How to deal with non-linear integral equations with singular kernels?

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- continuously varying scaling dimensions

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- derivation of asymptotical behaviour of energies by use of the singular kernel
- The $3\bar{3}$-network model, $sl(2|1)$ supersymmetric

Work in collaboration with Mouhcine Azhari
within DFG research group 2316 “Correlations in Integrable Quantum Many-Body Systems”

The problem

Hamiltonian derived from integrable staggered 6VM, see Sascha Gehrmann's talk, essence:

$$H=\sum_{j=1}^{2L}\left[-\frac{1}{2}\vec{\sigma}_j\vec{\sigma}_{j+2}+\sin^2\gamma\sigma_j^z\sigma_{j+1}^z-\frac{\mathrm{i}}{2}\left(\sigma_{j-1}^z-\sigma_{j+2}^z\right)\left(\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y\right)\right]$$

Jacobsen, Saleur 2006, Ikhlef, Jacobsen, Saleur 2008+12 (non-compact continuum limit, log-corrections)
Frahm, Martins 2011+12 (density functions, numerical solns.)
Candu, Ikhlef 2013, Frahm, Seel 2013 (non-linear integral eqs.)

Conformal spectrum

$$E(L)=Le_0+\frac{2\pi}{L}v_F\left(-\frac{1}{6}+\frac{\gamma}{2\pi}m^2+\frac{\pi}{2\gamma}w^2+\frac{2\gamma}{\pi-2\gamma}s^2\quad+\text{integers}\right),\qquad v_F=\sin(2\gamma)\frac{\pi}{\pi-2\gamma}$$

with "usual" integers m (magnetization), w (momentum) and "continuously" growing s for reallocating n BA-roots from one line to the other (see later).

$$s\simeq\frac{\pi n}{2\log L}\quad \text{large } L,\ n=0,1,2,\ldots\qquad\qquad \text{(Wiener-Hopf technique by IJS 12)}$$

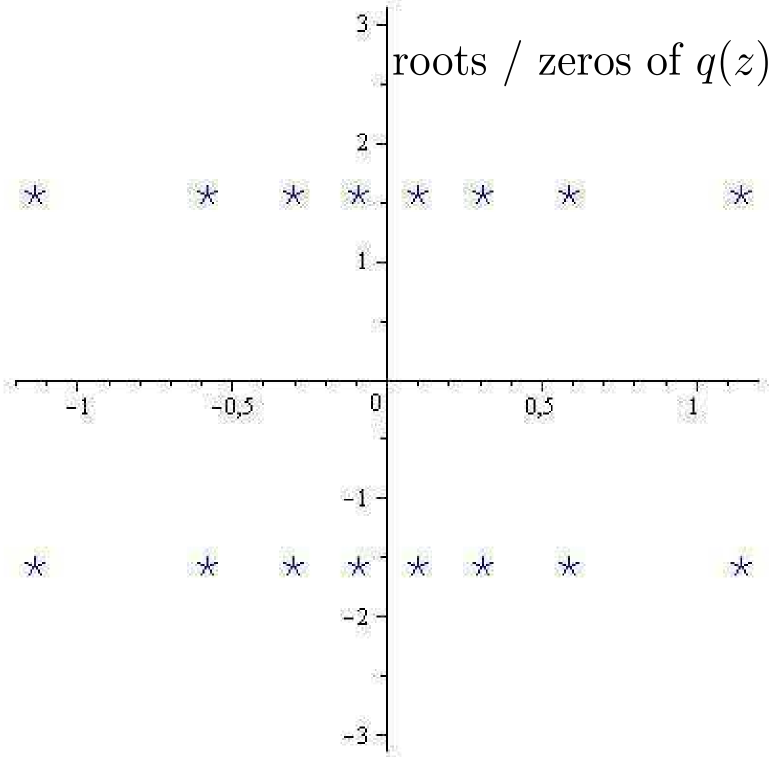
Phantastically accurate quantization condition for s valid even for quite finite systems by Bazhanov, Kotousov, Koval, Lukyanov 2019, 20, 21, $\Rightarrow SL(2,\mathbb{R})/U(1)$ NLSM on Euclidean $\to$ Lorentzian black hole.

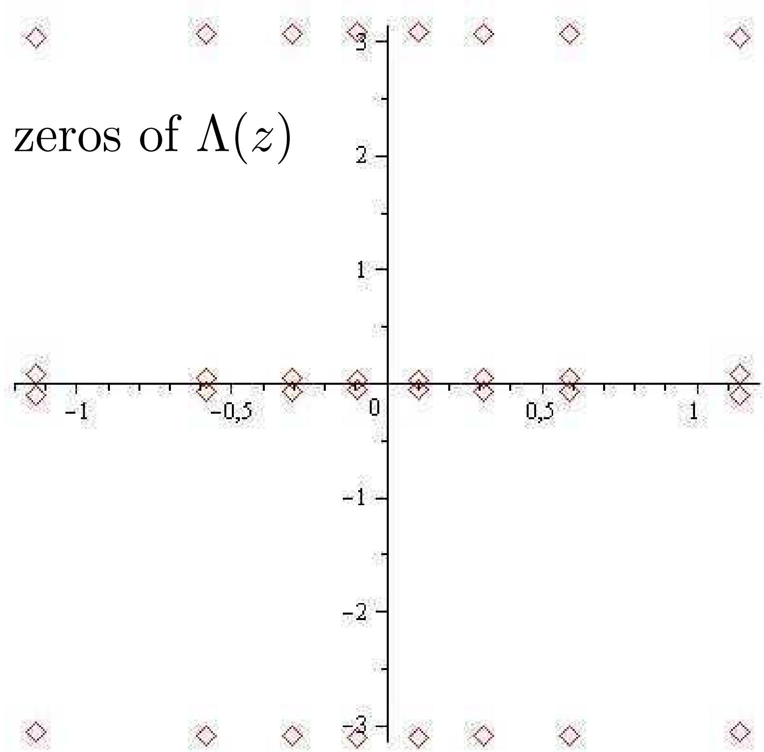
Bethe ansatz equations / TQ relations

$$\Lambda(z) = \Phi(z - \mathrm{i}\gamma)\frac{q(z+2\mathrm{i}\gamma)}{q(z)} + \Phi(z+\mathrm{i}\gamma)\frac{q(z-2\mathrm{i}\gamma)}{q(z)}$$

$$\Phi(z) = \sinh^L z, \qquad q(z) = \prod_j \sinh\tfrac{1}{2}(z - z_j)$$

Parameterization with $2\pi\mathrm{i}$-periodicity, 2 independent analyticity regions:





