SEPARATION OF VARIABLES AND CORRELATION FUNCTIONS FROM SPIN CHAINS TO CFT

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based on

2103.15800, 2011.08229, 1910.13442, 1907.03788, 1805.03927, 1610.08032

with Cavaglia, Gromov, Ryan, Sizov, Volin

+ [to appear] with Kazakov, Mishnyakov

Motivation

N=4 super Yang-Mills / strings on AdS5 x S5 is an integrable theory

For complete solution of N=4 SYM we need:

1) Exact spectrum

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{1}{|x-y|^{2\Delta}}$$

$$\mathcal{O}(x) = \mathrm{Tr}\,(\Phi_1\Phi_2\Phi_3\ldots)\,(x)$$

↗ Well understood

2) Exact 3pt functions

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}}$$

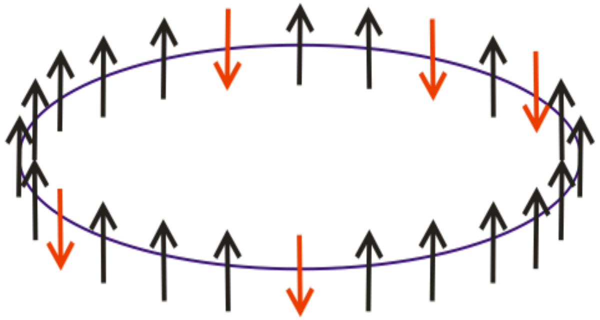
↗ Key open problem !

Solution for spectrum

integrable spin chains

Weak coupling: single trace operators

$$\mathrm{Tr}(\Phi_1(x)\Phi_2(x)\Phi_2(x)\Phi_1(x)\dots) \Longleftrightarrow$$



Finite coupling: Quantum Spectral Curve [Gromov, Kazakov, Leurent, Volin 13]

Difference equations on Baxter functions Q(u) + analytic requirements

$$g = \frac{\sqrt{\lambda}}{4\pi}$$

$$Q_A(u) \longrightarrow Q^A(u)$$

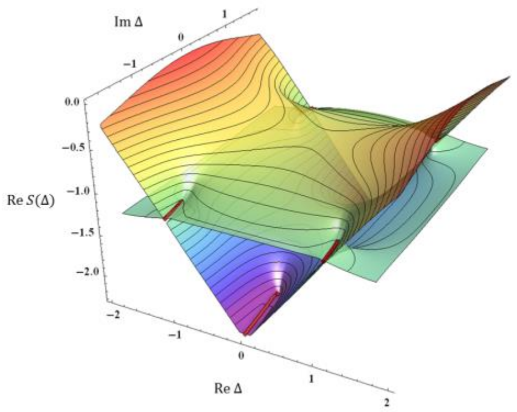
$$Q(u) \sim u^{\Delta}$$

Quantum Spectral Curve

Huge set of results for spectrum

Numerics
at finite coupling

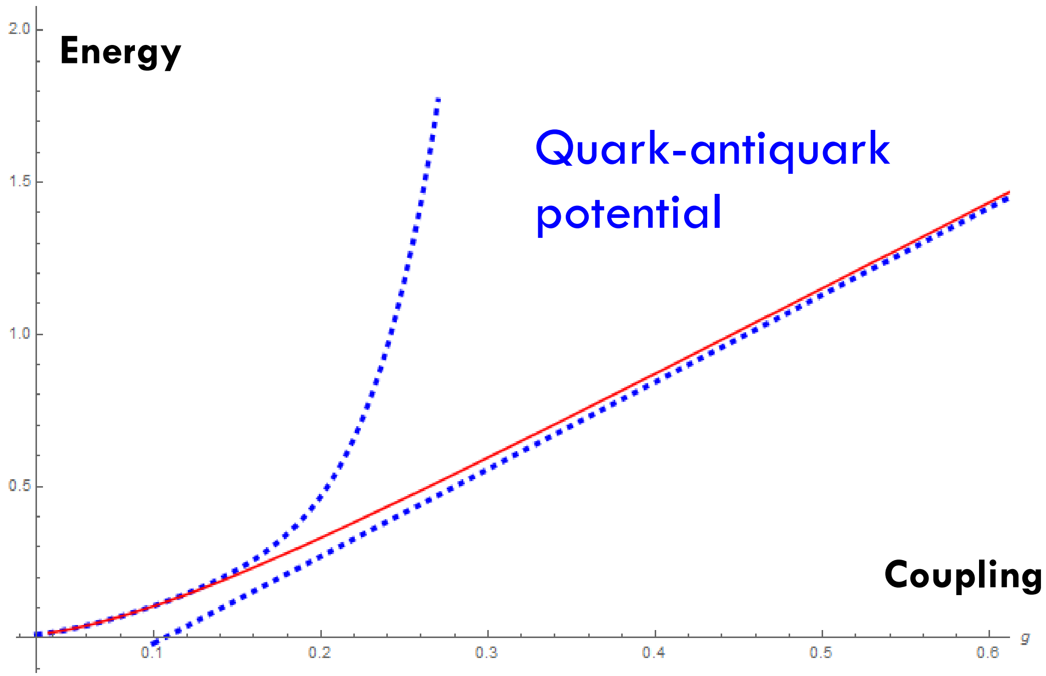
[Gromov, FLM, Sizov 15]



$$
\begin{aligned}
&\frac{1}{256}F_3 = \\
&-\frac{5S_{-5}}{8} - \frac{S_{-4,1}}{2} + \frac{S_1 S_{-3,1}}{2} + \frac{S_{-3,2}}{2} - \frac{5S_2 S_{-2,1}}{4} \\
&+\frac{S_{-4}S_1}{4} + \frac{S_{-3}S_2}{8} + \frac{3S_{3,-2}}{4} - \frac{3S_{-3,1,1}}{2} - S_1 S_{-2,1,1} \\
&+S_{2,-2,1} + 3S_{-2,1,1,1} - \frac{3S_{-2}S_3}{4} - \frac{S_5}{8} + \frac{S_{-2}S_1 S_2}{4} \\
&+\pi^2\left[\frac{S_{-2,1}}{8} - \frac{7S_{-3}}{48} - \frac{S_{-2}S_1}{12} + \frac{S_1 S_2}{48}\right] \\
&+\zeta_3\left[-\frac{7S_{-1,1}}{4} + \frac{7S_{-2}}{8} + \frac{7S_{-1}S_1}{4} - \frac{S_2}{16}\right] \\
&+\left[2\mathrm{Li}_4(\tfrac{1}{2}) - \frac{\pi^2\log^2 2}{12} + \frac{\log^4 2}{12}\right](S_{-1} - S_1) - \pi^4\left[\frac{2S_{-1}}{45} - \frac{S_1}{96}\right] \\
&+\frac{\log^5 2}{60} - \frac{\pi^2\log^3 2}{36} - \frac{2\pi^4\log 2}{45} - \frac{\pi^2\zeta_3}{24} + \frac{49\zeta_5}{32} - 2\mathrm{Li}_5(\tfrac{1}{2})
\end{aligned}
$$

BFKL limit

[Alfimov, Gromov, Kazakov 14]
[Gromov, FLM, Sizov 15]



[Gromov, FLM 16]

Perturbatively 10+ loops

$$\Delta = 4 + 12g^2 - 48g^4 + 336g^6 + \cdots + \#g^{20} + \mathcal{O}(g^{22})$$

[Marboe, Volin]

And much more

Spectrum is known

What about 3pt functions ???

Idea: use separation of variables (SoV) [Sklyanin]

We expect that in any integrable system wavefunctions factorise in a good basis

$$\langle x|\Psi\rangle \sim Q(x_1)Q(x_2)\dots Q(x_N)$$

Like $\Psi_{\text{Hydrogen}} = F_1(r)F_2(\theta)F_3(\varphi)$

Q's should be given by Quantum Spectral Curve at any coupling!

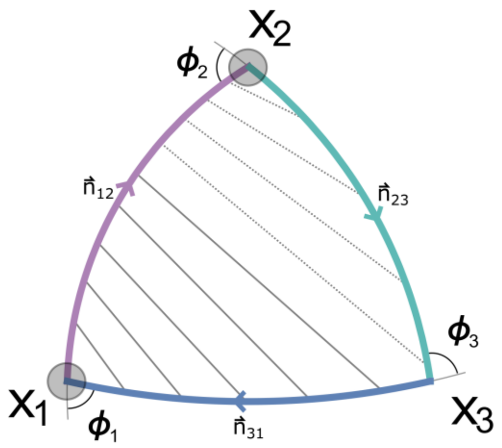
Very promising results for correlators already

[Cavaglia, Gromov, FLM 18]

See also [Giombi, Komatsu [18-20]

$$C_{123} = \frac{\langle Q_1 Q_2 Q_3\rangle}{\sqrt{\langle Q_1^2\rangle\langle Q_2^2\rangle\langle Q_3^2\rangle}}$$

$$\langle f\rangle := \int_{c-i\infty}^{c+i\infty} \frac{du}{2\pi i u} f(u), \quad c>0$$



+ very recent results [Basso, Georgoudis, Sueiro 22] [Bercini, Homrich, Vieira 22]
linking with hexagon expansion [Basso, Komatsu, Vieira 15]

