta, J}(u, v).$$

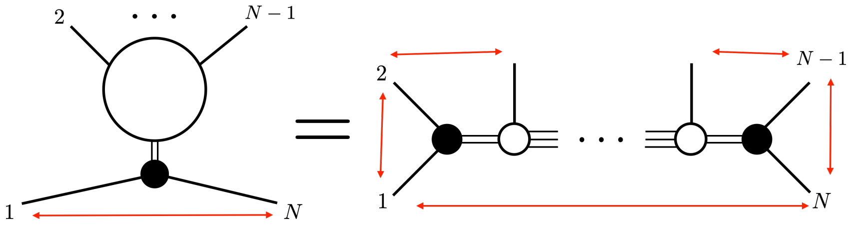
2. Large spin $\bar{h}^2 = \mathrm{O}(X_{14}^{-1})$ on RHS because $\mathcal{D}^2_{12}\mathrm{LHS} = (1-u)\partial_v\vartheta_v\mathrm{LHS}$ $= \mathrm{O}(X_{14}^{-1})\mathrm{LHS}$.
3. $X_{12} \to 0$: Block on RHS suppressed by u^h.

$$(u/v)^{\Delta_\phi}(1 + \frac{C^2_{\Delta_\star, J_\star}}{\mathrm{B}_{\bar{h}_\star}}\log u + \dots) = \int \frac{\mathrm{d}\bar{h}}{2} C^2_{h(\bar{h})+\bar{h}, \bar{h}-h}\mathcal{N}_{\bar{h}} u^{h(\bar{h})} K_0(2\bar{h}\sqrt{v}).$$

Double-twist CFT data at large $\bar{h}$ [Fit+13]; [KZ13]

$$\mathcal{N}_{\bar{h}} C^2_{h+\bar{h}, \bar{h}-h} = \frac{8\bar{h}^{2\Delta_\phi - 1}}{\Gamma(\Delta_\phi)^2} + \dots, \quad h(\bar{h}) = \Delta_\phi - \frac{\mathrm{B}_{\bar{h}_\star}}{C^2_{\Delta_\star, J_\star}}\frac{\Gamma(\Delta_\phi)^2}{\Gamma(\Delta_\phi - h_\star)^2}\bar{h}^{-2h_\star} + \dots$$

Natural generalization: crossing into comb

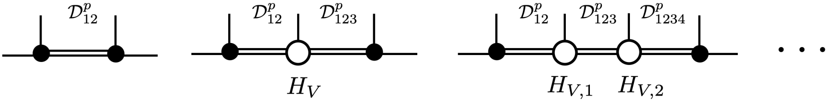


- Null polygon-type limits: $X_{i(i+1)} \to 0$ common in conformal gauge theory and studied in bootstrap, c.f. [BGV21]; [Ber+21]; [Ant+22].
- RHS: Multi-twist operators dominate $\phi \times \cdots \times \phi$ OPEs.
- LHS: OPE channel containing $\phi(x_i) \times \phi(x_{i+1})$.
- So far: only five and six points.

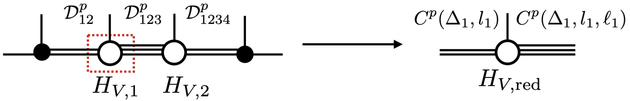
Plan

Multipoint block integrable systems

Commuting Hamiltonians associated to links (Casimir) and vertices.

$\mathcal{C} =$ 

Spectrum & boundary conditions given by spinning 3-point Gaudin models



Comb channels: $H_{V,\mathrm{red}} = \Theta_V^{-1}(\partial_\zeta^4 + V_{EFMV}(\zeta, \partial_\zeta) + E_{EFMV})\Theta_V$ **elliptic** $\mathbb{Z}/4\mathbb{Z}$ **Calogero-Moser** [Eti+11]; [Bur+21d].