The "auxiliary functions" ... destined to satisfy integral equations

Function related to counting function

$$a(z) := \frac{\Phi(z+i\gamma)q(z-2i\gamma)}{\Phi(z-i\gamma)q(z+2i\gamma)}, \qquad \text{BA eqns} \quad a(z_j) = -1$$

We use this function **off** the distribution lines like in

AK, Batchelor 90; AK, Batchelor, Pearce 91; AK 92; Destri, de Vega 92, 95; J. Suzuki 98

"It is convenient to consider":

$$a_1(x) := \frac{1}{a(x+i\gamma)} = \frac{\Phi(x)}{\Phi(x+2i\gamma)} \frac{q(x+3i\gamma)}{q(x-i\gamma)}$$

$$a_2(x) := a(x+i\pi-i\gamma) = \frac{\Phi(x)}{\Phi(x-2i\gamma)} \frac{q(x+i\pi-3i\gamma)}{q(x+i\pi+i\gamma)}$$

$$a_3(x) := a(x-i\gamma) = \frac{\Phi(x)}{\Phi(x-2i\gamma)} \frac{q(x-3i\gamma)}{q(x+i\gamma)}$$

$$a_4(x) := \frac{1}{a(x+i\gamma)} = \frac{\Phi(x)}{\Phi(x+2i\gamma)} \frac{q(x+i\pi+3i\gamma)}{q(x+i\pi-i\gamma)}$$

with x on or close to the real axis. TBA-correspondence: $a_i \equiv e^{\varepsilon_i/T}, \; \frac{\varepsilon_i}{T} = \frac{e_i}{T} + K * \log\left(1 + e^{\varepsilon_i/T}\right)$

The associated “auxiliary functions”

The analogues of the $1+\mathrm{e}^{\varepsilon_i/T}$ functions and their factorizations

$$A_1(x) := 1+a_1(x) = \frac{1}{\Phi(x+2\mathrm{i}\gamma)}\,\frac{q(x+\mathrm{i}\gamma)}{q(x-\mathrm{i}\gamma)}\Lambda(x+\mathrm{i}\gamma)$$

$$A_2(x) := 1+a_2(x) = \frac{1}{\Phi(x-2\mathrm{i}\gamma)}\,\frac{q(x+\mathrm{i}\pi-\mathrm{i}\gamma)}{q(x+\mathrm{i}\pi+\mathrm{i}\gamma)}\Lambda(x+\mathrm{i}\pi-\mathrm{i}\gamma)$$

$$A_3(x) := 1+a_3(x) = \frac{1}{\Phi(x-2\mathrm{i}\gamma)}\,\frac{q(x-\mathrm{i}\gamma)}{q(x+\mathrm{i}\gamma)}\Lambda(x-\mathrm{i}\gamma)$$

$$A_4(x) := 1+a_4(x) = \frac{1}{\Phi(x+2\mathrm{i}\gamma)}\,\frac{q(x+\mathrm{i}\pi+\mathrm{i}\gamma)}{q(x+\mathrm{i}\pi-\mathrm{i}\gamma)}\Lambda(x+\mathrm{i}\pi+\mathrm{i}\gamma)$$

Analyzing multiplicative functional equations by Fourier transform

What do we do with this? Definitions/equations like

$$f(x) = g(x + \mathrm{i}\alpha)/h(x + \mathrm{i}\beta)$$

after log-derivative and Fourier transform turn into

$$\mathrm{FT}\left[(\log f)'\right]_k = \mathrm{e}^{-\alpha k}\,\mathrm{FT}\left[(\log g)'\right]_k + \mathrm{e}^{-\beta k}\,\mathrm{FT}\left[(\log h)'\right]_k$$

Observation: We have 4 such equations "$A_i(x) = \ldots$" where q and Λ have two regions of analyticity in the complex plane. Therefore we have 2 different Fourier transforms for each.

The 4 linear equations for the $2+2$ Fourier transforms of $\log q$ and $\log\Lambda$ can be solved uniquely in terms of $\log A_1$,..., $\log A_4$.

The solution is inserted into the definitions of a_i and yields

The non-linear integral equations, version I – singular kernel

$$\begin{pmatrix} \log a_1 \\ \log a_2 \\ \log a_3 \\ \log a_4 \end{pmatrix} = d + K * \begin{pmatrix} \log A_1 \\ \log A_2 \\ \log A_3 \\ \log A_4 \end{pmatrix}, \qquad d(x) = L \log \mathrm{th}(\tfrac{1}{2} g x) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad g := \frac{\pi}{\pi - 2\gamma}$$

The kernel in Fourier transform notation

$$K = \begin{pmatrix} \sigma_1 & \sigma_2 \\ \sigma_2^\dagger & \sigma_1^T \end{pmatrix} \qquad (\dagger \text{ interchanges diagonal elements})$$

$$\sigma_1 = \frac{\cosh((\pi - 3\gamma)k)}{2\sinh(\gamma k)\sinh((\pi - 2\gamma)k)} \begin{pmatrix} -1 & e^{(\pi-2\gamma)k} \\ e^{(2\gamma-\pi)k} & -1 \end{pmatrix}$$

$$\sigma_2 = \frac{\cosh(\gamma k)}{2\sinh(\gamma k)\sinh((\pi - 2\gamma)k)} \begin{pmatrix} -e^{(\pi-2\gamma)k} & 1 \\ 1 & -e^{(2\gamma-\pi)k} \end{pmatrix}$$

which is highly singular: in real space with asymptotics $K_{i,j}(x) \simeq |x|$.