SoV should be very powerful

Yet almost undeveloped beyond GL(2) until recently

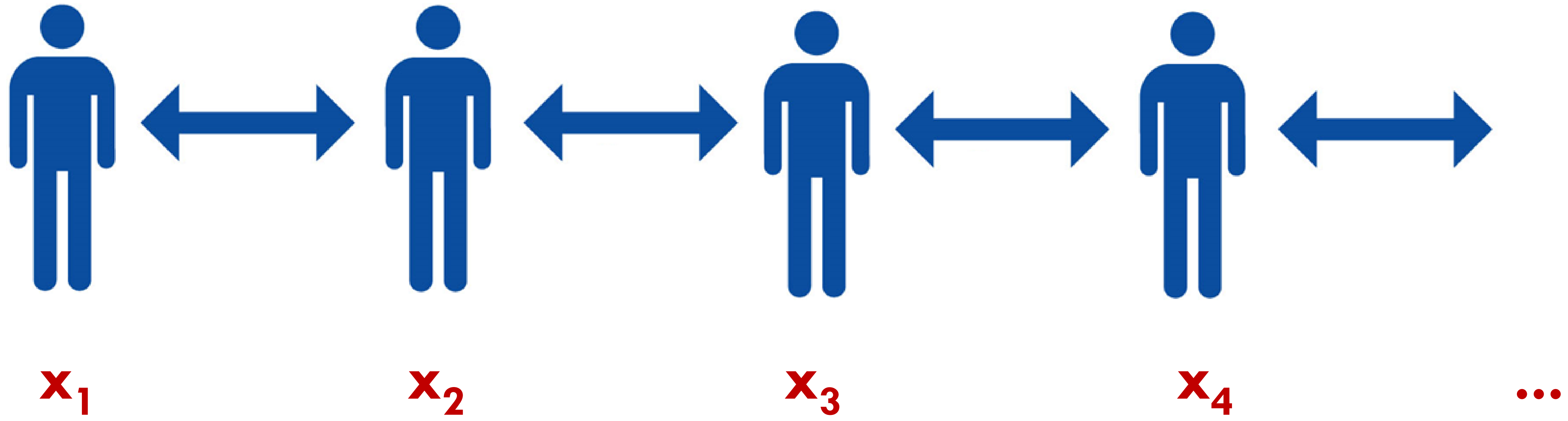
(for SYM we need PSU(2,2|4))

Also important for spin chains/cond-mat, seminal results for GL(2) models

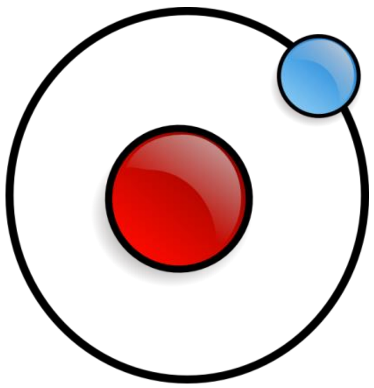
[Derkachov, Frahm, Kitanine, Korchemsky, Kozlowski, Maillet, Niccoli, Terras, Teschner, Smirnov, …]

Need to understand and develop SoV

PLEASE KEEP
2 METRES APART



Focus of this talk: SoV for GL(N) spin chains



hydrogen atom

$$\Psi_{nlm}(r,\theta,\phi) = R(r)P(\theta)F(\phi)$$

$$\langle \Psi_{nlm} | \Psi_{n'l'm'} \rangle = \int \mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta \; r^2 \sin^2\theta \; \Psi^*_{nml} \Psi_{n'm'l'} = \delta_{nn'}\delta_{mm'}\delta_{ll'}$$

Measure

Dual wave function

Wave function

Two main questions:

1) How to factorise wavefunctions?

2) What is the measure?

We will answer both

[review: **FLM** to appear, invited review for J Phys A]

Plan

- Construction of SoV basis
- Finding the measure
- Extensions to field theory and Yangian symmetry

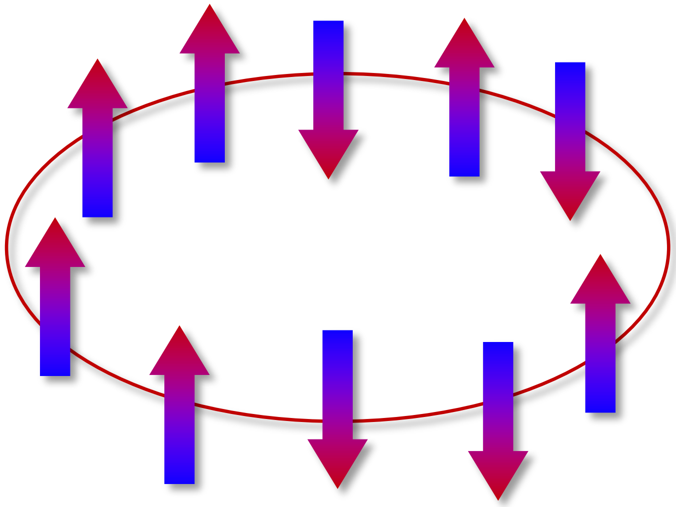
The SoV basis

SU(N) spin chains

Full Hilbert space for L sites is $\underbrace{\mathbb{C}^N \otimes \mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N}_{L \text{ times}}$

$$H = \sum_{n=1}^{L} (1 - P_{n,n+1})$$

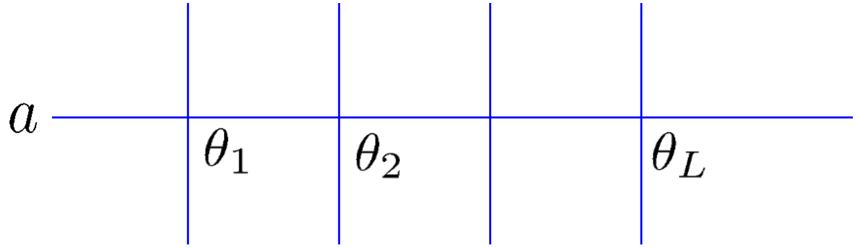
(+ boundary terms, i.e. twist)



Monodromy matrix:

$$T(u) = R_{a1}(u - \theta_1) \dots R_{aL}(u - \theta_L) g$$

$$R_{12}(u) = (u - \tfrac{i}{2}) + iP_{12}$$

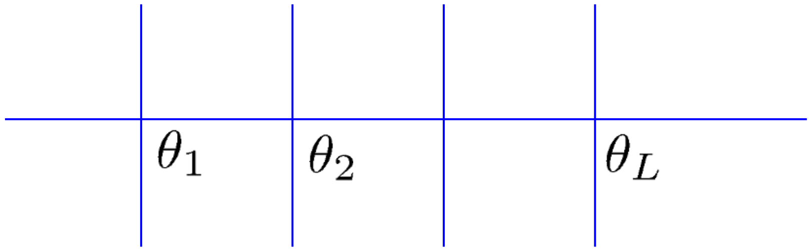


We take generic inhomogeneities θ_n and diagonal twist $g = \mathrm{diag}(\lambda_1, \dots, \lambda_{\mathrm{N}})$

Transfer matrix $\mathrm{Tr}_a\, T(u) = \sum_{n=0}^{L} T_n u^n$ gives commuting integrals of motion

Wavefunctions for spin chains – SU(2)

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$



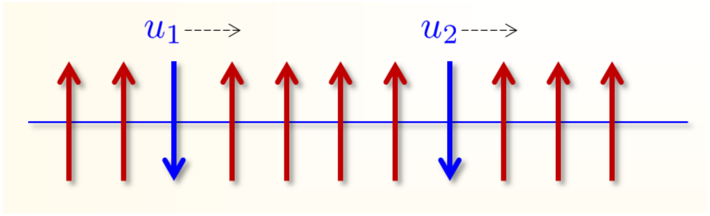
We wish to diagonalize

$$\tau_1(u) = \mathrm{tr}\, T(u) = A(u) + D(u)$$

States are created by operator $B(u)$

$$|\Psi\rangle = B(u_1)B(u_2)\dots B(u_M)|0\rangle$$

Bethe roots



Fixed by $\prod_{n=1}^{L} \frac{u_j - \theta_n + i/2}{u_j - \theta_n - i/2} = e^{2i\phi} \prod_{k \neq j}^{L} \frac{u_j - u_k + i}{u_j - u_k - i}$

Or by Baxter equation $Q_\theta^- Q_1^{++} + Q_\theta^+ Q_1^{--} - \tau_1 Q_1 = 0$

Impose τ_1, Q_1 are polynomials $\longrightarrow$ fix both

$$Q_1 = e^{u\phi} \prod_{k=1}^{M} (u - u_k)$$

$$Q_\theta = \prod_{n=1}^{L} (u - \theta_n)$$

$$f^\pm = f(u \pm i/2)$$

[Sklyanin 90-92]

$$|\Psi\rangle = B(u_1)B(u_2)\dots B(u_M)|0\rangle$$

Consider $\langle x|$ = eigenstates of operator $B(u) = \prod_{k=1}^{L}(u - x_k)$

$$Q_1 = e^{\phi_1 u}\prod_{j=1}^{N_u}(u - u_j)$$

Then wavefunctions factorize! $\langle x|\Psi\rangle = \prod_k Q_1(x_k)$

Proof: $\langle \mathbf{x}_1 \dots \mathbf{x}_L|\Psi\rangle = \prod_{k=1}^{L}\prod_{j=1}^{M}(u_j - \mathbf{x}_k) = \prod_{k=1}^{L} Q_1(\mathbf{x}_k)$

$\mathbf{x}_k = \theta_k \pm i/2, \quad k = 1, \dots L$ gives 2^L states, i.e. basis of the space – called SoV basis

In practice we need a slight modification $T \to T^{\text{good}} = KTK^{-1} \quad B \to B^{\text{good}}$

retains all nice properties

[Gromov, FLM, Sizov 16] [Belliard, Slavnov 18] [Sklyanin 90]

