Quantum local charges in chiral affine Gaudin models

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Introduction

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- Affine Gaudin models (AGM): physical systems associated with affine Lie algebras [Feigin Frenkel '07]
- Field theory built from Kac-Moody currents $\mathcal{J}^{(1)}(x), \dots, \mathcal{J}^{(N)}(x)$
- Clasically integrable: infinite familly of (local and non-local) Poisson-commuting charges built from $\mathcal{J}^{(r)}(x)$
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- Natural starting point for quantisation: chiral AGM
- All $\mathcal{J}^{(r)}(x)$ are left/right-moving fields of a 2d CFT
  $\rightarrow$ quantised as a current VOA (vertex operator algebra)
- This talk: first results and conjectures on the construction and diagonalisation of quantum local charges in this VOA
  (Integrable structure in CFT [Bazhanov Lukyanov Zamolodchikov '94, ...])

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- This talk: first results and conjectures on the construction and diagonalisation of quantum local charges in this VOA
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- Applications to integrable sigma-models (see talk by J. Teschner)

Contents

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Defining data:

- Simple complex Lie algebra $\mathfrak{g}$
- Parameters: punctures $z_1, \ldots, z_N \in \mathbb{C}$ and levels $\ell_1, \ldots, \ell_N \in \mathbb{C}^*$

Lie algebra conventions: basis $\{\mathrm{t}_a\}$

- Structure constants $f_{ab}{}^{c}$, with $[\mathrm{t}_a, \mathrm{t}_b] = f_{ab}{}^{c}\, \mathrm{t}_c$
- Minimal invariant bilinear form η_{ab}, with inverse η^{ab}

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Kac-Moody currents:

- $\mathfrak{g}$-valued fields $\mathcal{J}^{(r)}(x) = \mathcal{J}^{(r)}_a(x)\, \mathrm{t}^a \qquad (r \in \{1, \dots, N\}$ and $x \in [0, 2\pi])$
- Kac-Moody Poisson bracket:

$$\{\mathcal{J}^{(r)}_a(x), \mathcal{J}^{(s)}_b(y)\} = \delta_{rs}\Big(f_{ab}{}^c\, \mathcal{J}^{(r)}_c(x)\, \delta(x-y) - \ell_r\, \eta_{ab}\, \partial_x \delta(x-y) \Big)$$

Gaudin Lax matrix and twist function

- Gaudin Lax matrix and twist function: (z spectral parameter)

$$\Gamma(z,x) = \sum_{r=1}^{N} \frac{\mathcal{J}^{(r)}(x)}{z - z_r} \qquad \text{and} \qquad \varphi(z) = \sum_{r=1}^{N} \frac{\ell_r}{z - z_r}$$

- $\varphi(z)$ contains all the parameters
- Key formula: Poisson bracket of the Gaudin Lax matrix

$$\{\Gamma_a(z,x), \Gamma_b(w,y)\} =$$

$$- f_{ab}{}^{c}\, \frac{\Gamma_c(z,x) - \Gamma_c(w,x)}{z - w}\, \delta(x-y) + \eta_{ab}\, \frac{\varphi(z) - \varphi(w)}{z - w}\, \partial_x \delta(x-y)$$

Lax matrix and non-local charges

- Lax matrix:

$$\mathcal{L}(z,x) = \frac{\Gamma(z,x)}{\varphi(z)}$$

- $\{\mathcal{L}_a(z,x), \mathcal{L}_b(w,y)\}$ takes the form of a Maillet bracket [Maillet '85] with non-skew-symmetric $\mathcal{R}$-matrix

- Non-local charges in involution:

$$\Big\{ \text{Tr}_\rho\big(M(z)\big), \text{Tr}_{\rho'}\big(M(w)\big) \Big\} = 0\,, \qquad M(z) = \text{P}\overleftarrow{\text{exp}}\left(- \int_0^{2\pi} \mathcal{L}(z,x)\, \text{d}x \right)$$

  Infinitely many charges by Taylor-expansion in z

Hierarchies of local charges

- Infinite hierarchies of local charges $\mathcal{Q}_i^p$: [SL Magro Vicedo '17]
  - $i \in \{1, \dots, N-1\}$ associated to zeroes ζ_i of $\varphi(z)$
  - degrees $p+1$: $p \in E$, with $E \subset \mathbb{Z}_{\geq 1}$ infinite (depending on $\mathfrak{g}$)
  - Poisson-commuting $\{\mathcal{Q}_i^p, \mathcal{Q}_j^q\} = 0$
  - not contained in monodromy charges but Poisson-commute with them