The eigenvalue

For x (slightly below the real axis) the eigenvalue splits into purely bulk and finite size part

$$\log[\Lambda(x-\mathrm{i}\gamma)\Lambda(x+\mathrm{i}(\pi-\gamma))] = L\cdot\lambda(x) + \kappa * [\log A_1 + \log A_2 + \log A_3 + \log A_4]$$

$$\kappa(x) = -\mathrm{i}\frac{g}{\sinh(gx)}, \quad g = \frac{\pi}{\pi-2\gamma}$$

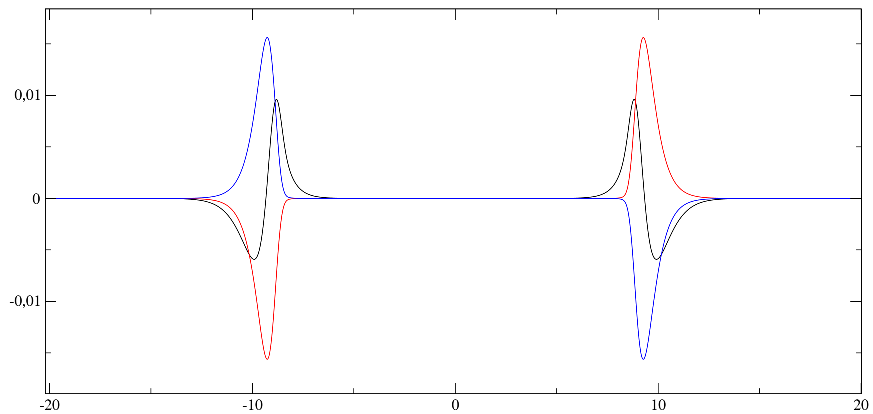
Energy expression from derivative at $x=0$

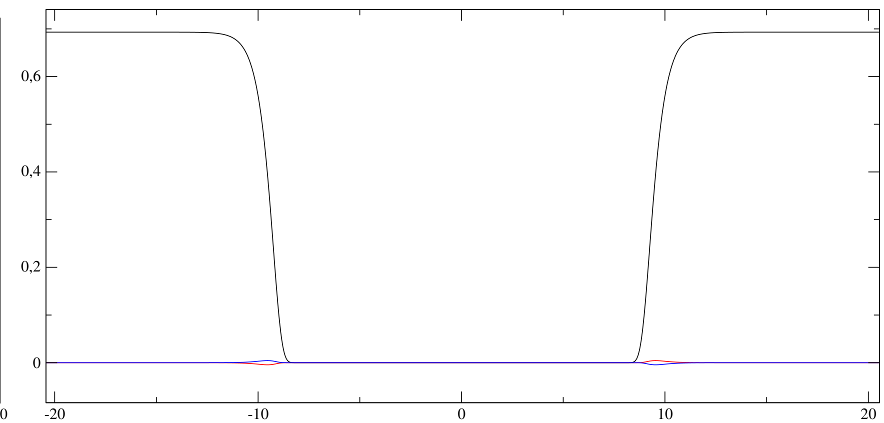
$$\begin{aligned} E &= \sin(2\gamma)\frac{d}{dx}\log[\Lambda(x-\mathrm{i}\gamma)\Lambda(x+\mathrm{i}(\pi-\gamma))] \\ &= Le_0 - \sin(2\gamma)\int_{-\infty}^{\infty} dx\,\frac{g^2\cosh gx}{(\sinh gx)^2}\,[\log A_1(x) + \log A_2(x) + \log A_3(x) + \log A_4(x)] \end{aligned}$$

where $g = \frac{\pi}{\pi-2\gamma}$.

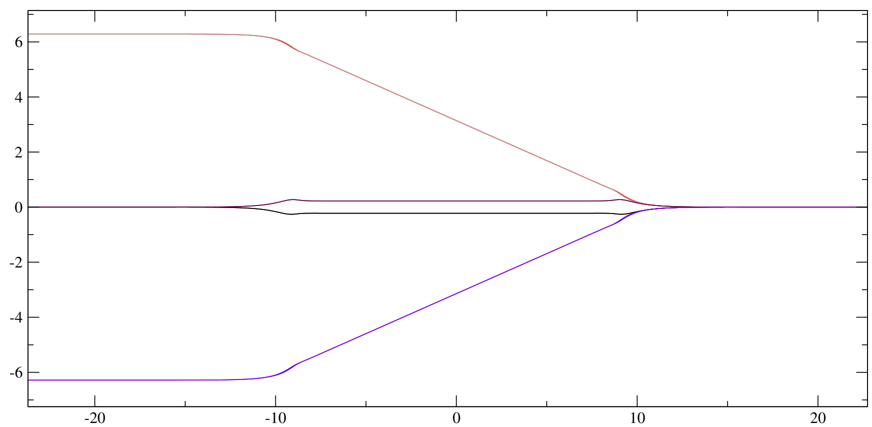
The ground-state solution and 1st excited state

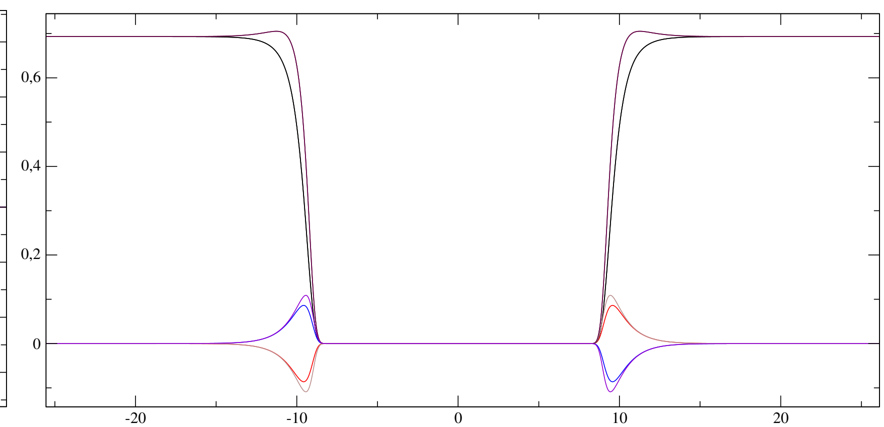
True ground state solution: $\log a_i - d$ and $\log A_i = \log(1 + a_i)$ for $L = 10^9$