SU(3) case

Sklyanin's proposal $B(u) = T_{13}(u)T_{12|13}(u-i) + T_{23}(u)T_{12|23}(u-i)$ [Sklyanin 92]

$$T_{j_1 j_2|k_1 k_2}(u) = \begin{vmatrix} T_{j_1 k_1}(u) & T_{j_1 k_2}(u+i) \\ T_{j_2 k_1}(u) & T_{j_2 k_2}(u+i) \end{vmatrix}$$

are the quantum minors

Like for SU(2) it creates states! $|\Psi\rangle = B^{\text{good}}(u_1)\dots B^{\text{good}}(u_M)|0\rangle$

[Gromov, FLM, Sizov 16]

$$T \to T^{\text{good}} = KTK^{-1}$$

$$B \to B^{\text{good}}$$

No nesting, surprisingly much simpler than usual BA

$$|\Psi\rangle = \sum_{a_i=2,3} F^{a_1 a_2 \dots a_M} T_{1a_1}(u_1) T_{1a_2}(u_2) \dots T_{1a_M}(u_M)|0\rangle$$

↑ wavefunction of auxiliary SU(2) chain

Kulish, Reshetikhin 83

$$\prod_{n=1}^{L} \frac{u_j - \theta_n + i/2}{u_j - \theta_n - i/2} = \frac{\lambda_2}{\lambda_1} \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i} \prod_{k=1}^{R} \frac{u_j - v_k - i/2}{u_j - v_k + i/2},$$

$$\prod_{n=1}^{M} \frac{v_j - u_n + i/2}{v_j - u_n - i/2} = \frac{\lambda_3}{\lambda_2} \prod_{k \neq j}^{R} \frac{v_j - v_k + i}{v_j - v_k - i}.$$

Factorisation of states follows $\langle \mathbf{x}|\Psi\rangle = \prod_k Q_1(\mathbf{x}_k)$ $\quad Q_1 = e^{\phi_1 u} \prod_{j=1}^{N_u} (u - u_j)$

All this extends to SU(N)

SU(N) case

B-operator is built from quantum minors

Inspired by classical SoV

$$B(u) = \sum_{j,\dots,p} T_{j|N}(u) T_{k|jN}(u-i) \dots T_{12\dots|pN}(u-(N-2)i)$$

[Smirnov 2000] [Chervov, Falqui, Talalaev 07] [Gromov, FLM, Sizov 16]

Creates states as $|\Psi\rangle = B^{\text{good}}(u_1)\dots B^{\text{good}}(u_M)|0\rangle$

For any SU(N) !

[Gromov, FLM, Sizov 16]

$$B(u) = \prod (u - x_k) \qquad \langle x|\Psi\rangle = \prod_k Q_1(x_k)$$

We also found spectrum of x's

States construction proven by [Liashyk, Slavnov 18] for SU(3) (heroic effort)

Then full proof for SU(N) [Ryan, Volin 18], who also showed equivalence with another way to build $\langle x|$

$$\langle x| \sim \langle 0|\hat{T}(\theta_1 + i/2)^{n_1} \dots \hat{T}(\theta_L + i/2)^{n_L}$$

[Maillet, Niccoli 18,19,20]

Analog of states construction found for super SU(1|2) [Gromov, FLM 17]

Computing the SoV measure

For scalar products we need measure

In GL(2)-type models:

$$\langle \Psi_B | \Psi_A \rangle = \int d^L \mathbf{x} \left(\underbrace{\prod_{i=1}^{L} Q^{(A)}(x_i)}_{\text{state } A} \right) \underbrace{M(\mathbf{x})}_{\text{measure}} \left(\underbrace{\prod_{i=1}^{L} Q^{(B)}(x_i)}_{\text{state } B} \right)$$

e.g. for s=-½ spin chain

$$M(\mathbf{x}) = \frac{\prod_{j<k} (e^{2\pi x_j} - e^{2\pi x_k})(x_j - x_k)}{\prod_{j,k} (1 + e^{2\pi (x_j - \theta_k)})}$$

[Sklyanin 90-92]
[Derkachov Korchemsky Manashov 02]

Higher rank GL(N) models are complicated

Measure was not known at all, except in classical limit [Smirnov Zeitlin 02]

To compute correlators

one inserts the complete basis

$$\mathbf{1} = \sum_x M_x \, |x\rangle\langle x|$$

measure $M_x = (\langle x|x\rangle)^{-1}$

Overlaps between these states look complicated

Can we find a way around this?

SU(2) spin chain

Idea: orthogonality of states must imply same for Qs

Baxter equation can be written as

$$\hat{O} \circ Q_1 = 0 \qquad \hat{O} = \frac{1}{Q_\theta^+} D^2 + \frac{1}{Q_\theta^-} D^{-2} - \frac{\tau_1}{Q_\theta^+ Q_\theta^-}$$

$$f^\pm = f(u \pm i/2)$$

$$D f(u) = f(u + i/2)$$

$$Q_\theta = \prod_n (u - \theta_n)$$

Key property: self-adjointness

$$\langle f \hat{O} g \rangle = \langle g \hat{O} f \rangle \qquad \langle f \rangle = \oint du \; f(u)$$

We can introduce L such brackets $\langle f\rangle_j = \oint du\,\mu_j\, f$ $\quad \mu_j = e^{2\pi(j-1)u}$ $\quad j=1,\dots,L$

$$\tau_1 = 2\cos\phi\, u^L + \sum_{k=1}^{L} I_k u^{k-1}$$

uniquely identifies the state

$$\hat{O} = \frac{1}{Q_\theta^+}D^2 + \frac{1}{Q_\theta^-}D^{-2} - \frac{\tau_1}{Q_\theta^+ Q_\theta^-}$$

This gives orthogonality!

$$\langle Q^B(\hat{O}^A - \hat{O}^B)Q^A\rangle_j = 0 \longrightarrow \sum_{k=1}^{L}(I_k^A - I_k^B)\left\langle \frac{u^{k-1}Q^A Q^B}{Q_\theta^+ Q_\theta^-}\right\rangle_j = 0$$

Nontrivial solution means det=0

Sum of residues at $u = \theta_n \pm i/2$
i.e. at x eigenvalues as expected

$$\det_{1\le j,k\le L}\left\langle \frac{u^{k-1}Q^A Q^B}{Q_\theta^+ Q_\theta^-}\right\rangle_j \propto \delta_{AB}$$

Scalar product in SoV

Matches known results [Sklyanin; Kitanine, Maillet, Niccoli, ...]
[Kazama, Komatsu, Nishimura, Serban, Jiang, ...]

SU(3) spin chain

Now we have 2 types of Bethe roots

$$\prod_{n=1}^{L}\frac{u_j-\theta_n+i/2}{u_j-\theta_n-i/2}=e^{i(\phi_1-\phi_2)}\prod_{k\neq j}^{N_u}\frac{u_j-u_k+i}{u_j-u_k-i}\prod_{l=1}^{N_v}\frac{u_j-v_l-i/2}{u_j-v_l+i/2}$$

$$1=e^{i(\phi_2-\phi_3)}\prod_{k\neq j}^{N_v}\frac{v_j-v_k+i}{v_j-v_k-i}\prod_{l=1}^{N_u}\frac{v_j-u_l-i/2}{v_j-u_l+i/2}$$

momentum-carrying $\{u_j\}_{j=1}^{N_u}$

auxiliary $\{v_j\}_{j=1}^{N_v}$

$$Q_1=e^{\phi_1 u}\prod_{j=1}^{N_u}(u-u_j) \qquad Q^2\equiv e^{(\phi_1+\phi_3)u}\prod_j(u-v_j)$$

Main new feature: should use Q^i in addition to Q_i to get simple measure

Other Qs give dual roots

Baxter equations:

$$\tau_a(u) = u^L \chi_a(G) + \sum_{j=1}^{L} u^{j-1} I_{a,j-1},$$

$$\bar{O} = \frac{1}{Q_\theta^-} D^{-3} - \frac{\tau_2}{Q_\theta^+ Q_\theta^-} D^{-1} + \frac{\tau_1}{Q_\theta^+ Q_\theta^-} D - \frac{1}{Q_\theta^+} D^{+3}$$

$$O = \frac{1}{Q_\theta^{++}} D^{+3} - \frac{\tau_2^+}{Q_\theta^{++} Q_\theta} D + \frac{\tau_1^-}{Q_\theta Q_\theta^{--}} D^{-1} - \frac{1}{Q_\theta^{--}} D^{-3}$$

$$\bar{O} \circ Q^a = 0 \qquad O \circ Q_a = 0$$

$$\langle f \rangle_j = \oint du \, \mu_j \, f$$

These two operators are conjugate! $\langle f O \circ g \rangle_j = \langle g \bar{O} \circ f \rangle_j$

$$\mu_j = e^{2\pi (j-1) u}$$

$$j = 1, \dots, L$$

$$\langle Q_b^B (\bar{O}^A - \bar{O}^B) Q^{a,A} \rangle_j = 0$$

$$\tau_a(u) = u^L \chi_a(G) + \sum_{j=1}^{L} u^{j-1}\, I_{a,j-1},$$

Linear system:

$$\sum_{\alpha=\{1,2\},\ k=1,\ldots,L} (I^A_{\alpha,k} - I^B_{\alpha,k})(-1)^\alpha \left\langle \frac{u^k Q^B_1 Q^{a,A[-3+2\alpha]}}{Q^+_\theta Q^-_\theta} \right\rangle_j = 0$$

We have 2L variables, and two choices of a give 2L equations

[Cavaglia, Gromov, FLM 19]

[Gromov, FLM, Ryan, Volin 19]

$$\langle \Psi_B | \Psi_A \rangle \propto \begin{vmatrix} \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_B^{2+} Q_1^A \right\rangle_j & \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_B^{2-} Q_1^A \right\rangle_j \\ \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_B^{3+} Q_1^A \right\rangle_j & \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_B^{3-} Q_1^A \right\rangle_j \end{vmatrix}$$

SU(3) scalar product

$$1 \leq j, k \leq L$$

Each bracket is a sum of residues at $u = \theta_n \pm i/2$

$$N_A^2 \delta_{AB} = \sum_{x,y} M_{x,y} \prod_{k=1}^{L} Q_1^A(X_{k,1}) Q_1^A(X_{k,2}) \prod_{k=1}^{L} \left[Q_B^2(Y_{k,1}) Q_B^3(Y_{k,2}) - Q_B^2(Y_{k,2}) Q_B^3(Y_{k,1}) \right]$$

matches spectrum of $B(u)$

Can we build the basis where these are the wavefunctions?

