Quantum local charges in chiral affine Gaudin models

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Based on [1804.01480, 1804.06751] with B. Vicedo and C.A.S. Young and [2204.06554] with G. Kotousov and J. Teschner
Introduction
**Introduction: affine Gaudin models**

- **Affine Gaudin models (AGM):** physical systems associated with affine Lie algebras [Feigin Frenkel ’07]
- Field theory built from Kac-Moody currents $\mathcal{J}^{(1)}(x), \ldots, \mathcal{J}^{(N)}(x)$
- Classically integrable: infinite family of (local and non-local) Poisson-commuting charges built from $\mathcal{J}^{(r)}(x)$
- **Quantum integrability still conjectural**
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Natural starting point for quantisation: chiral AGM

All $\mathcal{J}^{(r)}(x)$ are left/right-moving fields of a 2d CFT → quantised as a current VOA (vertex operator algebra)

This talk: first results and conjectures on the construction and diagonalisation of quantum local charges in this VOA (Integrable structure in CFT [Bazhanov Lukyanov Zamolodchikov '94, ...])
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All $\mathcal{J}^{(r)}(x)$ are left/right-moving fields of a 2d CFT → quantised as a current VOA (vertex operator algebra)

This talk: first results and conjectures on the construction and diagonalisation of quantum local charges in this VOA (Integrable structure in CFT [Bazhanov Lukyanov Zamolodchikov ’94, ...])

Applications to integrable sigma-models (see talk by J. Teschner)
Contents

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Classical affine Gaudin models
Classical affine Gaudin models

Defining data:
- Simple complex Lie algebra $\mathfrak{g}$
- Parameters: punctures $z_1, \ldots, z_N \in \mathbb{C}$ and levels $\ell_1, \ldots, \ell_N \in \mathbb{C}^*$

Lie algebra conventions: basis $\{t_a\}$
- Structure constants $f_{ab}^c$, with $[t_a, t_b] = f_{ab}^c t_c$
- Minimal invariant bilinear form $\eta_{ab}$, with inverse $\eta^{ab}$
Classical affine Gaudin models

Defining data:
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Kac-Moody currents:
- $g$-valued fields $\mathcal{J}^{(r)}(x) = \mathcal{J}_a^{(r)}(x) t^a$ \hspace{1cm} ($r \in \{1, \ldots, N\}$ and $x \in [0, 2\pi]$)
- Kac-Moody Poisson bracket:

$$\{\mathcal{J}_a^{(r)}(x), \mathcal{J}_b^{(s)}(y)\} = \delta_{rs} \left( f_{ab}^c \mathcal{J}_c^{(r)}(x) \delta(x - y) - \ell_r \eta_{ab} \partial_x \delta(x - y) \right)$$
Gaudin Lax matrix and twist function: (z spectral parameter)

\[ \Gamma(z, x) = \sum_{r=1}^{N} \frac{\mathcal{J}(r)(x)}{z - z_r} \quad \text{and} \quad \varphi(z) = \sum_{r=1}^{N} \frac{\ell_r}{z - z_r} \]

- \( \varphi(z) \) contains all the parameters

- **Key formula**: Poisson bracket of the Gaudin Lax matrix

\[ \{ \Gamma_a(z, x), \Gamma_b(w, y) \} = - f_{abc} \frac{\Gamma_c(z, x) - \Gamma_c(w, x)}{z - w} \delta(x - y) + \eta_{ab} \frac{\varphi(z) - \varphi(w)}{z - w} \partial_x \delta(x - y) \]
Lax matrix and non-local charges

- **Lax matrix:**

\[ \mathcal{L}(z, x) = \frac{\Gamma(z, x)}{\varphi(z)} \]

- \( \{\mathcal{L}_a(z, x), \mathcal{L}_b(w, y)\} \) takes the form of a Maillet bracket [Maillet '85] with non-skew-symmetric \( \mathcal{R} \)-matrix

- **Non-local charges in involution:**

\[ \left\{ \text{Tr}_\rho(M(z)), \text{Tr}_{\rho'}(M(w)) \right\} = 0 \]

\[ M(z) = \text{PExp} \left( -\int_0^{2\pi} \mathcal{L}(z, x) \, dx \right) \]

Infinitely many charges by Taylor-expansion in \( z \)
Hierarchies of local charges

- Infinite hierarchies of local charges $Q^p_i$: [SL Magro Vicedo '17]
  - $i \in \{1, \ldots, N - 1\}$ associated to zeroes $\zeta_i$ of $\varphi(z)$
  - degrees $p + 1$: $p \in E$, with $E \subset \mathbb{Z}_{\geq 1}$ infinite (depending on $\mathfrak{g}$)
  - Poisson-commuting $\{Q^p_i, Q^q_j\} = 0$
  - not contained in monodromy charges but Poisson-commute with them
Hierarchies of local charges