Reallocating 1 BA root from one line to the other ($n = \pm 1$) has solution



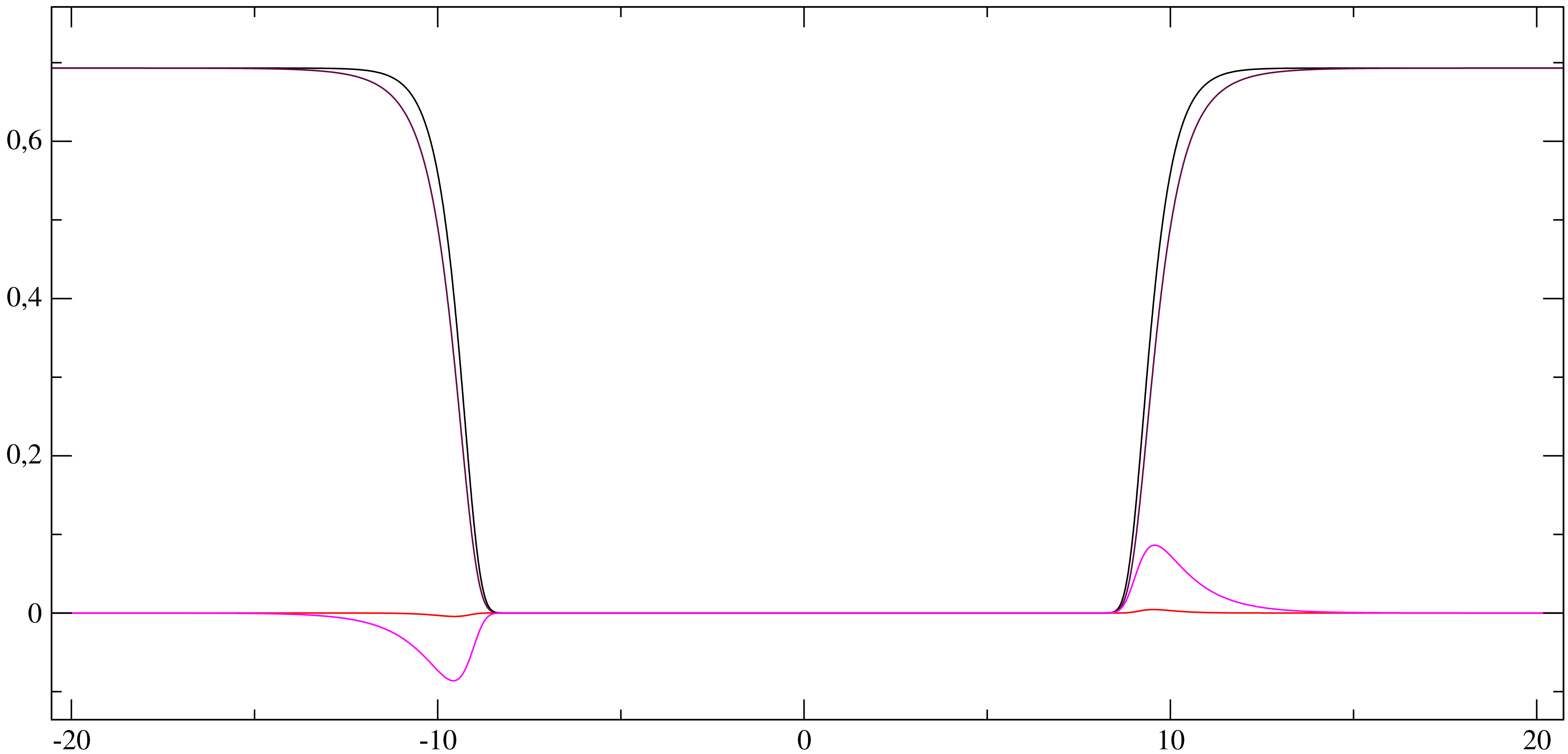


with huge changes in the $\log a_i$ functions, but only little in the $\log A_i$.

Why has the kernel to be singular and what are the alternatives I

...with huge changes in the $\log a_i$ functions, but only little in the $\log A_i$.

$\log A_1$ for ground state and excited state



Why has the kernel to be singular and what are the alternatives II

Claim / Theorem: All of us use the “same” functions and equivalent equations!

Candu, Ikhlef 2013: solve up to $L = 10^2$ (?)

use same functions on possibly slightly shifted contours, work with the singular kernel.

Frahm, Seel 2013: solve up to $L = 10^6$ (?)

use “practically” same functions, two replaced in the way $\tilde{a}_i = 1/a_j$, then

$$\log a_i = -\log \tilde{a}_i, \qquad \log A_i = \log(1 + a_i) = \log(1 + 1/\tilde{a}_i) = \log \tilde{A}_i - \log \tilde{a}_i$$

Difference in way of organizing of what is on the left and what is on the right hand side.

The optimal arrangement of the NLIE, version II – regular kernel

Super-great manipulation

$$a = d + K * A$$

$$2(a-d) = K * 2A = K * (2A - (a-d)) + K * (a-d)$$

$$(2-K) * (a-d) = K * (2A - (a-d))$$

$$a = d + K_r * (a - d - 2A) \qquad \text{with} \quad K_r := \frac{K}{K-2}$$

This kernel is regular! In Fourier transform notation

$$K_r = \begin{pmatrix} \kappa_1 & \kappa_2 \\ \kappa_2^\dagger & \kappa_1^T \end{pmatrix} \qquad \text{(}\dagger\text{ interchanges diagonal elements)}$$

$$\kappa_1 = \frac{\sinh((\pi-2\gamma)k)}{2\sinh(\pi k)} \begin{pmatrix} 1 & -e^{(\pi-2\gamma)k} \\ -e^{(2\gamma-\pi)k} & 1 \end{pmatrix}, \quad \kappa_2 = \frac{\sinh(2\gamma k)}{2\sinh(\pi k)} \begin{pmatrix} e^{(\pi-2\gamma)k} & -1 \\ -1 & e^{(2\gamma-\pi)k} \end{pmatrix}$$

Regular kernel K_r has one eigenvalue $+1$ for “momentum” $k=0$ with eigenstate $(1,-1,1,-1)$, and two eigenvalues 0 and one eigenvalue close to 0.

How to select the states?

Shifting n BA roots from one line to the other yields a winding of the $\log a_i(x)$ functions: $\log a_i(\infty) - \log a_i(-\infty) = \pm n 2\pi \mathrm{i}$. We use this winding number n instead of the quasi-momentum. Modifications for numerics necessary

$$a = d + K_r * (a - d - 2A) = d + nw + K_r * (a - d - n\tilde{w} - 2A)$$

where $n = 0, 1, 2...$ is the winding number and

$$w(x) = \begin{pmatrix} w_1(x) \\ w_2(x) \\ w_3(x) \\ w_4(x) \end{pmatrix}, \quad \tilde{w}(x) = 2\log \mathrm{th}\left(\frac{g}{2}x + \mathrm{i}\frac{\pi}{4}\right) \cdot \begin{pmatrix} +1 \\ -1 \\ +1 \\ -1 \end{pmatrix}$$

$$w_1(x) = -w_4(x) := \log \mathrm{th}\tfrac{1}{2}\left(x + \mathrm{i}\left(\frac{\pi}{2} - \gamma\right)\right) + \log \mathrm{th}\tfrac{1}{2}\left(x + \mathrm{i}\left(3\gamma - \frac{\pi}{2}\right)\right)$$
$$w_2(x) = -w_3(x) := \log \mathrm{th}\tfrac{1}{2}\left(x - \mathrm{i}\left(\frac{\pi}{2} - \gamma\right)\right) + \log \mathrm{th}\tfrac{1}{2}\left(x - \mathrm{i}\left(3\gamma - \frac{\pi}{2}\right)\right)$$