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- Zeroes of the twist function:

$$\varphi(z) = \sum_{r=1}^{N} \frac{\ell_r}{z - z_r} = K \frac{\prod_{i=1}^{N-1}(z-\zeta_i)}{\prod_{r=1}^{N}(z-z_r)}$$

Two equivalent sets of parameters: (z_r, ℓ_r) and (K, z_r, ζ_i)

Construction of the local charges

- Definition of the local charges:

$$\mathcal{Q}_i^p = \int_0^{2\pi} \mathcal{S}_{p+1}(\zeta_i, x)\,\mathrm{d}x\,, \qquad \mathcal{S}_{p+1}(z,x) = \tau_p^{a_1\cdots a_{p+1}}\,\Gamma_{a_1}(z,x)\cdots\Gamma_{a_{p+1}}(z,x)$$

- $\tau_p^{a_1\cdots a_{p+1}}$: invariant symmetric $(p+1)$–tensor on $\mathfrak{g}$

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- $\tau_p^{a_1\cdots a_{p+1}}$: invariant symmetric $(p+1)$–tensor on $\mathfrak{g}$ such that

$$\big\{\mathcal{S}_{p+1}(z,x), \mathcal{S}_{q+1}(w,y)\big\} = \mathcal{A}^{(0)}_{p,q}(z,w\,;y)\,\delta(x-y) + \mathcal{A}^{(1)}_{p,q}(z,w\,;y)\,\partial_x\delta(x-y)$$

with $\mathcal{A}^{(0)}_{p,q}(z,w\,;y) = \partial_y\big(\cdots\big) + \varphi(z)\big(\cdots\big) + \varphi(w)\big(\cdots\big)$

- Ensures that $\{\mathcal{Q}_i^p, \mathcal{Q}_j^q\} = 0$

$$\big\{\Gamma_a(z,x), \Gamma_b(w,y)\big\} = -f_{ab}{}^c\, \frac{\Gamma_c(z,x) - \Gamma_c(w,x)}{z-w}\,\delta(x-y) + \eta_{ab}\,\frac{\varphi(z)-\varphi(w)}{z-w}\,\partial_x\delta(x-y)$$

Invariant tensors

$$\mathcal{Q}_i^p = \int_0^{2\pi} \mathcal{S}_{p+1}(\zeta_i, x)\, \mathrm{d}x\,, \qquad \mathcal{S}_{p+1}(z,x) = \tau_p^{a_1\cdots a_{p+1}}\, \Gamma_{a_1}(z,x)\cdots\Gamma_{a_{p+1}}(z,x)$$

- Invariant symmetric tensors τ_p [Evans Hassan MacKay Mountain '99]
- p belongs to the set of affine exponents $E \subset \mathbb{Z}_{\geq 1}$, depending on $\mathfrak{g}$
- Always start with $p = 1$ (quadratic tensor)

$$\tau_1^{ab} = \eta^{ab}$$

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$$\tau_1^{ab} = \eta^{ab}$$

- For $\mathfrak{g} = \mathfrak{sl}(k)$, $E = \{1, \dots, k-1, k+1, \dots, 2k-1, 2k+1, \dots\}$

$$\tau_1^{ab} = \mathrm{Tr}\big(\mathrm{t}^a \mathrm{t}^b\big)\,, \qquad \tau_2^{abc} = \mathrm{Tr}\big(\mathrm{t}^{(a}\mathrm{t}^b\mathrm{t}^{c)}\big)$$

$$\tau_3^{abcd} = \mathrm{Tr}\big(\mathrm{t}^{(a}\mathrm{t}^b\mathrm{t}^c\mathrm{t}^{d)}\big) - \frac{3}{2k}\mathrm{Tr}\big(\mathrm{t}^{(a}\mathrm{t}^{b}\big)\mathrm{Tr}\big(\mathrm{t}^c\mathrm{t}^{d)}\big)\,, \quad \dots$$