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  - not contained in monodromy charges but Poisson-commute with them

- Zeroes of the twist function:

$$\varphi(z) = \sum_{r=1}^{N} \frac{\ell_r}{z - z_r} = K \frac{\prod_{i=1}^{N-1}(z - \zeta_i)}{\prod_{r=1}^{N}(z - z_r)}$$

Two equivalent sets of parameters: $(z_r, \ell_r)$ and $(K, z_r, \zeta_i)$
Construction of the local charges

- Definition of the local charges:
  \[ Q_i^p = \int_0^{2\pi} S_{p+1}(\zeta_i, x) \, dx, \quad S_{p+1}(z, x) = \tau_{a_1 \cdots a_{p+1}}^a \Gamma_{a_1}(z, x) \cdots \Gamma_{a_{p+1}}(z, x) \]

- \( \tau_{a_1 \cdots a_{p+1}}^a \): invariant symmetric \((p + 1)\)-tensor on \( g \)
Construction of the local charges

- **Definition of the local charges:**

  \[ Q^p_i = \int_0^{2\pi} S_{p+1}(\zeta, x) \, dx, \quad S_{p+1}(z, x) = \tau^{a_1 \cdots a_{p+1}} \Gamma_a(z, x) \cdots \Gamma_{a_{p+1}}(z, x) \]

- \( \tau^{a_1 \cdots a_{p+1}} \): invariant symmetric \((p + 1)\)-tensor on \( g \) such that

  \[ \{ S_{p+1}(z, x), S_{q+1}(w, y) \} = A_{p, q}^{(0)}(z, w ; y) \delta(x - y) + A_{p, q}^{(1)}(z, w ; y) \partial_x \delta(x - y) \]

  with \( A_{p, q}^{(0)}(z, w ; y) = \partial_y (\cdots) + \varphi(z)(\cdots) + \varphi(w)(\cdots) \)

- Ensures that \( \{ Q^p_i, Q^q_j \} = 0 \)

  \[ \{ \Gamma_a(z, x), \Gamma_b(w, y) \} = -f_{ab}^c \frac{\Gamma_c(z, x) - \Gamma_c(w, x)}{z - w} \delta(x - y) + \eta_{ab} \frac{\varphi(z) - \varphi(w)}{z - w} \partial_x \delta(x - y) \]
Invariant tensors

\[ Q_p^i = \int_0^{2\pi} S_{p+1}(\zeta, x) \, dx, \quad S_{p+1}(z, x) = \tau^{a_1 \cdots a_{p+1}}_p \Gamma_{a_1}(z, x) \cdots \Gamma_{a_{p+1}}(z, x) \]

- Invariant symmetric tensors \( \tau_p \) [Evans Hassan MacKay Mountain '99]
- \( p \) belongs to the set of affine exponents \( E \subset \mathbb{Z}_{\geq 1} \), depending on \( g \)
- Always start with \( p = 1 \) (quadratic tensor)

\[ \tau_{1}^{ab} = \eta^{ab} \]
Invariant tensors

\[ Q_i^p = \int_{0}^{2\pi} S_{p+1}(\zeta_i, x) \, dx , \quad S_{p+1}(z, x) = \tau_{p}^{a_1 \cdots a_{p+1}} \Gamma_{a_1}(z, x) \cdots \Gamma_{a_{p+1}}(z, x) \]

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- \( p \) belongs to the set of affine exponents \( E \subset \mathbb{Z}_{\geq 1} \), depending on \( g \)
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For \( g = sl(k) \), \( E = \{1, \ldots, k - 1, k + 1, \ldots, 2k - 1, 2k + 1, \ldots \} \)

\[ \tau_{1}^{ab} = \text{Tr}(t^a t^b) , \quad \tau_{2}^{abc} = \text{Tr}(t^a t^b t^c) \]

\[ \tau_{3}^{abcd} = \text{Tr}(t^a t^b t^c t^d) - \frac{3}{2k} \text{Tr}(t^a t^b) \text{Tr}(t^c t^d) , \quad \ldots \]
Other important property of $S_{p+1}(z, x)$:

$$\{S_{p+1}(z, x), J^{\text{diag}}(y)\} = \varphi(z) (\cdots), \quad \text{with} \quad J^{\text{diag}}(y) = \sum_{r=1}^{N} J^{(r)}(y)$$