Operator realization

[Gromov, FLM, Ryan, Volin 19]

$$\langle\Psi_B|\Psi_A\rangle = \int d^L\mathbf{x} \left(\underbrace{\prod_{i=1}^{L} Q^{(A)}(x_i)}_{\text{state } A}\right) \underbrace{M(\mathbf{x})}_{\text{measure}} \left(\underbrace{\prod_{i=1}^{L} Q^{(B)}(x_i)}_{\text{state } B}\right)$$

$$\langle x|\Psi_A\rangle \qquad \langle\Psi_B|y\rangle$$

Instead of integrals we have sums

$$\langle\Psi_B|\Psi_A\rangle = \sum_{x,y} M_{x,y}\langle\Psi_B|y\rangle\langle x|\Psi_A\rangle$$

Get scalar product from two SoV bases $|y\rangle$ and $\langle x|$

$\langle x|$ are eigenstates of Sklyanin's operator $B(u) = T_{13}(u)T_{12|13}(u-i) + T_{23}(u)T_{12|23}(u-i)$

$|y\rangle$ are eigenstates of new "dual" operator $C(u) = T_{13}(u-\frac{i}{2})T_{12|13}(u-\frac{i}{2}) + T_{23}(u-\frac{i}{2})T_{12|23}(u-\frac{i}{2})$

$$M_{x,y} = (\langle x|y\rangle)^{-1}$$

Measure matches what we got from Baxter!

$$M_{x,y} = (\langle x|y\rangle)^{-1}$$

$$\langle\Psi_B|\Psi_A\rangle = \sum_{x,y} M_{x,y}\langle\Psi_B|y\rangle\langle x|\Psi_A\rangle$$

For SU(2) this matrix is diagonal

For SU(3) it is not, but elements are still simple!

$$\langle\Psi_B|\Psi_A\rangle \propto \begin{vmatrix} \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_1^A Q_B^{2+} \right\rangle_j & \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_1^A Q_B^{2-} \right\rangle_j \\ \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_1^A Q_B^{3+} \right\rangle_j & \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_1^A Q_B^{3-} \right\rangle_j \end{vmatrix}$$

[Cavaglia, Gromov, FLM 19]

[Gromov, FLM, Ryan, Volin 19]

Alternative approach: [Maillet, Niccoli, Vignoli 20]

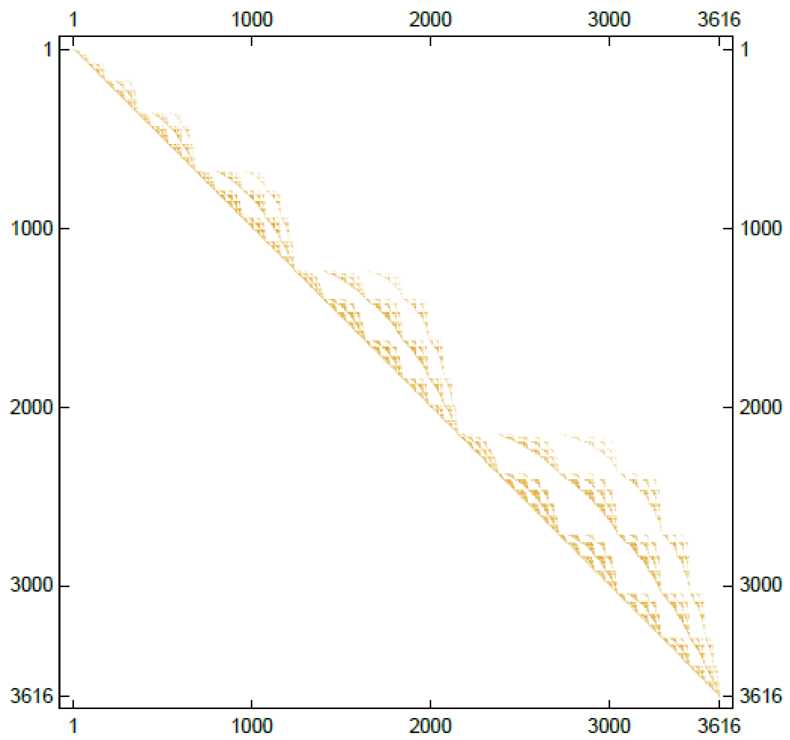
fix measure indirectly by deriving recursion relations for it

(+ another measure found in different basis)

Result should be same, would be interesting to prove

We also managed to compute measure for any SU(N) **explicitly** and for **any spin** [Gromov, **FLM**, Ryan 20]

Representation with weight [s, 0, … 0], including **infinite-dim** case



$$M_{\mathsf{y},\mathsf{x}} = \sum_{k=\text{perm}_\alpha n} \text{sign}(\sigma) \left(\prod_{a=1}^{N-1} \frac{\Delta(\mathsf{x}_{\sigma^{-1}(a)})}{\Delta(\{\theta_a\})}\right) \prod_{a=1}^{N-1} \frac{r_{\alpha,n_{\alpha,a}}}{r_{\alpha,0}} \Big|_{\sigma_{\alpha,a}=k_{\alpha,a}-m_{\alpha,a}+a}$$

$$r_{\alpha,n} = -\frac{1}{2\pi}\prod_{\beta=1}^{L}(n+1-i\theta_\alpha+i\theta_\beta)_{2s-1}$$

$$\langle\Psi_B|\Psi_A\rangle \propto \begin{vmatrix} \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_1^A Q_B^{2+} \right\rangle_j & \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_1^A Q_B^{2-} \right\rangle_j \\ \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_1^A Q_B^{3+} \right\rangle_j & \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_1^A Q_B^{3-} \right\rangle_j \end{vmatrix}$$

[Cavaglia, Gromov, FLM 19]

$$\langle\Psi_A|\Psi_B\rangle=\int d^L\mathbf{x}\left(\underbrace{\prod_{i=1}^{L}Q^{(A)}(x_i)}_{\text{state }A}\right)\underbrace{M(\mathbf{x})}_{\text{measure}}\left(\underbrace{\prod_{i=1}^{L}Q^{(B)}(x_i)}_{\text{state }B}\right)$$

state-independent operator,
contains shifts

$$\hat{M}(x) = \det\left|\underbrace{\left(\frac{\hat{x}^{j-1}}{1+e^{2\pi(\hat{x}-\theta_i)}}\right)}_{1\leqslant i,j\leqslant L}\otimes\underbrace{\begin{pmatrix}\mathcal{D}_x^{N-2} & \mathcal{D}_x^{N-4} & \dots & \mathcal{D}_x^{2-N}\\ \vdots & \vdots & \ddots & \vdots\\ \mathcal{D}_x^{N-2} & \mathcal{D}_x^{N-4} & \dots & \mathcal{D}_x^{2-N}\end{pmatrix}}_{(N-1)\times(N-1)}\right|$$

similar to conjecture of [Smirnov Zeitlin]
based on semi-classics
and quantization of alg curve

Alternatively to build SoV basis we act on reference state with transfer matrices

B(u) is diagonalized by [Maillet, Niccoli 18] [Ryan, Volin 18]

$$\langle x| \propto \langle 0| \prod_{k=1}^{L} [\hat{\tau}_2(\theta_k - i/2)]^{m_{k,1}+m_{k,2}} \qquad 0 \leq m_{k,1} \leq m_{k,2} \leq 1$$

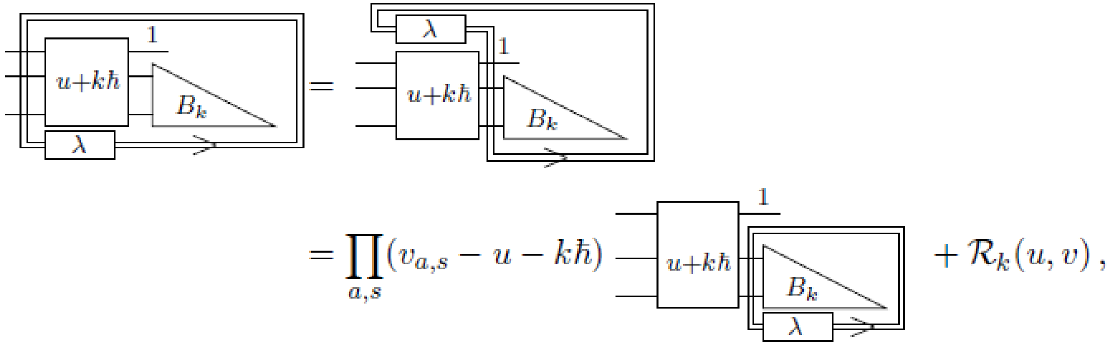
C(u) is diagonalized by [Ryan, Volin 18] [Gromov FLM, Ryan, Volin 19]

$$|y\rangle \propto \prod_{k=1}^{L} \hat{\tau}_1(\theta_k - i/2)^{n_{k,2}-n_{k,1}} \; \hat{\tau}_2(\theta_k - i/2)^{n_{k,1}} |0\rangle \qquad 0 \leq n_{k,1} \leq n_{k,2} \leq 1$$

see also another approach
[Derkachov, Valinevich 18]