Functional equations: Definition of auxiliary functions

Energy expression from derivative at $x=0$

$$E-Le_0 = -\sin(2\gamma)\int_{-\infty}^{\infty} dx\,\frac{g^2\cosh gx}{(\sinh gx)^2}\left[\log A_1(x)+\log A_2(x)+\log A_3(x)+\log A_4(x)\right]$$
$$= \frac{2\pi}{L}v_F\left[-\frac{1}{6}+\frac{2\gamma}{\pi-2\gamma}s^2\right], \qquad \text{where } g=\frac{\pi}{\pi-2\gamma}.$$

Results for $L=2,10,10^2,10^3,10^6,...,10^{15}$ and $N=2^{14}=16384$ ($N=2^{15}=32768$) grid points.
Computation time 40 s (80 s) for 1000 iterations (Intel i7 2.4 GHz), 16 decimals.

Comparison with Bazhanov, Kotousov, Koval, Lukyanov 2019 (ODE/IQFT correspondence)

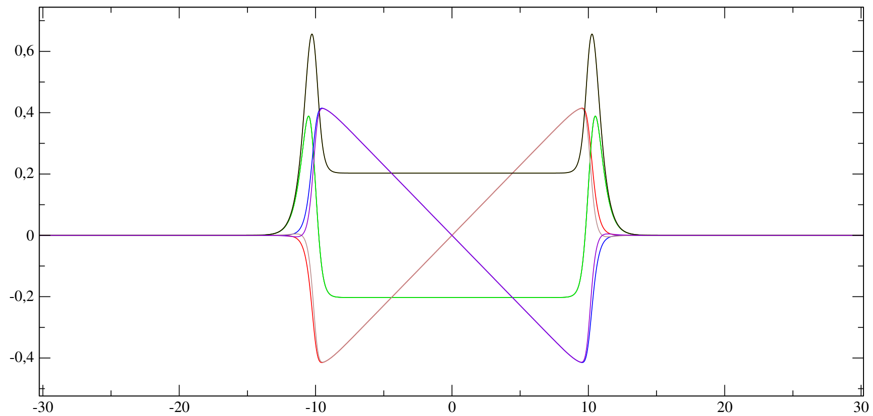
$$4\,s\log\left(\frac{L\Gamma\left(3/2+\frac{\gamma}{\pi-2\gamma}\right)}{\sqrt{\pi}\Gamma\left(1+\frac{\gamma}{\pi-2\gamma}\right)}\right)+8\,s\frac{\pi-\gamma}{\gamma}\log(2)-2\mathrm{i}\log\left(\frac{\Gamma(1/2-\mathrm{i}s)}{\Gamma(1/2+\mathrm{i}s)}\right)=n\,2\pi$$

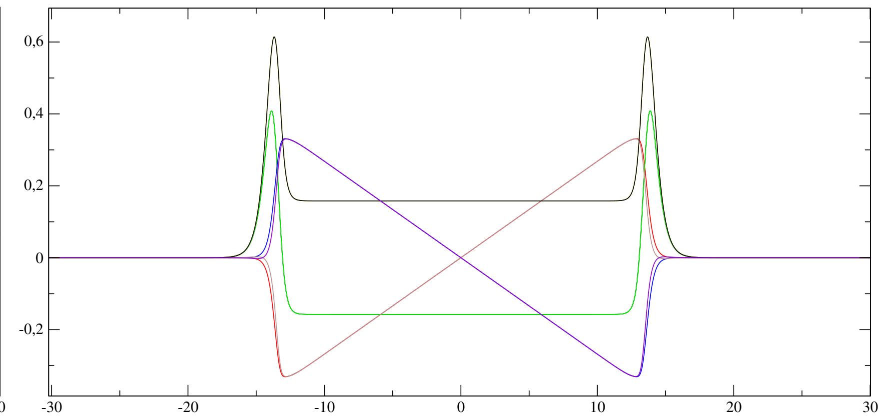
Results for $n=1$, $\gamma=0.8$: shown is square bracket above $[\ldots]$ $+1/6=\frac{2\gamma}{\pi-2\gamma}s^2$

L	2	10	10^2	10^3	10^6	10^9	10^{12}	10^{15}
NLIE	0.2533...	**0.078**2...	**0.0387**05...	**0.023349**53...	**0.00844098108**3...	**0.004323850**8...	**0.002622**512...	**0.001**67...
BKKL20	0.9002...	**0.077**5...	**0.0387**10...	**0.023349**63...	**0.00844098108**2...	**0.004323850**9...	**0.002622**535...	**0.001**75...

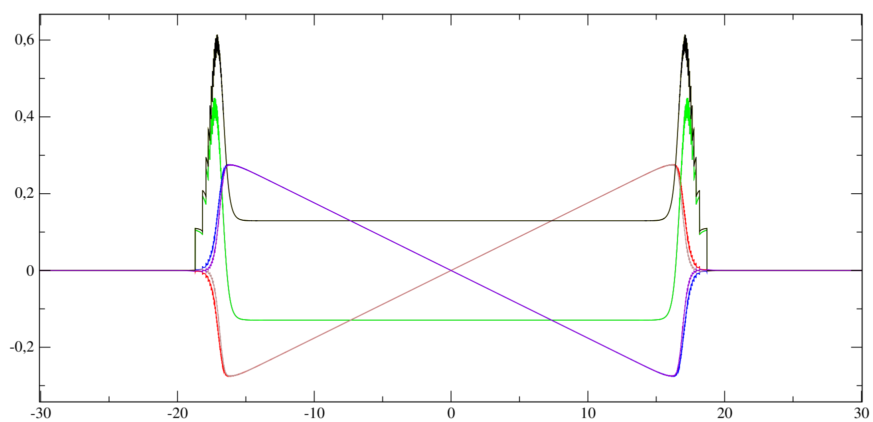
What limits the accuracy?