Diagonal symmetry

- Other important property of $\mathcal{S}_{p+1}(z,x)$:

$$\left\{\mathcal{S}_{p+1}(z,x), \mathcal{J}^{\text{diag}}(y)\right\} = \varphi(z)\big(\cdots\big)\,, \qquad \text{with} \quad \mathcal{J}^{\text{diag}}(y) = \sum_{r=1}^{N} \mathcal{J}^{(r)}(y)$$

- Implies

$$\left\{\mathcal{S}_{p+1}(\zeta_i,x), \mathcal{J}^{\text{diag}}_a(y)\right\} = 0$$

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- Implies

$$\big\{\mathcal{S}_{p+1}(\zeta_i,x),\mathcal{J}^{\text{diag}}_a(y)\big\}=0$$

- Densities $\mathcal{S}_{p+1}(\zeta_i,x)$ invariant under diagonal gauge symmetry

$$\mathcal{J}^{(r)}(x)\longmapsto h(x)^{-1}\mathcal{J}^{(r)}(x)h(x)+\ell_r\,h(x)^{-1}\partial_x h(x)\,,\qquad h(x)\in G$$

- Belong to the classical $\dfrac{\widehat{\mathfrak{g}}_{\ell_1}\oplus\cdots\oplus\widehat{\mathfrak{g}}_{\ell_N}}{\widehat{\mathfrak{g}}^{\,\text{diag}}_{\ell_1+\cdots+\ell_N}}$ coset $\mathcal{W}$-algebra:

$$\mathcal{W}=\Big\{\text{gauge-invariant differential polynomials in } \mathcal{J}^{(r)}_a(x)\Big\}$$

GKO coset construction

- Densities $\mathcal{S}_{p+1}(\zeta_i, x)$ belong to the classical coset $\mathcal{W}$-algebra:

$$\mathcal{W} = \Big\{ \text{gauge-invariant differential polynomials in } \mathcal{J}_a^{(r)}(x) \Big\}$$

- Important element of $\mathcal{W}$, linear combination of $\mathcal{S}_2(\zeta_i, x)$:

$$\mathcal{T}(x) = \frac{\eta^{ab}}{2} \left(\sum_{r=1}^{N} \frac{1}{\ell_r} \mathcal{J}_a^{(r)}(x) \mathcal{J}_b^{(r)}(x) - \frac{1}{\ell_{\text{diag}}} \mathcal{J}_a^{\text{diag}}(x) \mathcal{J}_b^{\text{diag}}(x) \right)$$

- Recovers the energy-momentum tensor of GKO coset construction: $\mathcal{T} = \sum_r \mathcal{T}^{(r)} - \mathcal{T}^{\text{diag}}$, with $\mathcal{T}^{(r)}$ the Segal-Sugawara tensor of $\mathcal{J}^{(r)}$

- $\mathcal{T}$ generates x-diffeomorphisms in $\mathcal{W}$ and satisfies classical Virasoro:

$$\{\mathcal{T}(x), \mathcal{T}(y)\} = -\big(\mathcal{T}(x) + \mathcal{T}(y)\big)\, \partial_x \delta(x-y)$$

Quantum chiral affine Gaudin models

Quantum chiral affine Gaudin models

- Chiral affine Gaudin models: currents $\mathcal{J}^{(r)}(x)$ all left-moving (or right-moving) fields of a 2d CFT $\rightarrow$ quantised as a current VOA
- Quantum Kac-Moody currents $\mathsf{J}^{(r)}(x)$:

$$\Big[\mathsf{J}_a^{(r)}(x), \mathsf{J}_b^{(s)}(y)\Big] = 2\pi\, \delta_{rs}\Big(f_{ab}{}^{c}\, \mathsf{J}_c^{(r)}(y)\, \delta(x-y) + \mathrm{i}\, k_r\, \eta_{ab}\, \partial_x \delta(x-y) \Big)$$

- Classical limit: $\big[\mathcal{J}_a^{(r)}(x), \mathcal{J}_b^{(s)}(y)\big] = \mathrm{i}\hbar\, \big\{\mathcal{J}_a^{(r)}(x), \mathcal{J}_b^{(s)}(y)\big\} + O(\hbar^2)$ with

$$\mathsf{J}^{(r)}(x) = \frac{2\pi\, \mathcal{J}^{(r)}(x) + O(\hbar)}{\mathrm{i}\hbar}\,, \qquad k_r = \frac{2\pi\, \ell_r + O(\hbar)}{\hbar}$$