Implies

$$\{S_{p+1}(\zeta_i, x), J^{\text{diag}}_a(y)\} = 0$$
Diagonal symmetry

- Other important property of $S_{p+1}(z, x)$:

$$\{S_{p+1}(z, x), J^{\text{diag}}(y)\} = \varphi(z)(\cdots), \quad \text{with} \quad J^{\text{diag}}(y) = \sum_{r=1}^{N} J^{(r)}(y)$$

- Implies

$$\{S_{p+1}(\zeta_i, x), J^{\text{diag}}_a(y)\} = 0$$

- Densities $S_{p+1}(\zeta_i, x)$ invariant under diagonal gauge symmetry

$$J^{(r)}(x) \mapsto h(x)^{-1} J^{(r)}(x) h(x) + \ell_r h(x)^{-1} \partial_x h(x), \quad h(x) \in G$$

- Belong to the classical $\hat{g}_{\ell_1} \oplus \cdots \oplus \hat{g}_{\ell_N}$ coset $\mathcal{W}$-algebra:

$$\mathcal{W} = \left\{\text{gauge-invariant differential polynomials in } J^{(r)}_a(x)\right\}$$
Densities $S_{p+1}(\zeta, x)$ belong to the classical coset $\mathcal{W}$-algebra:

$$\mathcal{W} = \left\{ \text{gauge-invariant differential polynomials in } J_a^{(r)}(x) \right\}$$

Important element of $\mathcal{W}$, linear combination of $S_2(\zeta, x)$:

$$T(x) = \frac{\eta^{ab}}{2} \left( \sum_{r=1}^{N} \frac{1}{\ell_r} J_a^{(r)}(x) J_b^{(r)}(x) - \frac{1}{\ell_{\text{diag}}} J_a^{\text{diag}}(x) J_b^{\text{diag}}(x) \right)$$

Recovers the energy-momentum tensor of GKO coset construction:

$$T = \sum_r T^{(r)} - T^{\text{diag}}, \text{ with } T^{(r)} \text{ the Segal-Sugawara tensor of } J^{(r)}$$

$T$ generates $x$-diffeomorphisms in $\mathcal{W}$ and satisfies classical Virasoro:

$$\{T(x), T(y)\} = -(T(x) + T(y)) \partial_x \delta(x - y)$$
Quantum chiral affine Gaudin models
Quantum chiral affine Gaudin models

- **Chiral affine Gaudin models**: currents $\mathcal{J}^{(r)}(x)$ all left-moving (or right-moving) fields of a 2d CFT $\rightarrow$ quantised as a current VOA

- **Quantum Kac-Moody currents $J^{(r)}(x)$**: 

  $$
  \left[ J^{(r)}_a(x), J^{(s)}_b(y) \right] = 2\pi \delta_{rs} \left( f_{ab}^c J^{(r)}_c(y) \delta(x - y) + i k_r \eta_{ab} \partial_x \delta(x - y) \right)
  $$

- **Classical limit**: 

  $$
  \left[ \mathcal{J}^{(r)}_a(x), \mathcal{J}^{(s)}_b(y) \right] = i\hbar \left\{ \mathcal{J}^{(r)}_a(x), \mathcal{J}^{(s)}_b(y) \right\} + O(\hbar^2)
  $$

  with

  $$
  J^{(r)}(x) = \frac{2\pi \mathcal{J}^{(r)}(x) + O(\hbar)}{i\hbar}, \quad k_r = \frac{2\pi \ell_r + O(\hbar)}{\hbar}
  $$
Chiral affine Gaudin models: currents $\mathcal{J}^{(r)}(x)$ all left-moving (or right-moving) fields of a 2d CFT $\rightarrow$ quantised as a current VOA

Quantum Kac-Moody currents $J^{(r)}(x)$:

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$$J^{(r)}(x) = \frac{2\pi \mathcal{J}^{(r)}(x) + O(\hbar)}{i\hbar}, \quad k_r = \frac{2\pi \ell_r + O(\hbar)}{\hbar}$$

Operator Product Expansion (OPE):

$$J^{(r)}_a(x)J^{(s)}_b(y) = \delta_{rs} \left(i f_{ab}^\ c J^{(r)}_c(y) \frac{1}{x - y} + \frac{k_r \eta_{ab}}{(x - y)^2}\right) + \text{reg}$$
Quantum $\mathcal{W}$-algebra

\[ J_a^{(r)}(x)J_b^{(s)}(y) = \delta_{rs} \left( \frac{i f_{ab} c J_c^{(r)}(y)}{x - y} + \frac{k_r \eta_{ab}}{(x - y)^2} \right) + \text{reg} \]

- Diagonal current:

\[ J^{\text{diag}}(x) = \sum_{r=1}^{N} J^{(r)}(x), \quad \text{with level} \quad k_{\text{diag}} = \sum_{r=1}^{N} k_r \]