To solve $\quad a = d + nw + K_r * (a - d - n\tilde{w} - 2A) \quad$ where terms in brackets for $L = 10^9, 10^{12}$ look like

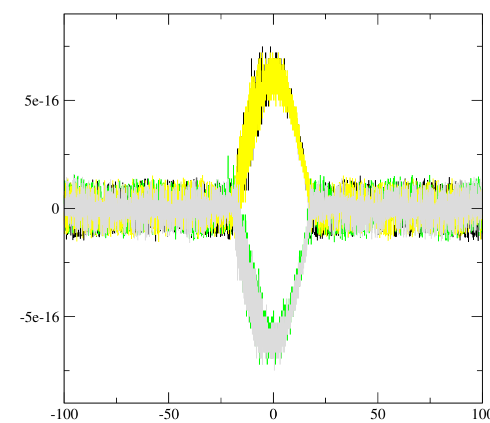




Wiggles appear for $L = 10^{15}$.



Yet the equations are solved: LHS-RHS=0



Analytical derivation of correction terms from NLIE version I

We use the NLIE with singular (!) kernel and differentiate it once

$$(\log a_i)' = d' + \sum_{j=1}^{4} K'_{ij} * \log(1+a_j)$$

then we multiply from left and...

$$\int_0^\infty dx \sum_{i=1}^{4} \log(1+a_i(x))(\log a_i(x))' =$$

$$\int_0^\infty dx \sum_{i=1}^{4} \log(1+a_i(x))d'(x) + \int_0^\infty dx \int_{-\infty}^\infty dy \sum_{i,j=1}^{4} \log(1+a_i(x))K'_{ij}(x-y)\log(1+a_j(y))$$

LHS: change of variable gives dilogarithmic integral, only data $a_i(0) =, a_i(\infty) = 1$ enter $\to \pi^2/3$.

RHS: 1st term is the wanted object, 2nd term – double integral – can be massaged

$$\int_0^\infty dx \int_{-\infty}^\infty dy \ldots = \underbrace{\int_0^\infty dx \int_0^\infty dy \ldots}_{=0} + \int_0^\infty dx \int_{-\infty}^0 dy \ldots$$

the first term is zero by antisymmetry of the kernel, $K'_{ij}(x-y) = -K'_{ji}(y-x)$.

Analytical derivation of correction terms from NLIE version I

In the second term the kernel K is linear and K'_{ij} can be replaced by constants

$$\ldots = -\frac{1}{4\gamma(\pi-2\gamma)}\int_0^\infty dx \int_{-\infty}^0 dy \sum_{i,j=1}^4 (-1)^{i+j}\log(1+a_i(x))\log(1+a_j(y)) = \frac{|I|^2}{4\gamma(\pi-2\gamma)}$$

where

$$I := \int_0^\infty dx \log\frac{(1+a_1(x))(1+a_3(x))}{(1+a_2(x))(1+a_4(x))}$$

such an integral from $-\infty$ to 0 gives $-I$ (and is purely imaginary).

What is I? From the NLIE we derive

$$n\,2\pi\mathrm{i} = \log a_1(+\infty) - \log a_1(-\infty) = \frac{1}{4\gamma(\pi-2\gamma)}\,\frac{2\log L}{g}\cdot I$$

Now we have for the double integral

$$\ldots = 2\pi^2\frac{2\gamma}{\pi-2\gamma}\left(\frac{\pi n}{2\log L}\right)^2$$

without having solved the NLIE or having applied Wiener-Hopf techniques.

The supersymmetric $sl(2|1)$ supersymmetric $3\bar{3}$ model

Derivation of staggered vertex model and proof of integrability by R. Gade (1998)
extensive investigations of spectrum by Essler, Frahm, Saleur (2005)

Bethe ansatz equations as for the QTM of the supersymmetric tJ model

$$\frac{\Phi_-(u_j+\mathrm{i})}{\Phi_-(u_j-\mathrm{i})} = -\mathrm{e}^{\mathrm{i}\varphi}\frac{q_\gamma(u_j+\mathrm{i})}{q_\gamma(u_j-\mathrm{i})}, \qquad j=1,\ldots,N$$

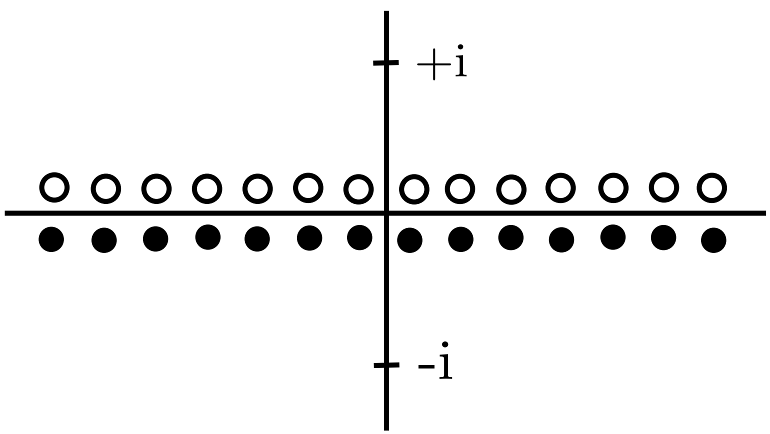
$$\frac{\Phi_+(\gamma_\alpha+\mathrm{i})}{\Phi_+(\gamma_\alpha-\mathrm{i})} = -\mathrm{e}^{\mathrm{i}\varphi}\frac{q_u(\gamma_\alpha+\mathrm{i})}{q_u(\gamma_\alpha-\mathrm{i})}, \qquad \alpha=1,\ldots,M$$

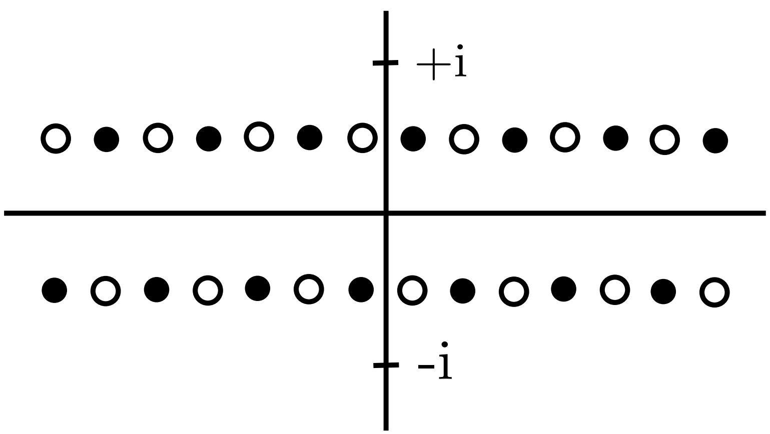
These equations are the same for the QTM of the tJ model and for the supersymmetric network model.