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- Operator Product Expansion (OPE):

$$\mathrm{J}^{(r)}_a(x)\mathrm{J}^{(s)}_b(y) = \delta_{rs}\left(\frac{\mathrm{i}\,f_{ab}{}^c\,\mathrm{J}^{(r)}_c(y)}{x-y} + \frac{k_r\,\eta_{ab}}{(x-y)^2}\right) + \text{reg}$$

Quantum $\mathcal{W}$-algebra

$$\mathrm{J}_a^{(r)}(x)\mathrm{J}_b^{(s)}(y) = \delta_{rs}\left(\frac{\mathrm{i}\, f_{ab}{}^{c}\,\mathrm{J}_c^{(r)}(y)}{x-y} + \frac{k_r\,\eta_{ab}}{(x-y)^2}\right) + \text{reg}$$

- Diagonal current:

$$\mathrm{J}^{\text{diag}}(x) = \sum_{r=1}^{N}\mathrm{J}^{(r)}(x)\,, \qquad \text{with level} \qquad k_{\text{diag}} = \sum_{r=1}^{N} k_r$$

- Quantum $\dfrac{\widehat{\mathfrak{g}}_{k_1}\oplus\cdots\oplus\widehat{\mathfrak{g}}_{k_N}}{\widehat{\mathfrak{g}}_{k_{\text{diag}}}}$ coset $\mathcal{W}$-algebra:

$$\mathrm{W} = \left\{ \begin{array}{c} \text{normal ordered differential polynomials in } \mathrm{J}_a^{(r)}(x) \\ \text{having regular OPE with } \mathrm{J}^{\text{diag}}(y) \end{array} \right\}$$

- Algebra of extended conformal symmetry, closed under OPEs

GKO energy-momentum tensor

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$$\mathrm{T}(x) = \frac{\eta^{ab}}{2} \left(\sum_{r=1}^{N} \frac{1}{k_r + h^\vee} :\mathrm{J}^{(r)}_a(x)\mathrm{J}^{(r)}_b(x): - \frac{1}{k_{\text{diag}} + h^\vee} :\mathrm{J}^{\text{diag}}_a(x)\mathrm{J}^{\text{diag}}_b(x): \right)$$

with $h^\vee$ the dual Coxeter number, defined by $f_{ac}{}^{d}\, f_{bd}{}^{c} = 2h^\vee\, \eta_{ab}$

- Satisfies Virasoro OPEs:

$$\mathrm{T}(x)\mathrm{T}(y) = \frac{\partial \mathrm{T}(y)}{x-y} + \frac{2\mathrm{T}(y)}{(x-y)^2} + \frac{c}{2(x-y)^4} + \text{reg}$$

with central charge $c = \left(\sum_r \frac{k_r}{k_r + h^\vee} - \frac{k_{\text{diag}}}{k_{\text{diag}} + h^\vee} \right) \dim \mathfrak{g}$

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- In terms of Fourier modes:

$$\mathrm{T}(x) = \sum_{n\in\mathbb{Z}} L_n\, e^{-\mathrm{i}nx} - \frac{c}{24}\,, \qquad [L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}$$

Quantum local charges?

- Question: infinite family of commuting charges in W, quantising $\mathcal{Q}_i^p$?

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- Reminder on classical local charges:

$$\mathcal{Q}_i^p = \int_0^{2\pi} \mathcal{S}_{p+1}(\zeta_i, x)\, \mathrm{d}x \,, \qquad \mathcal{S}_{p+1}(z,x) = \tau_p^{a_1 \cdots a_{p+1}}\, \Gamma_{a_1}(z,x) \cdots \Gamma_{a_{p+1}}(z,x)$$

- Satisfy $\{\mathcal{Q}_i^p, \mathcal{Q}_j^q\} = 0$ and $\mathcal{S}_{p+1}(\zeta_i, x) \in \mathcal{W}$ using

  1. $\big\{\mathcal{S}_{p+1}(z,x), \mathcal{S}_{q+1}(w,y)\big\} = \sum_{k=0,1} \mathcal{A}^{(k)}_{p,q}(z,w\,;y)\, \partial_x^k \delta(x-y)$
     with $\mathcal{A}^{(0)}_{p,q}(z,w\,;y) = \partial_y(\cdots) + \varphi(z)(\cdots) + \varphi(w)(\cdots)$
  2. $\big\{\mathcal{S}_{p+1}(z,x), \mathcal{J}^{\text{diag}}(y)\big\} = \varphi(z)(\cdots)$