- Quantum $\hat{g}_{k_1} \oplus \cdots \oplus \hat{g}_{k_N}$ coset $\mathcal{W}$-algebra:

\[ \mathcal{W} = \left\{ \text{normal ordered differential polynomials in } J_a^{(r)}(x) \right\} \]

\[ \text{having regular OPE with } J^{\text{diag}}(y) \]

- Algebra of extended conformal symmetry, closed under OPEs
GKO energy-momentum tensor:

\[ T(x) = \frac{\eta_{ab}}{2} \left( \sum_{r=1}^{N} \frac{1}{k_r + h^\vee} :J^{(r)}_a(x)J^{(r)}_b(x): - \frac{1}{k_{\text{diag}} + h^\vee} :J^{\text{diag}}_a(x)J^{\text{diag}}_b(x): \right) \]

with \( h^\vee \) the dual Coxeter number, defined by \( f_{ac}^d f_{bd}^c = 2h^\vee \eta_{ab} \)

- Satisfies Virasoro OPEs:

\[ T(x)T(y) = \frac{\partial T(y)}{x - y} + \frac{2T(y)}{(x - y)^2} + \frac{c}{2(x - y)^4} + \text{reg} \]

with central charge \( c = \left( \sum_r \frac{k_r}{k_r + h^\vee} - \frac{k_{\text{diag}}}{k_{\text{diag}} + h^\vee} \right) \text{dim } g \).
**GKO energy-momentum tensor**

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\]

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with **central charge** \( c = \left( \sum_r \frac{k_r}{k_r + h^\vee} - \frac{k_{\text{diag}}}{k_{\text{diag}} + h^\vee} \right) \dim g \)

- **In terms of Fourier modes:**

\[
T(x) = \sum_{n \in \mathbb{Z}} L_n e^{-inx} - \frac{c}{24}, \quad [L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} n(n^2 - 1) \delta_{n+m,0}
\]
Question: infinite family of commuting charges in $W$, quantising $Q^p_i$?
Quantum local charges?

- **Question:** infinite family of commuting charges in $W$, quantising $Q^p_i$?

- **Reminder on classical local charges:**

  $$Q^p_i = \int_0^{2\pi} S_{p+1}(\zeta_i, x) \, dx, \quad S_{p+1}(z, x) = \tau_p^{a_1\cdots a_{p+1}} \Gamma_{a_1}(z, x) \cdots \Gamma_{a_{p+1}}(z, x)$$

- Satisfy $\{Q^p_i, Q^q_j\} = 0$ and $S_{p+1}(\zeta_i, x) \in \mathcal{W}$ using

  1. $\{S_{p+1}(z, x), S_{q+1}(w, y)\} = \sum_{k=0,1} A_{p,q}^{(k)}(z, w ; y) \partial_x^k \delta(x - y)$
  
  with $A_{p,q}^{(0)}(z, w ; y) = \partial_y (\cdots) + \varphi(z)(\cdots) + \varphi(w)(\cdots)$
  2. $\{S_{p+1}(z, x), J^{\text{diag}}(y)\} = \varphi(z)(\cdots)$
Quantum local charges?

- Question: infinite family of commuting charges in $W$, quantising $Q_{i}^{p}$?

- Reminder on classical local charges:

$$Q_{i}^{p} = \int_{0}^{2\pi} S_{p+1}(\zeta_{i}, x) dx,$$

$$S_{p+1}(z, x) = \tau_{p}^{a_{1} \cdots a_{p+1}} \Gamma_{a_{1}}(z, x) \cdots \Gamma_{a_{p+1}}(z, x)$$

- Satisfy $\{Q_{i}^{p}, Q_{j}^{q}\} = 0$ and $S_{p+1}(\zeta_{i}, x) \in W$ using

1. $\{S_{p+1}(z, x), S_{q+1}(w, y)\} = \sum_{k=0,1} A_{p, q}^{(k)}(z, w ; y) \partial_{x}^{k} \delta(x - y)$

   with $A_{p, q}^{(0)}(z, w ; y) = \partial_{y} (\cdots) + \varphi(z)(\cdots) + \varphi(w)(\cdots)$

2. $\{S_{p+1}(z, x), J^{\text{diag}}(y)\} = \varphi(z)(\cdots)$

- Quantum equivalent of conditions (1) and (2)?
Quantum Gaudin Lax matrix and twist function:

- Quantum Gaudin Lax matrix and twist function:
  \[
  \Gamma^{(qt)}(z, x) = \sum_{r=1}^{N} \frac{J(r)(x)}{z - z_r} \quad \text{and} \quad \varphi^{(qt)}(z) = \sum_{r=1}^{N} \frac{k_r}{z - z_r}
  \]