Comb channel coordinates and BCs [Bur+21c]

Polynomial cross-ratios

$$\mathcal{D}_{\text{Gaudin}} = \text{polynomial}\left(u_i, v_i, U_s^m, \partial_{u_i,v_i,U_s^m}\right)$$

OPE cross-ratios

$$(u_i, v_i, U_s^5, U_i^6) =$$
$$(z_i\bar{z}_i, (1-z_i)(1-\bar{z}_i), \text{poly}(z, \bar{z}, \mathcal{X}_s, \Upsilon_i))$$

Boundary conditions

$$\psi^{(12),(45)}_{(\Delta,J);t}(z,\bar{z},\Upsilon,\mathcal{X}) \overset{z,\bar{z},\Upsilon\to 0}{\sim} \prod_{i=1}^{N-3} \bar{z}_i^{h_i} z_i^{\bar{h}_i} \prod_{i=2}^{N-4} \Upsilon_i^{\kappa_i} \prod_{s=2}^{N-3} t_s(\mathcal{X}_s), \quad H_{V,\text{red}}\, t = E_t\, t.$$

Singularities in null polygon-type limits $X_{i(i+1)} \to 0$

Definition

$$\mathcal{D} = (X_{ij}^n) \iff \mathcal{D}|_{X_{ij}\to\epsilon X_{ij}} = \epsilon^n \mathcal{D}^{(n)} + \epsilon^{n+1}\mathcal{D}^{(n+1)} + \dots$$

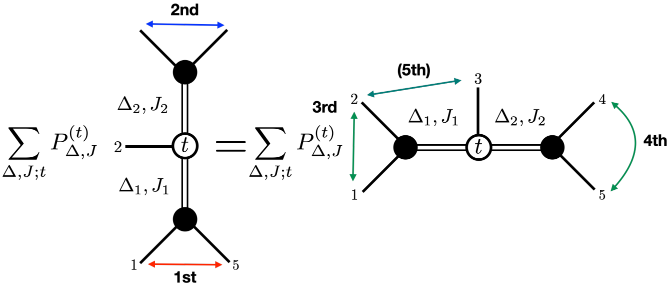
- c.f. τ_α^{-2} singularity of BC_2 Calogero-Sutherland potential: $\mathcal{D}_{12}^2(u, v, \partial_u, \partial_v) = \partial_v \vartheta_v + \cdots = \mathcal{O}(X_{14}^{-1})$.
- Higher point comb:

$$\mathcal{D}_{12}^2 = \mathrm{O}(X_{1N}^{-1} X_{23}^{-1}), \quad \mathcal{D}_{123}^2 = \mathrm{O}(X_{1N}^{-1} X_{34}^{-1}), \dots$$
$$H_V = \mathrm{O}(X_{1N}^{-1} X_{23}^{-1} X_{34}^{-1}), \dots$$

- Polynomial CRs $\Rightarrow$ factorization by ∂'s. Very simple to solve!
- 5-point example: $\Theta^{-1}\mathcal{D}_{12}^2\Theta = v_2 \partial_{U^5}\partial_{v_1}$, $\Theta^{-1}\mathcal{D}_{123}^2\Theta = v_1 \partial_{U^5}\partial_{v_2}$. $\Theta^{-1} H_V \Theta = v_1 v_2 \partial_{U^5}\partial_{v_1}\partial_{v_2}\mathcal{L}_1(\vartheta_u, \vartheta_v, \vartheta_{U^5})$.

Plan

Five-point crossing equation [Kav+22]



Results

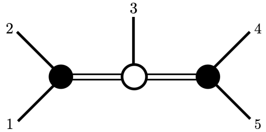
- LHS: isolated sum of leading-twist operators $\mathcal{O}_\star = 1, T_{\mu\nu}, \phi^2, \ldots$ [FGG71]
- RHS: large spin OPE coefficients of two leading double-twist operators $[\phi\phi]_{0,J_{1,2}}$ [BGV21]; [Ant+22]
- Higher resolution of OPE coefficients when $X_{23} = \text{finite}$: discrete basis of tensor structures $H_V \psi_{(\Delta,J);t_\nu} = E_t(\nu)\psi_{(\Delta,J);t_\nu}$, $\nu = 0, 1, \ldots$

Relevant limits of the 5-point integrable system [Kav+22]

Cyclic cross-ratios

$$(u_{12}, u_{23}, u_{34}, u_{45}, u_{15}) = \left(\frac{u_1}{v_2}, v_1, v_2, \frac{u_2}{v_1}, \frac{U^5}{v_1 v_2}\right), \quad \mathrm{u}_{ij} = \mathcal{O}(X_{ij}).$$

$$\Theta_5^{-1}\mathcal{D}_{45}^2\Theta_5 = (1-u_{45})\partial_{u_{15}}\partial_{u_{34}} + \dots$$



$$\Theta_5 \psi_{\mathcal{O};t}(u_{i(i+1)}) \xrightarrow{u_{34}\to 0} F_h(u_{12}, u_{45})\psi^{rat}_{h;t}\left(\frac{v}{1-u}, u_{15}\right)$$