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     with $\mathcal{A}_{p,q}^{(0)}(z,w\,;y) = \partial_y(\cdots) + \varphi(z)(\cdots) + \varphi(w)(\cdots)$
  2. $\big\{\mathcal{S}_{p+1}(z,x), \mathcal{J}^{\text{diag}}(y)\big\} = \varphi(z)(\cdots)$
- Quantum equivalent of conditions (1) and (2)?

Quantum Gaudin Lax matrix and twist function

- Quantum Gaudin Lax matrix and twist function:

$$\Gamma^{(\mathrm{qt})}(z,x) = \sum_{r=1}^{N} \frac{\mathsf{J}^{(r)}(x)}{z - z_r} \qquad \text{and} \qquad \varphi^{(\mathrm{qt})}(z) = \sum_{r=1}^{N} \frac{k_r}{z - z_r}$$

- Classical limit:

$$\Gamma^{(\mathrm{qt})}(z,x) = \frac{2\pi\,\Gamma(z,x) + O(\hbar)}{\mathrm{i}\hbar}\,, \qquad \varphi^{(\mathrm{qt})}(z) = \frac{2\pi\,\varphi(z) + O(\hbar)}{\hbar}$$

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- OPE of the quantum Gaudin Lax matrix:

$$\Gamma_a^{(\mathrm{qt})}(z,x)\Gamma_b^{(\mathrm{qt})}(w,y) =$$

$$- \frac{\mathrm{i}\, f_{ab}{}^{c}}{x-y} \frac{\Gamma_c^{(\mathrm{qt})}(z,y) - \Gamma_c^{(\mathrm{qt})}(w,y)}{z-w} - \frac{\eta_{ab}}{(x-y)^2} \frac{\varphi^{(\mathrm{qt})}(z) - \varphi^{(\mathrm{qt})}(w)}{z-w} + \text{reg}$$

Quantum quadratic density

- Quantum quadratic density:

$$\mathsf{S}_2(z,x) = \eta^{ab} :\Gamma_a^{(\mathrm{qt})}(z,x)\Gamma_b^{(\mathrm{qt})}(z,x):$$

- From OPE $\Gamma^{(\mathrm{qt})}(z,x)\Gamma^{(\mathrm{qt})}(w,y)$ we get

$$\mathsf{S}_2(z,x)\mathsf{S}_2(w,y) = \sum_{k=0}^{3} \frac{\mathsf{A}_{1,1}^{(k)}(z,w\,;y)}{(x-y)^{k+1}} + \mathrm{reg}$$

with $\mathsf{A}_{1,1}^{(0)}(z,w\,;y) = \partial_y\big(\cdots\big) + D_{z,1}\big(\cdots\big) + D_{w,1}\big(\cdots\big)$ and twisted derivative

$$D_{z,p}f(z) = \partial_z f(z) - \frac{p}{h^\vee}\varphi^{(\mathrm{qt})}(z)f(z)$$

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$$S_2(z,x)S_2(w,y) = \sum_{k=0}^{3} \frac{A_{1,1}^{(k)}(z,w\,;y)}{(x-y)^{k+1}} + \mathrm{reg}$$

with $A_{1,1}^{(0)}(z,w\,;y) = \partial_y(\cdots) + D_{z,1}(\cdots) + D_{w,1}(\cdots)$ and twisted derivative

$$D_{z,p}f(z) = \partial_z f(z) - \frac{p}{h^\vee}\varphi^{(\mathrm{qt})}(z)f(z) = -\frac{2\pi p}{h^\vee \hbar}\Big(\varphi(z)f(z) + O(\hbar)\Big)$$

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$$\mathsf{S}_2(z,x)\mathsf{S}_2(w,y) = \sum_{k=0}^{3} \frac{\mathsf{A}_{1,1}^{(k)}(z,w\,;y)}{(x-y)^{k+1}} + \mathrm{reg}$$

with $\mathsf{A}_{1,1}^{(0)}(z,w\,;y) = \partial_y(\cdots) + D_{z,1}(\cdots) + D_{w,1}(\cdots)$ and twisted derivative

$$D_{z,p}f(z) = \partial_z f(z) - \frac{p}{h^\vee}\varphi^{(\mathrm{qt})}(z)f(z) = -\frac{2\pi p}{h^\vee \hbar}\Big(\varphi(z)f(z) + O(\hbar)\Big)$$

- Similarly, $\mathsf{S}_2(z,x)\mathsf{J}^{\mathrm{diag}}(y) = D_{z,1}(\cdots) + \mathrm{reg}$

- How to build gauge-invariant commuting charges from $\mathsf{S}_2(z,x)$?