Proof is direct generalization of
highly nontrivial methods from [Ryan, Volin 18]

Based on commutation relations +
identifying Gelfand-Tsetlin patterns



Correlators from SoV

Diagonal form factors of type $\frac{\langle\Psi|\frac{\partial \hat{I}_n}{\partial p}|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{\partial I_n}{\partial p}$ are computable, give ratios of determinants

From self-adjoint property:

$$0 = \langle Q(\hat{O} + \delta O) \circ (Q + \delta Q)\rangle = \underbrace{\langle QO \circ \delta Q\rangle}_{=0} + \langle Q\delta O \circ Q\rangle$$

$$\tau_1 = 2\cos\phi\, u^L + \sum_{k=0}^{L-1} I_k u^k$$

Link δI_n with $\delta\phi$

So $$\partial_\phi I_k = \frac{1}{2\sin\phi} \frac{\det\limits_{i,j=1,\dots,L} m_{ij}^{(k)}}{\det\limits_{i,j=1,\dots,L} m_{ij}}$$

norm

From $\partial/\partial\theta_i$ we get local operators on i-th site [Gromov, **FLM**, Ryan 20]

All this generalizes to SU(N)

Can also compute many other correlators in det form

E.g. overlaps with different twists $\langle \Psi^{\tilde{\lambda}_a} | \Psi^{\lambda_a} \rangle = \left[\!\left[\tilde{Q}_{12}, \tilde{Q}_{13} \middle| Q_1 \right]\!\right]$ [Gromov, **FLM**, Ryan 20]

Also on-shell and off-shell overlaps involving B and C operators

$$|\Psi\rangle_{\text{off shell}} \equiv \mathbf{b}(v_1)\ldots\mathbf{b}(v_k)|\Omega\rangle$$

$$\frac{\langle \Phi | \mathbf{c}_{\gamma_1}(v_1)\ldots\mathbf{c}_{\gamma_K}(v_K)\mathbf{b}_{\beta_1}(w_1)\ldots\mathbf{b}_{\beta_J}(w_J)|\Theta\rangle}{\langle \Phi | \Psi \rangle}$$

Likely this gives a complete set of operators

Very recently – all matrix elements for simple complete set of operators in determinant form!

[Gromov, Primi, Ryan 22]

Key idea – SoV basis can be chosen to be twist-independent

Usual choice – diagonal twist

$$g = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

GL(2) covariance lets us choose any twist we like with the same eigenvalues

Much more convenient

$$G = \begin{pmatrix} \chi_1 & -\chi_2 \\ 1 & 0 \end{pmatrix}$$

$$\operatorname{tr} G = \chi_1 = \lambda_1 + \lambda_2$$

$$\det G = \chi_2 = \lambda_1 \lambda_2$$

$$t(u) = T_{12}(u) + \chi_1 T_{11}(u) - \chi_2 T_{21}(u)$$

SoV bases independent of twist

[Ryan, Volin]

Serve to factorise wave functions of different Hamiltonians

[Gromov, FLM, Ryan]

Principal operators

[Gromov, Primi, Ryan 22]

$$t(u) = T_{12}(u) + \chi_1 T_{11}(u) - \chi_2 T_{21}(u)$$

$$t(u) = \chi_1 u^L + \sum_{\beta=1}^{L} \hat{I}_\beta u^{\beta-1}$$

Now integrals of motion admit character expansion

$$\hat{I}_\beta \longrightarrow \hat{I}_\beta^{(0)} + \chi_1 \hat{I}_\beta^{(1)} + \chi_2 \hat{I}_\beta^{(2)}$$

$$t(u) \longrightarrow \mathbb{P}_0(u) + \chi_1 \mathbb{P}_1(u) + \chi_2 \mathbb{P}_2(u)$$

$\mathbb{P}_r(u)$ - Principal operators

[Gromov, Primi, Ryan]

→ Generate remaining operator $T_{22}(u)$

Their form factors (including off-diagonal) have simple det form!

Expect lots of applications [in progress]

Non-compact spin chains

[Cavaglia, Gromov, FLM 19]

Infinite-dim highest weight representation of SL(N) on each site

Now we have integrals instead of sums $\langle f\rangle_j = \int_{-\infty}^{\infty} du\, \mu_j\, f$ $\qquad \mu_j = \frac{1}{1+e^{2\pi(u-\theta_j)}}$

$\bar{O}\circ Q^a = 0 \qquad O\circ Q_a = 0$

$$\bar{O} = Q_\theta^- D^{-3} - \tau_2 D^{-1} + \tau_1 D - Q_\theta^+ D^{+3}$$

$$O = Q_\theta^{++} D^{+3} - \tau_2^+ D + \tau_1{}^- D - Q_\theta^{--} D^{-3}$$

We would like $\langle g\bar{O}\circ f\rangle = \langle fO\circ g\rangle$

Now when we shift the contour we cross poles of the measure

$$\langle g\bar{O}\circ f\rangle = \int \mu\, g\left[Q_\theta^- f^{[-3]} - \tau_2 f^- + \tau_1 f^+ - Q_\theta^+ f^{[+3]}\right] = \langle fO\circ g\rangle + \text{pole contributions}$$

$$Q_1(\theta_j+\tfrac{i}{2})\tau_1(\theta_j+\tfrac{i}{2}) - Q_1(\theta_j+\tfrac{3i}{2})Q_\theta(\theta_j+\tfrac{i}{2}) = 0$$

Poles cancel when $g = Q_1$! Then everything works as before

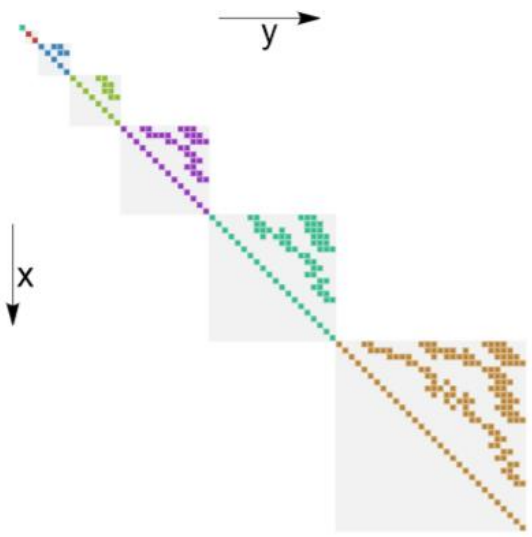
We also generalized to any spin s of the representation [Gromov FLM, Ryan 20]

$$\langle f \rangle_n = \int_{-\infty}^{\infty} du\ \mu_n\ f \qquad \mu_n = \frac{1}{1+e^{2\pi(u-\theta_n)}} \Rightarrow \mu_n = \frac{\Gamma(s-i(u-\theta_n))\Gamma(s+i(u-\theta_n))}{e^{\pi(u-\theta_n)}}$$

For SL(2) we reproduce [Derkachov, Manashov, Korchemsky]

To build SoV basis we need more involved T's in non-rectangular reps see [Ryan, Volin 20]

$$|y\rangle \propto \hat{T}_{\{m_1,m_2\}}\left(\theta_n + is + i\frac{m_1-\mu_1'}{2}\right)|0\rangle$$



Integral = sum over infinite set of poles in lower half-plane

The measure we get from Baxters again matches the one from building the basis!

Comment on chronology:

Such tricks with Baxters were used in [Cavaglia, Gromov, FLM 18] for N=4 SYM

Then in [Cavaglia, Gromov, FLM 19] for SL(N) spin chain

And then in [Gromov, FLM, Ryan, Volin 19] for SU(N) spin chain

Extensions to field theory

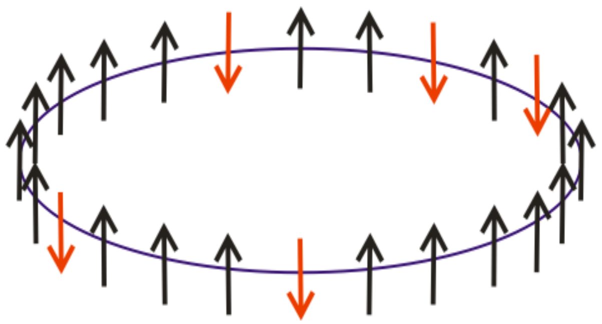
Integrability in N=4 super Yang-Mills

single trace operators

$$\mathrm{Tr}(\Phi_1(x)\Phi_2(x)\Phi_2(x)\Phi_1(x)\ldots)$$

$\Longleftrightarrow$

integrable spin chains



$$\Psi \sim Q(x_1)Q(x_2)\ldots Q(x_n)$$

Q-functions are known at any coupling
from Quantum Spectral Curve

[Gromov, Kazakov, Leurent, Volin 13]