Characterization of largest eigenvalue differs:

tJ: maximum value of Λ

network model: maximum value(s) of $\Lambda\cdot\bar{\Lambda}$





“strange strings” (Essler, Frahm, Saleur 2005)

Compact notation for NLIEs: network model (version I)

Supersymmetric network model: 6 non-linear integral equations, version I

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} d \\ d \end{pmatrix} + \begin{pmatrix} K - K_s & K_s \\ K_s & K - K_s \end{pmatrix} * \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

where a_1 and a_2 are two copies of the 3d vector a, and A_1 and A_2 are two copies of the 3d vector A.
Driving terms

$$d := \begin{pmatrix} L \log \mathrm{th} \frac{\pi}{2} x - \mathrm{i}\varphi/2 \\ L \log \mathrm{th} \frac{\pi}{2} x + \mathrm{i}\varphi/2 \\ 0 \end{pmatrix},$$

and kernel matrices (in Fourier representation)

$$K(k) = \frac{1}{2\cosh k/2} \begin{pmatrix} \mathrm{e}^{-|k|/2} & -\mathrm{e}^{-|k|/2-k} & 1 \\ -\mathrm{e}^{-|k|/2+k} & \mathrm{e}^{-|k|/2} & 1 \\ 1 & 1 & 0 \end{pmatrix}, \; K_s(k) = \begin{pmatrix} \frac{1}{2\sinh|k|} & -\frac{\mathrm{e}^{-k}}{2\sinh|k|} & -\frac{\mathrm{e}^{-k/2}}{2\sinh(k)} \\ -\frac{\mathrm{e}^{k}}{2\sinh|k|} & \frac{1}{2\sinh|k|} & \frac{\mathrm{e}^{k/2}}{2\sinh(k)} \\ \frac{\mathrm{e}^{k/2}}{2\sinh(k)} & -\frac{\mathrm{e}^{-k/2}}{2\sinh(k)} & 0 \end{pmatrix}$$

Good properties: symmetry $K(-k)^T = K(k)$, $K_s(-k)^T = K_s(k)$ may allow for analytic calculations of CFT
bad properties: K_s is very singular! Kernel of integral equations not integrable!

NLIEs version II: regular kernels

Most compact notation of NLIE as two weakly coupled 3×3 systems

$$a_i = d \pm \tilde{d} + K * A_i, \qquad i = 1,2 \quad \text{for which } +,- \text{ applies}$$

and additional driving term

$$\tilde{d} := \frac{1}{2}(\tilde{K} - K) * (A_1 - A_2) - \frac{1}{2}\tilde{K} * (a_1 - a_2)$$

Regular kernels

$$K(k) = \frac{1}{2\cosh k/2} \begin{pmatrix} \mathrm{e}^{-|k|/2} & -\mathrm{e}^{-|k|/2-k} & 1 \\ -\mathrm{e}^{-|k|/2+k} & \mathrm{e}^{-|k|/2} & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad K(k) = K^T(-k)$$

$$\tilde{K}(k>0) = \begin{pmatrix} -\frac{1}{\mathrm{e}^k+1} & \mathrm{e}^{-k} - \mathrm{e}^{-2k} + \frac{\mathrm{e}^{-k}}{\mathrm{e}^k+1} & \mathrm{e}^{-k/2} - \mathrm{e}^{-3k/2} \\ \frac{\mathrm{e}^k}{\mathrm{e}^k+1} & -\frac{1}{\mathrm{e}^k+1} & 0 \\ 0 & \mathrm{e}^{-k/2} - \mathrm{e}^{-3k/2} & -\mathrm{e}^{-k} \end{pmatrix}, \qquad \tilde{K}(k<0) := \tilde{K}^T(-k)$$

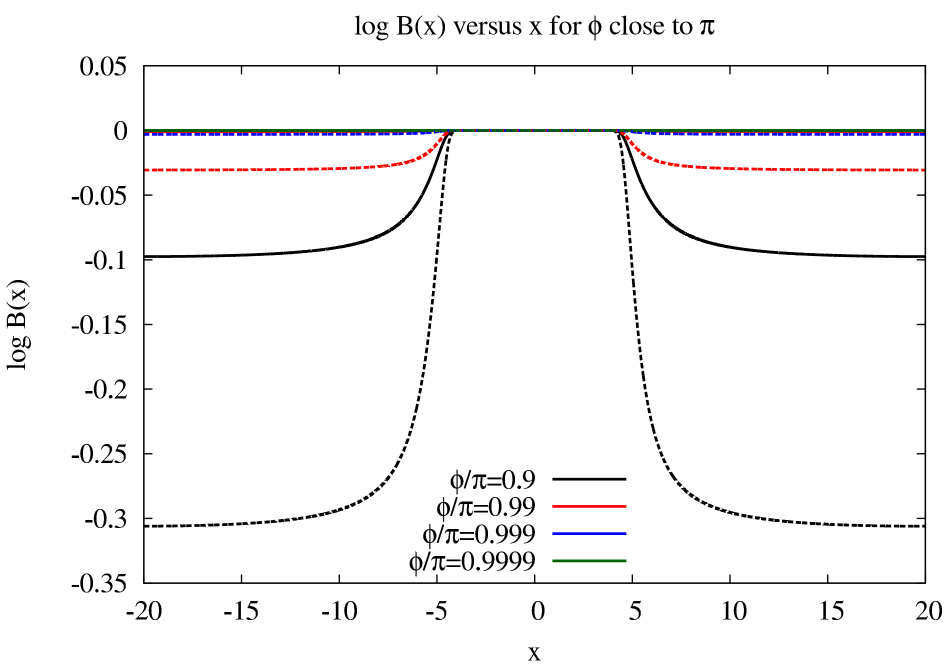
Numerical solution to NLIE: ground-state