- Classical limit:
  \[
  \Gamma^{(qt)}(z, x) = \frac{2\pi \Gamma(z, x) + O(\hbar)}{i\hbar}, \quad \varphi^{(qt)}(z) = \frac{2\pi \varphi(z) + O(\hbar)}{\hbar}
  \]
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  \]

- **OPE of the quantum Gaudin Lax matrix:**

  \[
  \Gamma^{(qt)}_a(z, x) \Gamma^{(qt)}_b(w, y) = -i f_{ab}^c \frac{\Gamma^{(qt)}_c(z, y) - \Gamma^{(qt)}_c(w, y)}{x - y} - \frac{\eta_{ab}}{(x - y)^2} \frac{\varphi^{(qt)}(z) - \varphi^{(qt)}(w)}{z - w} + \text{reg}
  \]
Quantum quadratic density

- Quantum quadratic density:

\[ S_2(z, x) = \eta^{ab} : \Gamma^{(qt)}_a(z, x) \Gamma^{(qt)}_b(z, x) : \]

- From OPE \( \Gamma^{(qt)}(z, x) \Gamma^{(qt)}(w, y) \) we get

\[ S_2(z, x)S_2(w, y) = \sum_{k=0}^{3} \frac{A_{1,1}^{(k)}(z, w ; y)}{(x - y)^{k+1}} + \text{reg} \]

with \( A_{1,1}^{(0)}(z, w ; y) = \partial_y (\cdots) + D_{z,1}(\cdots) + D_{w,1}(\cdots) \) and twisted derivative

\[ D_{z,p}f(z) = \partial_z f(z) - \frac{p}{h^\vee} \varphi^{(qt)}(z)f(z) \]
Quantum quadratic density

- **Quantum quadratic density:**

  \[ S_2(z, x) = \eta^{ab} \cdot \Gamma^{(qt)}_a(z, x) \Gamma^{(qt)}_b(z, x) : \]

- From OPE \( \Gamma^{(qt)}(z, x) \Gamma^{(qt)}(w, y) \) we get

  \[
  S_2(z, x)S_2(w, y) = \sum_{k=0}^{3} \frac{A^{(k)}_{1,1}(z, w ; y)}{(x - y)^{k+1}} + \text{reg}
  \]

  with

  \[
  A^{(0)}_{1,1}(z, w ; y) = \partial_y (\cdots) + D_{z,1}(\cdots) + D_{w,1}(\cdots) \]

  and twisted derivative

  \[
  D_{z,p}f(z) = \partial_z f(z) - \frac{p}{\hbar^\vee} \varphi^{(qt)}(z)f(z) = -\frac{2\pi p}{\hbar^\vee \hbar} \left( \varphi(z)f(z) + O(\hbar) \right)
  \]
Quantum quadratic density

- Quantum quadratic density:

\[ S_2(z, x) = \eta^{ab} : \Gamma_{a}^{(qt)}(z, x) \Gamma_{b}^{(qt)}(z, x) : \]

- From OPE \( \Gamma^{(qt)}(z, x) \Gamma^{(qt)}(w, y) \) we get

\[
S_2(z, x)S_2(w, y) = \sum_{k=0}^{3} \frac{A_{1,1}^{(k)}(z, w ; y)}{(x - y)^{k+1}} + \text{reg}
\]

with \( A_{1,1}^{(0)}(z, w ; y) = \partial_y (\cdots) + D_{z,1} (\cdots) + D_{w,1} (\cdots) \) and twisted derivative

\[
D_{z,p} f(z) = \partial_z f(z) - \frac{p}{\hbar^{\nu}} \varphi^{(qt)}(z) f(z) = -\frac{2\pi p}{\hbar^{\nu} \hbar} \left( \varphi(z) f(z) + O(\hbar) \right)
\]

- Similarly, \( S_2(z, x)J_{\text{diag}}(y) = D_{z,1} (\cdots) + \text{reg} \)

- How to build gauge-invariant commuting charges from \( S_2(z, x) \)?
Function $\mathcal{P}(z)$ and Pochhammer integrals

- Introduce multi-valued function

$$\mathcal{P}(z) = \prod_{r=1}^{N} (z - z_r)^{k_r}, \quad \partial_z \mathcal{P}(z) = \varphi^{(qt)}(z) \mathcal{P}(z)$$

- Satisfies $\mathcal{P}(z)^{-p/h^\vee} D_{z,p} f(z) = \partial_z (\mathcal{P}(z)^{-p/h^\vee} f(z))$
Function $\mathcal{P}(z)$ and Pochhammer integrals