Gives exact spectrum very efficiently !
All-loop, numerical, perturbative, …

[Marboe, Volin 14,16,17] [Alfimov, Gromov, Kazakov 14]
[Gromov, FLM, Sizov 13,14] [Gromov, FLM, Sizov 15 x2]
[Gromov, FLM 15, 16]
[FLM, Preti 20] …

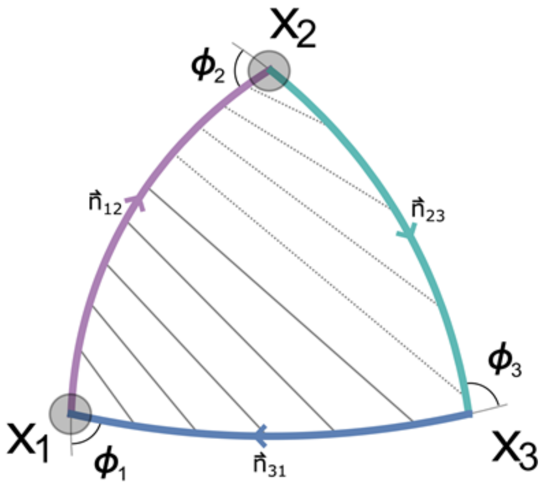
Hope to link with exact 3-pt functions
which are much less understood

Goal: write correlators in terms of Q's

First all-loop example:
3 Wilson lines + scalars
in ladders limit

$$C_{123} = \frac{\langle Q_1 Q_2 Q_3 \rangle}{\sqrt{\langle Q_1^2 \rangle \langle Q_2^2 \rangle \langle Q_3^2 \rangle}}$$

[Cavaglia, Gromov, FLM 18]



Similar structures seen in very different regime via localization
[Komatsu, Giombi 18,19]

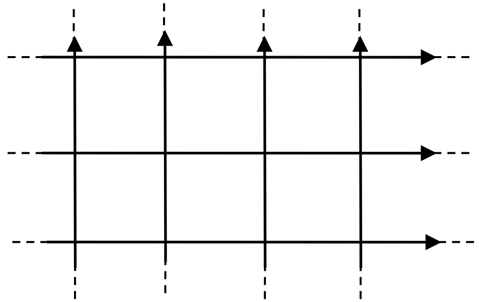
Extension to fishnet CFT

$$S = \frac{N}{2}\int d^4x \; \mathrm{tr}\; \left(\partial^\mu \phi_1^\dagger \partial_\mu \phi_1 + \partial^\mu \phi_2^\dagger \partial_\mu \phi_2 + 2\xi^2\, \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2\right)$$

[Gurdogan, Kazakov 15] [Volodya's and Enrico's talks]

Baby version of N=4 SYM, no inherits integrability

Integrability visible
directly from Feynman graphs



We find very similar
structures

$$C_{\mathcal{O}\mathcal{O}\mathcal{L}} \propto \frac{d\Delta}{d\xi^2} = \frac{\int_{|} \frac{q\bar{q}}{u}\frac{du}{2\pi i}}{\int_{|} i\left(q^+\bar{q}^- - q^-\bar{q}^+\right)\frac{du}{2\pi i}}$$

[Cavaglia, Gromov, **FLM** 21
+ with Sever in progress]

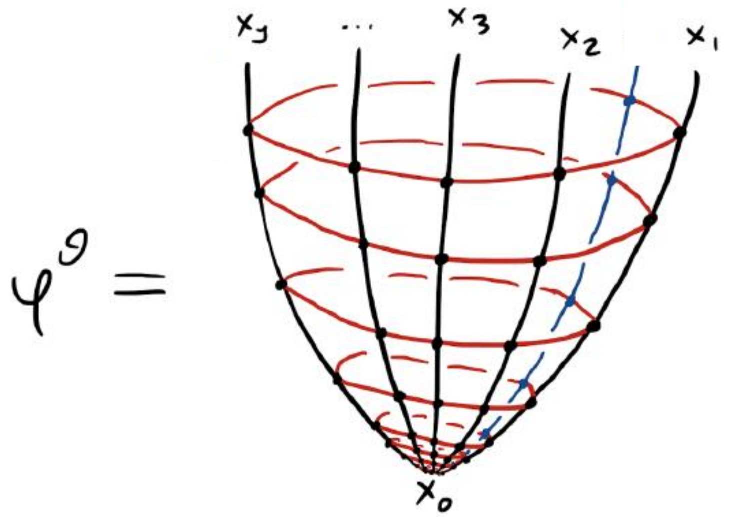
Spin chain picture

Get SO(4,2) spin chain in principal series rep

$$\varphi_{\mathcal{O}}(x_1,\dots,x_J) = \langle \mathcal{O}(x_0)\, \mathrm{tr}\, [\phi_1^\dagger(x_1)\dots\phi_J^\dagger(x_J)]\rangle \,.$$

[Gromov, Sever 19]

Spin chain wavefunction = CFT correlator

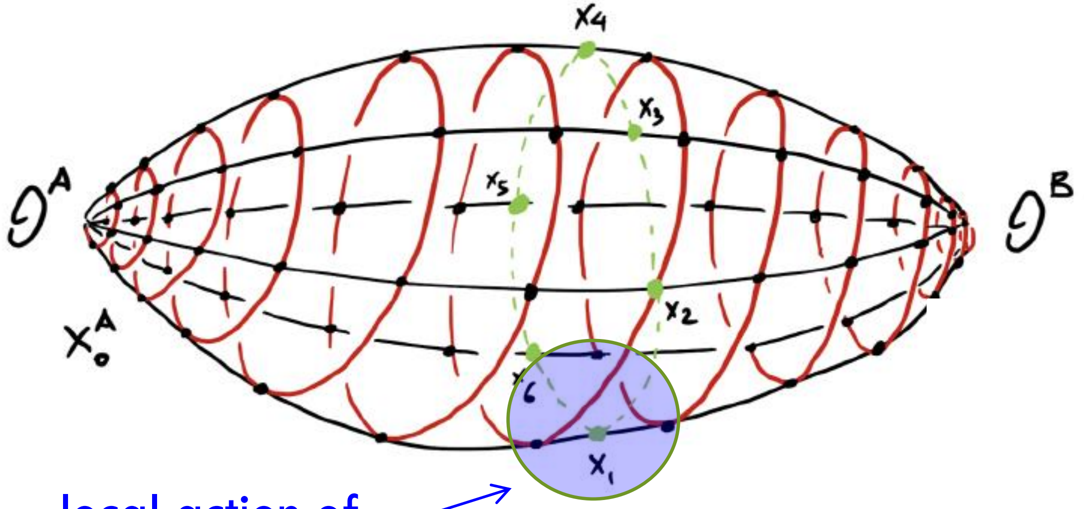


Spin chain form factors ⟶ more involved correlators

Can compute them via SoV! [Cavaglia, Gromov, **FLM** 21]

E.g. $\partial I/\partial p$ gives 2pt function with insertions to all loops

$$\frac{\partial \hat H}{\partial h_\alpha}\hat H^{-1} = -8\left[-\frac{x^2_{\alpha,\alpha-1}+x^2_{\alpha,\alpha+1}}{2}\left(1+x^\mu_\alpha\frac{\partial}{\partial x^\mu_\alpha}\right)+(x^2_{\alpha,\alpha-1}x^\mu_{\alpha+1}+x^2_{\alpha,\alpha+1}x^\mu_{\alpha-1})\frac{\partial}{\partial x^\mu_\alpha}\right] \times \Box_\alpha^{-1}\frac{1}{x^2_{\alpha,\alpha-1}}\frac{1}{x^2_{\alpha,\alpha+1}} \,. \tag{5.36}$$



local action of diff operator

Extensions in progress

Proposal for g-function

[Cavaglia, Gromov, **FLM** 21]

Typical structure for g-function:

$$g \equiv \sqrt{\frac{\langle B|\Psi\rangle \langle\Psi|B\rangle}{\langle\Psi|\Psi\rangle}} = \underbrace{\exp\left(\int_0^\infty \Theta(u)\log(1+Y(u))du\right)}_{\text{boundary-dependent, simple}} \times \underbrace{\sqrt{\frac{\det\left[1-\hat{G}_-\right]}{\det\left[1-\hat{G}_+\right]}}}_{\text{universal factor, hard}}$$

we will guess it from norm

Like for GL(N) spin chains we conjecture the scalar product in SoV

$$\langle\Psi_A|\Psi_B\rangle \propto \det M_{AB}$$

built from integrals of Q-functions

For parity-symmetric states $M_{AA} = \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix} \Rightarrow \det M = \det M_+ \det M_-$

We propose universal part of g-function $(g_{\text{universal}})^2 \propto \frac{|M_-|}{|M_+|_*}$ nontrivially satisfies selection rules!

inspired by spin chain/sin-Gordon results
[Gombor, Pozsgay 20, 21] [Caetano, Komatsu 20]