Function $\mathcal{P}(z)$ and Pochhammer integrals

- Introduce multi-valued function

$$\mathcal{P}(z) = \prod_{r=1}^{N} (z - z_r)^{k_r}\,, \qquad \partial_z \mathcal{P}(z) = \varphi^{(\mathrm{qt})}(z)\,\mathcal{P}(z)$$

- Satisfies $\mathcal{P}(z)^{-p/h^\vee} D_{z,p} f(z) = \partial_z \big(\mathcal{P}(z)^{-p/h^\vee} f(z) \big)$

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- Introduce multi-valued function

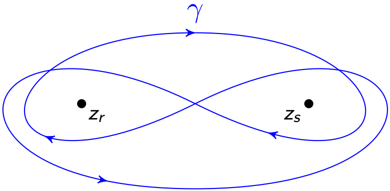
$$\mathcal{P}(z) = \prod_{r=1}^{N} (z - z_r)^{k_r}\,, \qquad \partial_z \mathcal{P}(z) = \varphi^{(\mathrm{qt})}(z)\, \mathcal{P}(z)$$

- Satisfies $\mathcal{P}(z)^{-p/h^\vee} D_{z,p} f(z) = \partial_z \big(\mathcal{P}(z)^{-p/h^\vee} f(z) \big)$

- γ in $P = \{$closed contours on which $\mathcal{P}(z)$ has a single-valued branch$\}$:

$$\oint_\gamma \mathcal{P}(z)^{-p/h^\vee} D_{z,p} f(z)\, \mathrm{d}z = 0$$

- Pochhammer contours:



Quantum quadratic charges

- Quantum quadratic densities and charges: $\gamma \in P$

$$\mathsf{Q}^1_\gamma = \int_0^{2\pi} \mathsf{W}_{2,\gamma}(x)\,\mathrm{d}x\,, \qquad \mathsf{W}_{2,\gamma}(x) = \oint_\gamma \mathcal{P}(z)^{-1/h^\vee}\,\mathsf{S}_2(z,x)\,\mathrm{d}z$$

- Since $\mathsf{S}_2(z,x)\mathsf{J}^{\mathrm{diag}}(y) = D_{z,1}(\cdots) + \mathrm{reg}$:

$$\mathsf{W}_{2,\gamma}(x)\mathsf{J}^{\mathrm{diag}}(y) = \mathrm{reg}\,, \qquad \text{hence} \qquad \mathsf{W}_{2,\gamma}(x) \in \mathsf{W}$$

- From OPE $\mathsf{S}_2(z,x)\mathsf{S}_2(w,y)$:

$$\big[\mathsf{Q}^1_\gamma, \mathsf{Q}^1_{\gamma'}\big] = 0\,, \qquad \forall\, \gamma,\gamma' \in P$$

- Pochhammer integrals reminiscent of [Lukyanov '13, Bazhanov Lukyanov '13] (ODE/IQFT correspondence for the Fateev integrable structure)

Higher-degree quantum charges: conjecture

- **Conjecture:** for every $p \in E$, there exist

$$\mathsf{S}_{p+1}(z,x) = \tau_p^{a_1\cdots a_{p+1}} \; :\Gamma^{(\mathrm{qt})}_{a_1}(z,x)\cdots\Gamma^{(\mathrm{qt})}_{a_{p+1}}(z,x): \; + \; \dots$$

with quantum corrections built from $\partial_z^\alpha \partial_x^\beta \Gamma^{(\mathrm{qt})}_a(z,x)$, such that