$$\Theta_5 \psi_{\mathcal{O};t}(u_{i(i+1)}) \xrightarrow{u_{12,45}\to 0} u^h(1-v)^{\bar{h}} u_{12}^{h_1} u_{45}^{h_2} F_{hyp}(v, u_{15})$$

$$F_h(u_{12}, u_{45})\psi^{rat}_{h;t}\left(\frac{v}{1-u}, u_{15}\right) \longrightarrow u_{12}^{h_1} u_{45}^{h_2} \psi^{rat}_{h;t}(v, u_{15})$$

$$u_{12}^{h_1} u_{45}^{h_2}(1-v)^{\bar{h}-h} t(\mathcal{X}) \xleftarrow[v_2\to 1]{} u^h(1-v)^{\bar{h}} u_{12}^{h_1} u_{45}^{h_2} F_{hyp}(v, u_{15}) \longrightarrow u_{12}^{h_1} u_{45}^{h_2} \psi^{rat}_{h;t}(v, u_{15})$$

We can solve $F_{hyp}, \psi^{rat}_{\bar{h}}$:

- F_{hyp}: Hypergeometric integral formula, c.f. [FGG71]
- $\psi^{rat}_{\bar{h}}(v_1, v_2, u_{15})$: Hypergeometric power series ($\bar{h}_2^2 = \mathrm{O}(1)$) or sum of Bessel functions ($\bar{h}_2^2 = \mathrm{O}(X_{15}^{-1}X_{34}^{-1})$).

Relevant limits of the 5-point integrable system [Kav+22]

Most singular example:

- $\bar{h}_1^2 = \mathrm{O}(X_{15}^{-1}X_{23}^{-1})$, $\Theta_5^{-1}\mathcal{D}_{12}^2\Theta_5 = \partial_{u_{23}}\partial_{u_{15}}$,
- $\bar{h}_2^2 = \mathrm{O}(X_{15}^{-1}X_{34}^{-1})$, $\Theta_5^{-1}\mathcal{D}_{12}^2\Theta_5 = \partial_{u_{34}}\partial_{u_{15}}$,
- $E_t = \mathrm{O}(X_{15}^{-1}X_{23}^{-1}X_{34}^{-1})$, $\Theta_5^{-1}H_V\Theta_5 = \partial_{u_{23}}\partial_{u_{34}}\partial_{u_{15}}\mathcal{L}_1(\vartheta_u, \vartheta_v, \vartheta_{U^5})$

Solution $\psi^{rat}_{\bar{h};t} = \mathcal{N}_t \dfrac{\mathcal{K}_{h_2-2h_\phi-h_1-\frac{d-2}{2}}((\bar{h}_1^2 u_{23}+\bar{h}_2^2 u_{34})(u_{15}+\bar{h}_1^{-2}\bar{h}_2^{-2}E_t))}{(\bar{h}_1^2 u_{23}+\bar{h}_2^2 u_{34})^{h_2-2h_\phi-h_1-\frac{d-2}{2}}}$

Less singular example: $\bar{h}_1^2 = \mathrm{O}(X_{15}^{-1}) = E_t$. Solution in terms of $\mathcal{K}_\alpha$ and spectrum given by

$$E_t(\nu) = \bar{h}_2^2\left(\left(-\frac{d}{4}+h_1+\nu\right)^2 - \frac{1}{4}d\left(1-\frac{d}{12}\right)\right), \quad \nu \in \mathbb{Z}_{\geq 0}.$$

$\Longrightarrow$ Vertex operators analytically solvable near singularities.
$\Longrightarrow$ We have used these asymptotics to obtain new large spin 5-point CFT data from lightcone limits.

Conclusion & Outlook

- We can relate large-spin, multi-twist CFT data to low-spin CFT data by solving the crossing equation near lightcone singularities.
- All tools to solve it contained in behavior of blocks' integrable system near lightcone limits.
- Applications to OPE coefficients of two double-twist operators [KSS15] and triple-twist operators (Ongoing) in five- and six-point functions.

Open questions:

- Normalization of blocks near lightcone singularities: Interpolation between OPE BCs and singularity similar to integrable scattering problem.
- Systematic classification of conformal blocks' singularities at higher points and their rational degenerations for higher point crossing equations.
- Integrable subsystems in asymptotic plane wave limits $\psi \sim u^h F_{hyp}$ related to GKZ hypergeometric systems.