N=4 SYM

Still have the key starting point! [Cavaglia, Gromov, **FLM** 21]

$$\langle \bar{Q}_B(\mathcal{O}_A - \mathcal{O}_B) Q_A \rangle_\alpha = 0$$

Main difference with spin chains/fishnets:
infinitely many degrees of freedom

Implies infinitely many integrals of motion

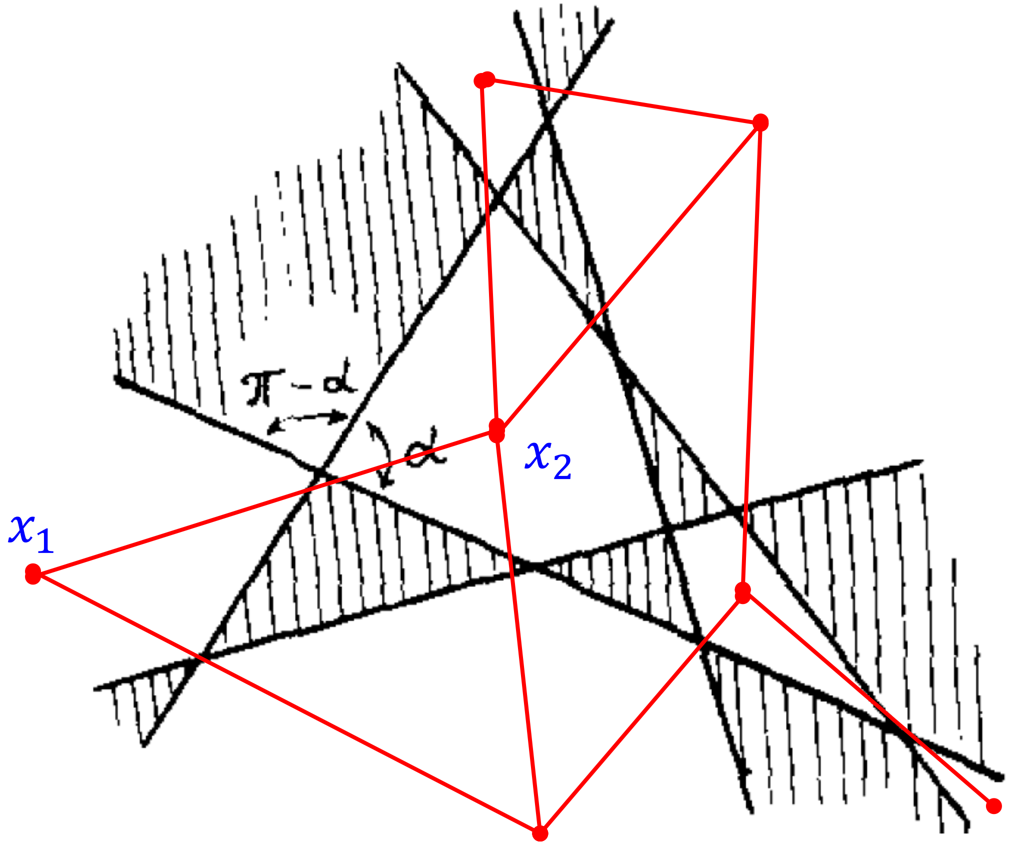
Transfer matrix is not polynomial anymore, need to find a good basis of IoM's

Yangian symmetry for correlators

[Kazakov, **FLM**, Mishnyakov
to appear]

Study conformal Feynman integrals

arising in the most general fishnet CFT ('Loom CFT') [Kazakov, Olivucci 22] [Kazakov's talk]



Start from 'Baxter lattice' (set of intersecting lines)

propagator $= \dfrac{1}{|x_1 - x_2|^{\Delta}}$ $\Delta = D\left(2 - \dfrac{\alpha}{\pi}\right)$

[Zamolodchikov]

These Feynman graphs should be integrable in any D

Feynman graphs with n external legs $\longleftrightarrow$ $\langle \mathrm{Tr}\, [\Phi_1(x_1)\Phi_2(x_2)\ldots\Phi_n(x_n)]\rangle$

We find they are Yangian invariant !

$$(L_1 L_2 \ldots L_n)_{\alpha\beta}\, |\text{graph}\rangle = \lambda(u)\delta_{\alpha\beta}|\text{graph}\rangle$$

[Kazakov, **FLM**, Mishnyakov to appear]

Conformal Laxes act on external legs

$$L(u_+, u_-) = \begin{pmatrix} u_+ - \mathbf{p}\mathbf{x} & \mathbf{p} \\ \mathbf{x}(u_+ - u_-) - \mathbf{x}\mathbf{p}\mathbf{x} & \mathbf{x}\mathbf{p} + u_- \end{pmatrix}$$

[Chicherin, Derkachov, Isaev 12]

Generalization of [Chicherin, Kazakov, Loebbert, Muller, Zhong 17] [..]

- any $\Delta' s$
- any graph geometry
- any (even) D

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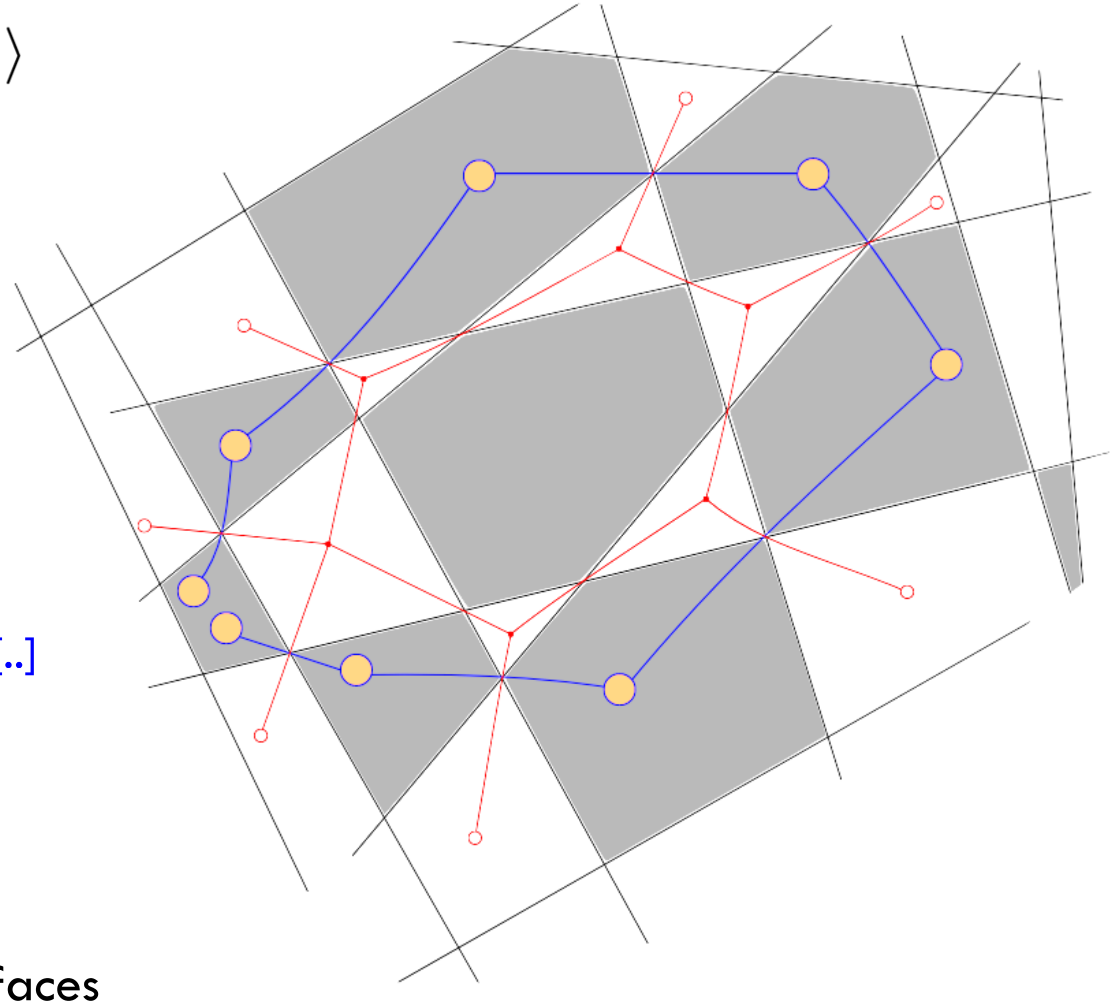


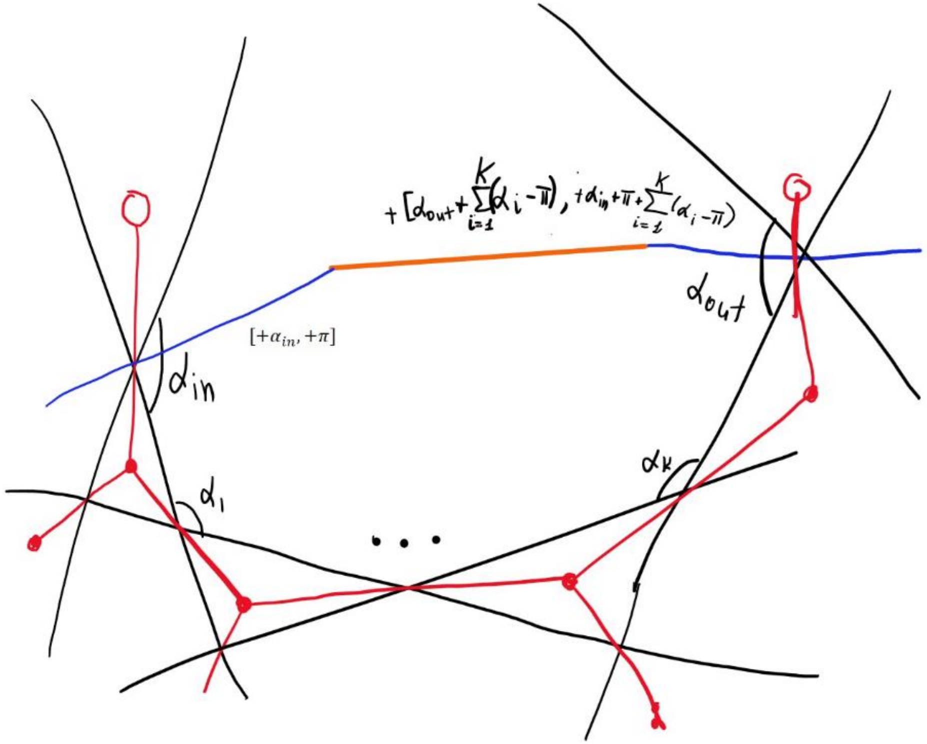
Generalization of [Chicherin, Kazakov, Loebbert, Muller, Zhong 17] [..]