1. $\mathsf{S}_{p+1}(z,x)\mathsf{S}_{q+1}(w,y) = \sum_{k\geq 0} \dfrac{\mathsf{A}^{(k)}_{p,q}(z,w\,;y)}{(x-y)^{k+1}} + \text{reg}$
   with $\mathsf{A}^{(0)}_{p,q}(z,w\,;y) = \partial_y(\cdots) + D_{z,p}(\cdots) + D_{w,q}(\cdots)$
2. $\mathsf{S}_{p+1}(z,x)\mathsf{J}^{\text{diag}}(y) = D_{z,p}(\cdots) + \text{reg}$

- **Consequence:** commuting charges with gauge-invariant densities

$$\mathsf{Q}^p_\gamma = \int_0^{2\pi} \mathsf{W}_{p+1,\gamma}(x)\,\mathrm{d}x\,, \qquad \mathsf{W}_{p+1,\gamma}(x) = \oint_\gamma \mathcal{P}(z)^{-p/h^\vee}\,\mathsf{S}_{p+1}(z,x)\,\mathrm{d}z$$

Higher-degree quantum charges: first checks

- First checks of the conjecture:
  - quadratic $\mathsf{S}_2(z,x)$ for any $\mathfrak{g}$
  - cubic $\mathsf{S}_3(z,x)$ for $\mathfrak{g}=\mathfrak{sl}(k)$, $k>2$ [SL Vicedo Young '18]
  - quartic $\mathsf{S}_4(z,x)$ for $\mathfrak{g}=\mathfrak{sl}(2)$ [Kotousov SL Teschner '22, Franzini Young '22]

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$$\mathsf{S}_4(z) = \tau_3^{abcd} \; :\Gamma_a(z)\Gamma_b(z)\Gamma_c(z)\Gamma_d(z): + \frac{5\mathrm{i}}{4} f^{abc} \; :\partial_x\Gamma_a(z)\partial_z\Gamma_b(z)\Gamma_c(z):$$
$$+ \frac{\eta^{ab}}{48}\Big(45\,\varphi(z)^2 \; :\partial_x\Gamma_a(z)\partial_x\Gamma_b(z): - 140 \; :\partial_z^2\partial_x\Gamma_a(z)\partial_x\Gamma_b(z): - 30 \; :\partial_z\partial_x\Gamma_a(z)\partial_z\partial_x\Gamma_b(z): \Big)$$
$$+ \frac{5\eta^{ab}}{12}\Big(3 \; :\partial_z\partial_x^2\Gamma_a(z)\partial_z\Gamma_b(z): - \; :\partial_z^2\partial_x^2\Gamma_a(z)\Gamma_b(z): \Big)$$

Classical limit

$$\mathrm{W}_{p+1,\gamma}(x) = \oint_\gamma \mathcal{P}(z)^{-p/h^\vee} \mathrm{S}_{p+1}(z,x)\,\mathrm{d}z \qquad \text{with}$$

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- Classical limit: with $\rho'(z) = \varphi(z)$

$$\Gamma^{(\mathrm{qt})}(z,x) = \frac{2\pi\,\Gamma(z,x) + O(\hbar)}{\mathrm{i}\hbar}, \qquad \mathcal{P}(z) = \exp\left(\frac{2\pi}{\hbar}\big(\rho(z) + O(\hbar)\big)\right)$$

$$\mathrm{W}_{p+1,\gamma}(x) \propto \oint_\gamma \exp\left(-\frac{2\pi p}{h^\vee\hbar}\big(\rho(z) + O(\hbar)\big)\right)\Big(\mathcal{S}_{p+1}(z,x) + O(\hbar)\Big)\mathrm{d}z$$

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- In the limit $\hbar \to 0$, saddle-point approximation: integral localises at the extrema of $\rho(z)$, *i.e.* the zeroes ζ_i of $\varphi(z)$

  $\rightarrow$ $\mathsf{W}_{p+1,\gamma}(x)$ yields a linear combination of $\mathcal{S}_{p+1}(\zeta_i, x)$

Spectrum of the local charges

- Spectrum of the charges Q^p_γ on representations of $\widehat{\mathfrak{g}}^{\oplus N}$?
- Quantisation of chiral AGMs related to conjectured "affine" Langlands geometric correspondence [Feigin Frenkel '07, Frenkel Hernandez '16, SL Vicedo Young '18, Gaiotto Lee Vicedo Wu '20, Kotousov Lukyanov '21]