- Introduce multi-valued function
  \[ \mathcal{P}(z) = \prod_{r=1}^{N} (z - z_r)^{k_r}, \quad \partial_z \mathcal{P}(z) = \varphi^{(q)}(z) \mathcal{P}(z) \]

- Satisfies $\mathcal{P}(z)^{-p/h^\vee} D_{z,p} f(z) = \partial_z (\mathcal{P}(z)^{-p/h^\vee} f(z))$

- $\gamma$ in $P = \{\text{closed contours on which } \mathcal{P}(z) \text{ has a single-valued branch}\}$:
  \[ \oint_{\gamma} \mathcal{P}(z)^{-p/h^\vee} D_{z,p} f(z) \, dz = 0 \]

- Pochhammer contours:
Quantum quadratic charges

- Quantum quadratic densities and charges: $\gamma \in P$

$$Q^1_\gamma = \int_0^{2\pi} W_{2,\gamma}(x) \, dx, \quad W_{2,\gamma}(x) = \oint \mathcal{P}(z)^{-1/h} S_2(z, x) \, dz$$

- Since $S_2(z, x) J^{\text{diag}}(y) = D_{z,1}(\cdots) + \text{reg}$:

$$W_{2,\gamma}(x) J^{\text{diag}}(y) = \text{reg}, \quad \text{hence} \quad W_{2,\gamma}(x) \in W$$

- From OPE $S_2(z, x) S_2(w, y)$:

$$[Q^1_\gamma, Q^1_{\gamma'}] = 0, \quad \forall \gamma, \gamma' \in P$$

- Pochhammer integrals reminiscent of [Lukyanov '13, Bazhanov Lukyanov '13] (ODE/IQFT correspondence for the Fateev integrable structure)
Higher-degree quantum charges: conjecture

- **Conjecture:** for every \( p \in E \), there exist

\[
S_{p+1}(z, x) = \tau_{p}^{a_1 \cdots a_{p+1}} : \Gamma_{a_1}^{(q_t)}(z, x) \cdots \Gamma_{a_{p+1}}^{(q_t)}(z, x) : + \ldots
\]

with quantum corrections built from \( \partial_z^\alpha \partial_x^\beta \Gamma_{a}^{(q_t)}(z, x) \), such that

1. \[
S_{p+1}(z, x)S_{q+1}(w, y) = \sum_{k \geq 0} \frac{A_{p,q}^{(k)}(z, w ; y)}{(x - y)^{k+1}} + \text{reg}
\]
   with \( A_{p,q}^{(0)}(z, w ; y) = \partial_y (\cdots) + D_{z,p} (\cdots) + D_{w,q} (\cdots) \)

2. \[
S_{p+1}(z, x)J_{\text{diag}}^\gamma (y) = D_{z,p} (\cdots) + \text{reg}
\]

- **Consequence:** commuting charges with gauge-invariant densities

\[
Q_{\gamma}^p = \int_{0}^{2\pi} W_{p+1,\gamma}^{}(x) \, dx , \quad W_{p+1,\gamma}^{}(x) = \int_{\gamma} \mathcal{P}(z)^{-p/h^\gamma} S_{p+1}(z, x) \, dz
\]
First checks of the conjecture:

- **quadratic** $S_2(z, x)$ for any $g$
- **cubic** $S_3(z, x)$ for $g = sl(k), \ k > 2$ [SL Vicedo Young '18]
- **quartic** $S_4(z, x)$ for $g = sl(2)$ [Kotousov SL Teschner '22, Franzini Young '22]
First checks of the conjecture:

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Higher-degree quantum charges: first checks

- **First checks of the conjecture:**
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\[
S_4(z) = \tau_3^{abcd} : \Gamma_a(z)\Gamma_b(z)\Gamma_c(z)\Gamma_d(z) : + \frac{5i}{4} f^{abc} : \partial_x\Gamma_a(z)\partial_z\Gamma_b(z)\Gamma_c(z) : \\
+ \frac{\eta^{ab}}{48} \left( 45 \phi(z)^2 : \partial_x\Gamma_a(z)\partial_x\Gamma_b(z) : - 140 : \partial_z^2\partial_x\Gamma_a(z)\partial_x\Gamma_b(z) : - 30 : \partial_z\partial_x\Gamma_a(z)\partial_z\partial_x\Gamma_b(z) : \right) \\
+ \frac{5\eta^{ab}}{12} \left( 3 : \partial_z\partial_x^2\Gamma_a(z)\partial_z\Gamma_b(z) : - : \partial_z^2\partial_x^2\Gamma_a(z)\Gamma_b(z) : \right)
\]
Classical limit

\[
W_{p+1,\gamma}(x) = \oint_{\gamma} \mathcal{P}(z)^{-p/h^\gamma} S_{p+1}(z, x) \, dz \quad \text{with}
\]

\[
S_{p+1}(z, x) = \tau_p^{a_1 \cdots a_{p+1}} : \Gamma_{a_1}^{(q_t)}(z, x) \cdots \Gamma_{a_{p+1}}^{(q_t)}(z, x) : + \ldots
\]