- any $\Delta' s$
- any graph geometry
- any (even) D

Key new idea – draw chain of Lax operators on dual faces

$$L(u_+, u_-) = \begin{pmatrix} u_+ - \mathbf{p}\mathbf{x} & \mathbf{p} \\ \mathbf{x}(u_+ - u_-) - \mathbf{x}\mathbf{p}\mathbf{x} & \mathbf{x}\mathbf{p} + u_- \end{pmatrix}$$

Read off parameters from local geometry



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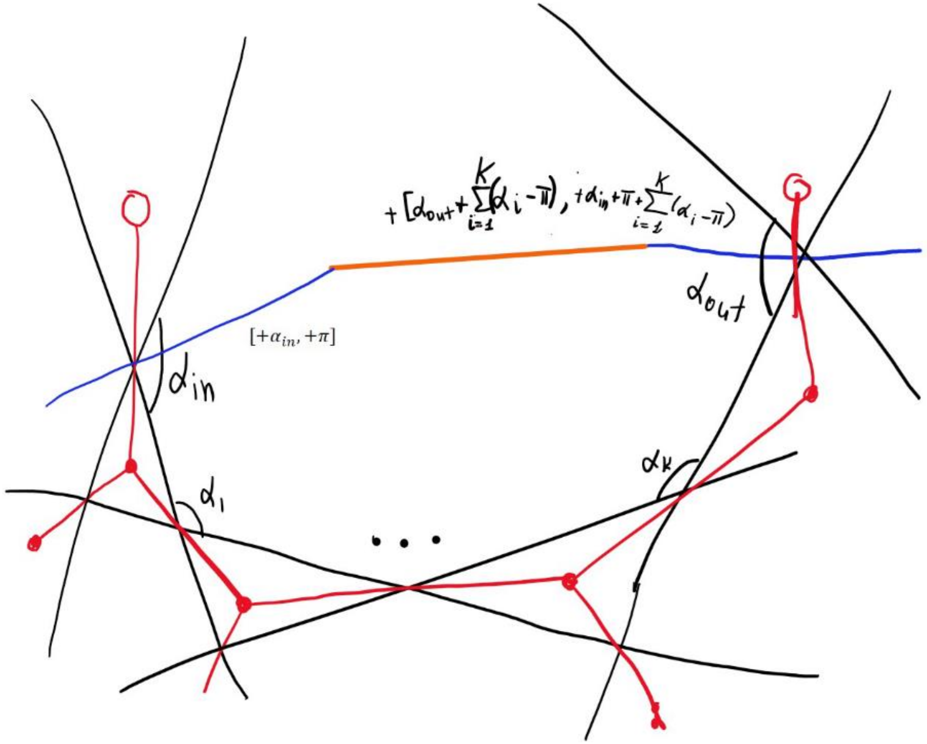
Read off parameters from local geometry



Get new differential equations for these integrals!

[Kazakov, **FLM**, Mishnyakov to appear]

Yangian invariance already led to powerful results for correlators/amplitudes

[Corcoran, Loebbert, Miczajka, Muller, Munkler, Staudacher ,... 18-22]

Relations with SoV to be explored

e.g. for 2d graphs linked with Calabi-Yau geometry

[Duhr, Klemm Loebbert, Nega, Porkert 22]

Hope to bootstrap new integrals

OUTLOOK

Finally we have SoV basis and measure for higher-rank spin chain

Longstanding problem solved

- Expect many applications: super case [Gromov, **FLM** 18 ; Maillet, Niccoli, Vignoli 20], SO(N) [Ferrando, Frassek, Kazakov; Ekhamar, Shu, Volin 20]; long range [Ferrando, Lamers, **FLM**, Serban to appear] [Jules's talk] principal series rep for fishnet, Slavnov scalar products, TD limit, …

- Algebraic meaning of $\int Q_1 Q_2 Q_3$?

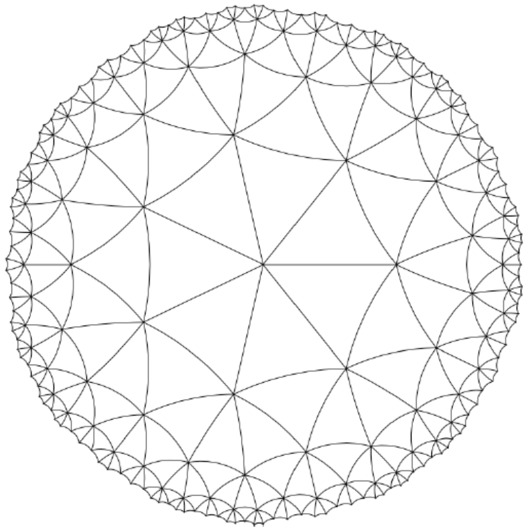
- SoV for Gaudin models and conformal blocks [cf Volker's talk]

- AdS/CFT: other correlators, beyond ladders/fishnets, … Many hints of hidden SoV structures! [Cavaglia, Gromov, **FLM** 18] [Giombi, Komatsu 18, 19] [Bercini, Homrich, Vieira 22]…

Thank you!

Matrix models and gravity

vertex $\rightarrow t_n$
face $\rightarrow t_n{}^*$



$$Z(t,t^*) = \lambda^{-\frac{N^2}{2}} \int \mathcal{D}M \; e^{-\frac{N}{2\lambda}\mathrm{Tr}M^2 \,+\, \sum_{k=1}^{\infty} \frac{1}{k}\mathrm{Tr}B^k \;\mathrm{Tr}(MA)^k}$$

[Kazakov, Staudacher, Wynter 96]

Matrix model = sum over graphs

→ Critical regime

2d gravity

Z_{disk} matches gravity prediction! [Kazakov, **FLM** 21]

With these couplings we control curvature, how to get AdS / JT gravity? [Witten '20] [Jackiw-Teitelboim] [Mirzakhani]

Another problem: forests on random graphs [Gorsky, Kazakov, **FLM**, Mishnyakov 22]

$$\sum_{\text{graphs } G} \lambda^{|G|} \det(L+m^2) = \sum_{F=(F_1\ldots F_l)\in G} \lambda^{|G|} m^{2l} \prod_{i=1}^{l} |F_i| = Z_{\text{matrix model}}$$

Gives massive fermions coupled to 2d gravity

Generalisation of
[Bondesan, Caracciolo, Saleur, Sportiello, ... 04, 09, 16]

Hope to better understand holography in 2d /1d /0d

Algebraic picture

Generating functional for transfer matrices in antisymmetric reps

$$W = (1-\Lambda_1(u)D^2)(1-\Lambda_2(u)D^2)\dots(1-\Lambda_N(u)D^2) = \sum_{k=1}^{N}(-1)^k\tau_k(u)D^k$$

Define left and right action $\overrightarrow{D}f(u) = f(u+i/2), \quad f\overleftarrow{D} = f(u-i/2)$

Then $Q_a\overleftarrow{W} = 0$ and $\overrightarrow{W}Q^a = 0$

Using that for any operator $\oint g\overrightarrow{O}f = \oint f\overleftarrow{O}g$ we get $\oint Q_a^A(\overrightarrow{W}_A - \overrightarrow{W}_B)Q_B^b = 0$

The two Baxter equations are 'conjugate' to each other! [Cavaglia, Gromov, FLM 19]

$$\hat{O} \circ Q_1 \equiv Q_\theta^{++} Q_1^{[+3]} - \tau_1^+ Q_1^+ + \tau_2^- Q_1^- - Q_\theta^{--} Q_1^{[-3]} = 0$$
$$\hat{\bar{O}} \circ Q_{\bar{a}} \equiv Q_\theta^- Q_{\bar{a}}^{[-3]} - \tau_1 Q_{\bar{a}}^- + \tau_2 Q_{\bar{a}}^+ - Q_\theta^+ Q_{\bar{a}}^{[+3]} = 0$$

$$\langle g\, f \rangle_j \equiv \int_{-\infty}^{\infty} \mu_j(x) g(x)\, f(x)$$

Analog of self-adjointness property: $\langle Q_1\, \hat{\bar{O}} \circ f \rangle_j = 0$

$$\mu_j(u) = \frac{1}{1 + e^{2\pi(u-\theta_j)}}$$

$$\begin{aligned}\langle\; g\, \hat{\bar{O}} \circ f \rangle_j \;&= \int_{-\infty}^{+\infty} \mu_j(u) g(u) \left[Q_\theta^- f^{[-3]} - \tau_1 f^- + \tau_2 f^+ - Q_\theta^+ f^{[+3]} \right] du \\ &= \int_{-\infty+i0}^{+\infty+i0} \mu_j(u + \tfrac{i}{2}) \left[\underbrace{Q_\theta^{++} g^{[+3]} - \tau_1^+ g^+ + \tau_2^- g^- - Q_\theta^{--} g^{[-3]}}_{\hat{O} \circ g} \right] f(u)\, du \\ &\quad + \text{residues from poles}\,,\end{aligned}$$

Poles cancel if $g \equiv Q_1$! Use nontrivial relations between T's and Q's