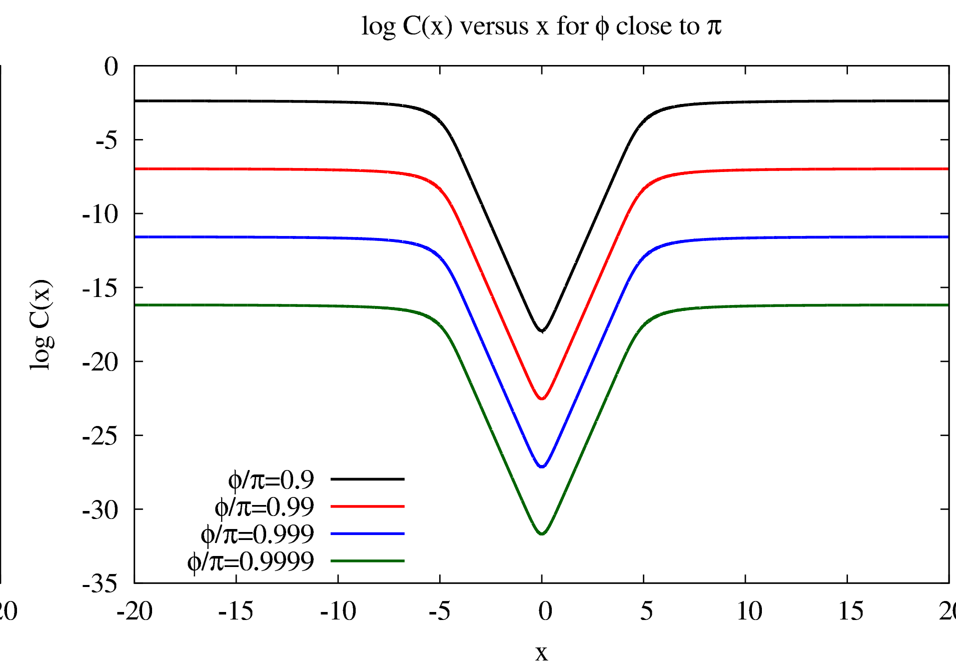
Ground state of model with $\varphi = \pi$ completely degenerate, but not for $\varphi \neq \pi$.

$$a_j := \begin{pmatrix} \log b_j \\ \log \bar{b}_j \\ \log c_j \end{pmatrix}, \qquad A_j := \begin{pmatrix} \log B_j \\ \log \bar{B}_j \\ \log C_j \end{pmatrix}$$

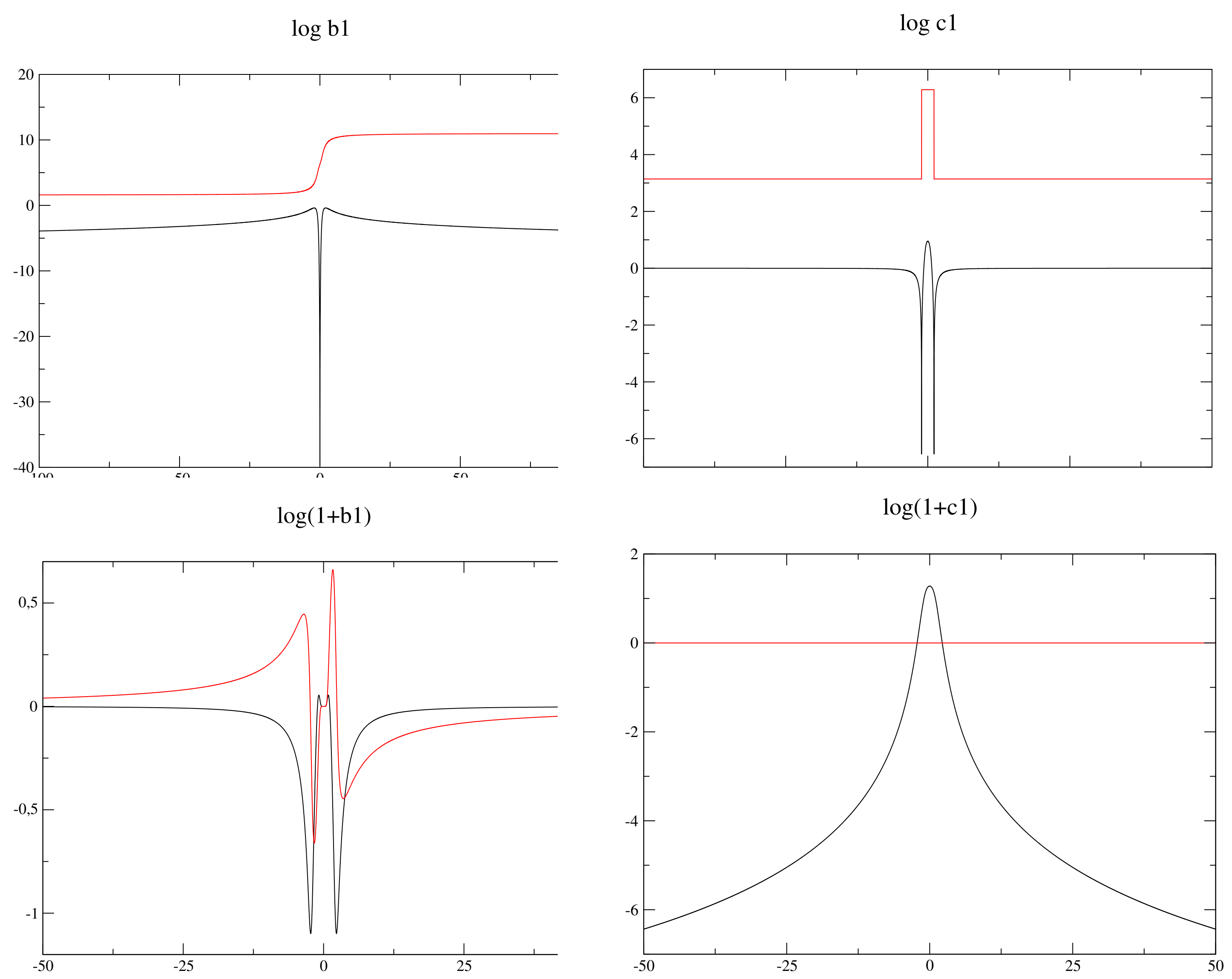
For $\varphi = \pi$ we know $\quad b_j = \bar{b}_j = 0,\ B_j = \bar{B}_j = 1, \quad c_j = -1, C_j = 0.$

For $\varphi \neq \pi$ with $\tilde{d} = 0$ we find numerically ($L = 10^6$)





Numerical solution to NLIE: excited states, $\varphi = \pi$



Summary

Results:

- Quick derivation of NLIEs
- Understanding of all published NLIE equations from one "master set" of NLIE
- Transformation of the singular form into a regular version
- Numerics by use of regular NLIE up to L^{15}
- Asymptotics analytically derived from singular version of NLIE
- Some results for the $3\bar{3}$ model with $sl(2|1)$ symmetry: finite size correction $O(1/\log L)$

To do:

- increase accuracy for numerics, go to $L > 10^{15}$
- treat the $3\bar{3}$ model to same level of understanding