- Classical limit: with \( \rho'(z) = \varphi(z) \)

\[
\Gamma^{(q_t)}(z, x) = \frac{2\pi \Gamma(z, x) + O(\hbar)}{i\hbar}, \quad \mathcal{P}(z) = \exp \left( \frac{2\pi}{\hbar} (\rho(z) + O(\hbar)) \right)
\]

\[
W_{p+1,\gamma}(x) \propto \oint_{\gamma} \exp \left( -\frac{2\pi p}{\hbar \sqrt{\hbar}} (\rho(z) + O(\hbar)) \right) \left( S_{p+1}(z, x) + O(\hbar) \right) \, dz
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Classical limit

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\[ W_{p+1,\gamma}(x) \propto \oint_{\gamma} \exp \left( -\frac{2\pi}{h^\gamma \hbar} \left( \rho(z) + O(\hbar) \right) \right) \left( S_{p+1}(z, x) + O(\hbar) \right) \, dz \]

- **In the limit** \( \hbar \to 0 \), saddle-point approximation: integral localises at the extrema of \( \rho(z) \), i.e. the zeroes \( \zeta_i \) of \( \varphi(z) \)

\[ \to W_{p+1,\gamma}(x) \text{ yields a linear combination of } S_{p+1}(\zeta_i, x) \]
Spectrum of the local charges

- Spectrum of the charges $Q^p_\gamma$ on representations of $\hat{g} \oplus N$?

- Quantisation of chiral AGMs related to conjectured “affine” Langlands geometric correspondence [Feigin Frenkel ’07, Frenkel Hernandez ’16, SL Vicedo Young ’18, Gaiotto Lee Vicedo Wu ’20, Kotousov Lukyanov ’21]

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- ODE/IQFT correspondence [Dorey Tateo ’99, Bazhanov Lukyanov Zamolodchikov ’03, Lukyanov ’13, Bazhanov Lukyanov ’13, ...]
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- Algebraic Bethe ansatz for chiral AGMs (acting on highest-weight representations): precise conjectures for the eigenvectors and eigenvalues of $Q^p_\gamma$ and first checks
  [Schechtman Varchenko ’91, Feigin Frenkel ’07, SL Vicedo Young ’18 ’18]
Example: Fateev integrable structure [Fateev ’96]

- Two punctures \( N = 2 \) and \( g = \mathfrak{sl}(2) \)

\[
\frac{\mathfrak{sl}(2)_{k_1} \oplus \mathfrak{sl}(2)_{k_2}}{\mathfrak{sl}(2)_{k_1 + k_2}^{\text{diag}}} \text{ coset}
\]

- Quadratic and quartic charges \( Q^1_\gamma \) and \( Q^3_\gamma \) [Kotousov SL Teschner ’22]

\[
W_{2,\gamma} \propto T^{\text{GKO}} = \frac{\eta^{ab}}{2} \left( \frac{J^{(1)}_a J^{(1)}_b}{k_1 + 2} + \frac{J^{(2)}_a J^{(2)}_b}{k_2 + 2} - \frac{J^{\text{diag}}_a J^{\text{diag}}_b}{k_1 + k_2 + 2} \right)
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- **Various comparisons with** [Fateev '96, Feigin Semikhatov '01, Lukyanov Zamolodchikov '12, Bazhanov Lukyanov '13, Bazhanov Kotousov Lukyanov '18]:
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- Recover various integrable hierarchies: Fateev, KdV, Drinfeld-Sokolov, ...
- Generalisations: higher-order poles, cyclotomic models, ...
- Relations to toroidal algebras, affine Yangians, 4d-Chern-Simons, ...
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Thank you for your attention!
Spectrum of the local charges
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- Common eigenvector of $Q^p_\gamma \leftrightarrow$ affine $L^{\hat{g}}$-oper (differential operator depending on $\text{rk}(\mathfrak{g})$ functions $\nu_k(z)$, e.g. $-\partial_z^2 + \nu(z) + \chi \mathcal{P}(z)$ for $\mathfrak{g} = \mathfrak{sl}_2$)

- Conjectured eigenvalue of $Q^p_\gamma$: $u_p(z)$ built from $\nu_k(z)$

$$I^p_\gamma = \oint_\gamma \mathcal{P}(z)^{-p/\hbar^\vee} u_p(z) \, dz$$

- ODE/IQFT correspondence [Dorey Tateo ’99, Bazhanov Lukyanov Zamolodchikov ’03, Lukyanov ’13, Bazhanov Lukyanov ’13, ...]
Bethe ansatz

[Schechtman Varchenko ’91, Feigin Frenkel ’07, SL Vicedo Young ’18 ’18]

- Hilbert space: tensor product of $N$ highest-weight representations of $\widehat{g}$
- Bethe vector with $M$ excitations: $\Psi_{\alpha_1,\ldots,\alpha_M}(w_1,\ldots,w_M)$
  ($\{\alpha_j\}$ are simple roots of $\widehat{g}$, $\{w_j\} \subset \mathbb{C}$ are the Bethe roots)

- Conjecture:
  - eigenvector of $Q_p^\gamma \Leftrightarrow \{w_j\}$ satisfy Bethe ansatz equations
  - associated affine oper $\Leftrightarrow$ Miura oper with certain regularity properties

- First checks:
  - proven for quadratic $Q_1^\gamma$, for any $g$, any $N$ and any $M$
  - proven for cubic $Q_2^\gamma$ for $g = \mathfrak{sl}(k)$ ($k > 2$), any $N$ and $M = 0, 1, 2$
Example: the Fateev integrable structure
Chiral AGM with $N = 2$ currents

- Take $N = 2$, i.e. 2 punctures in the complex plane
- By translation and dilation of $z$, we can fix $z_1 = 0$ and $z_2 = 1$

$$\Gamma^{(qt)}(z, x) = \frac{J^{(1)}(x)}{z} + \frac{J^{(2)}(x)}{z - 1} \quad \text{and} \quad P(z) = z^{k_1}(z - 1)^{k_2}$$

- Local observables: $(\hat{g}_{k_1} \oplus \hat{g}_{k_2})/\hat{g}_{k_1+k_2}$ coset $\mathcal{W}$-algebra
- One Pochhamer contour $\gamma$ and Euler $B$-function integrals:

$$B(a, b) = \oint_{\gamma} z^a(z - 1)^b \, dz$$
Quadratic density

$\Gamma^{(qt)}(z, x) = \frac{J^{(1)}(x)}{z} + \frac{J^{(2)}(x)}{z - 1}$ and $\mathcal{P}(z) = z^{k_1}(z - 1)^{k_2}$

- Quadratic density:

$$W_{2, \gamma}(x) = \eta^{ab} \int_\gamma \mathcal{P}(z)^{-1/h^\vee} : \Gamma^{(qt)}_a(z, x) \Gamma^{(qt)}_b(z, x) : \, dz$$

- Explicitely computed using Euler $B$-function:

$$W_{2, \gamma}(x) \propto \frac{\eta^{ab}}{2} \left( \frac{\mathcal{J}_a^{(1)}(x) \mathcal{J}_b^{(1)}(x)}{k_1 + h^\vee} + \frac{\mathcal{J}_a^{(2)}(x) \mathcal{J}_b^{(2)}(x)}{k_2 + h^\vee} - \frac{\mathcal{J}^{\text{diag}}(x) \mathcal{J}^{\text{diag}}(x)}{k_1 + k_2 + h^\vee} \right)$$

- Recover the GKO coset energy-momentum tensor (quantum corrections by $h^\vee$ come from Pochhammer integrals)
Quartic density for $g = sl_2$

- Further specialise to $g = sl_2$
- $\hat{sl}(2)_{k_1} \oplus \hat{sl}(2)_{k_2}/\hat{sl}(2)_{k_1+k_2}^\text{diag}$ coset: corner-brane $\mathcal{W}$-algebra
  
  [Fateev ’96, Feigin Semikhatov ’01, Lukyanov Zamolodchikov ’12]:
  - one spin-2 field: energy-momentum tensor $T$
  - one spin-3 field: descendant $\partial T$
  - three spin-4 fields: descendants $\partial^2 T$, $:T^2:$ and primary $W_{4,P}$
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- Quartic density from AGM construction: [Kotousov SL Teschner '22]
  
  $$W_{4,\gamma}(x) \propto W_{4,P}(x) + \delta_1 :T^2(x): + \delta_2 \partial^2 T(x)$$

  $$\delta_1 = \frac{15(k_1 + 2)(3k_1 + 4)(k_2 + 2)(3k_2 + 4)(k_1 + k_2 + 2)(3k_1 + 3k_2 + 8)}{176 - 44(k_1 + 2)^2 - 44(k_2 + 2)^2 - k_1 k_2(37k_1 + 37k_2 + 192)}$$

- Agrees with screening-charge computation in Fateev integrable structure [Fateev '96, Feigin Semikhatov '01, Lukyanov Zamolodchikov '12]