$\Updownarrow$ conjecture

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- Algebraic Bethe ansatz for chiral AGMs (acting on highest-weight representations): precise conjectures for the eigenvectors and eigenvalues of Q_γ^p and first checks
[Schechtman Varchenko '91, Feigin Frenkel '07, SL Vicedo Young '18 '18]

Example: Fateev integrable structure

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- Two punctures $N = 2$ and $\mathfrak{g} = \mathfrak{sl}(2)$

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- Recover various integrable hierarchies: Fateev, KdV, Drinfeld-Sokolov, ...
- Generalisations: higher-order poles, cyclotomic models, ...
- Relations to toroidal algebras, affine Yangians, 4d-Chern-Simons, ...
- Applications to integrable sigma-models [Teschner's talk]

Thank you for your attention!

Spectrum of the local charges

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- Quantisation of chiral AGMs related to conjectured "affine" Langlands geometric correspondence [Feigin Frenkel '07, Frenkel Hernandez '16, SL Vicedo Young '18, Gaiotto Lee Vicedo Wu '20, Kotousov Lukyanov '21]
- Common eigenvector of $\mathcal{Q}^p_\gamma$ $\Leftrightarrow$ affine ${}^L\widehat{\mathfrak{g}}$-oper (differential operator depending on $\text{rk}(\mathfrak{g})$ functions $v_k(z)$, *e.g.* $-\partial_z^2 + v(z) + \chi\,\mathcal{P}(z)$ for $\mathfrak{g} = \mathfrak{sl}_2$)
- Conjectured eigenvalue of $\mathcal{Q}^p_\gamma$: $u_p(z)$ built from $v_k(z)$

$$I^p_\gamma = \oint_\gamma \mathcal{P}(z)^{-p/h^\vee}\, u_p(z)\, \mathrm{d}z$$

- ODE/IQFT correspondence [Dorey Tateo '99, Bazhanov Lukyanov Zamolodchikov '03, Lukyanov '13, Bazhanov Lukyanov '13, ...]

Bethe ansatz

[Schechtman Varchenko '91, Feigin Frenkel '07, SL Vicedo Young '18 '18]

- Hilbert space: tensor product of N highest-weight representations of $\widehat{\mathfrak{g}}$
- Bethe vector with M excitations: $\Psi_{\alpha_1,\ldots,\alpha_M}(w_1,\ldots,w_M)$
  ($\{\alpha_j\}$ are simple roots of $\widehat{\mathfrak{g}}$, $\{w_j\} \subset \mathbb{C}$ are the Bethe roots)
- Conjecture:
  - eigenvector of $\mathrm{Q}^p_\gamma \Leftrightarrow \{w_j\}$ satisfy Bethe ansatz equations
  - associated affine oper $\Leftrightarrow$ Miura oper with certain regularity properties
- First checks:
  - proven for quadratic Q^1_γ, for any $\mathfrak{g}$, any N and any M
  - proven for cubic Q^2_γ for $\mathfrak{g} = \mathfrak{sl}(k)$ $(k > 2)$, any N and $M = 0, 1, 2$

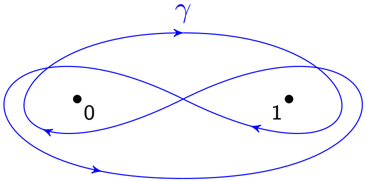
Example: the Fateev integrable structure

Chiral AGM with $N = 2$ currents

- Take $N = 2$, *i.e.* 2 punctures in the complex plane
- By translation and dilation of z, we can fix $z_1 = 0$ and $z_2 = 1$

$$\Gamma^{(\mathrm{qt})}(z,x) = \frac{\mathsf{J}^{(1)}(x)}{z} + \frac{\mathsf{J}^{(2)}(x)}{z-1} \qquad \text{and} \qquad \mathcal{P}(z) = z^{k_1}(z-1)^{k_2}$$

- Local observables: $(\widehat{\mathfrak{g}}_{k_1} \oplus \widehat{\mathfrak{g}}_{k_2}) / \widehat{\mathfrak{g}}^{\text{diag}}_{k_1+k_2}$ coset $\mathcal{W}$-algebra
- One Pochhamer contour γ and Euler B-function integrals:



$$B(a,b) = \oint_\gamma z^a (z-1)^b \,\mathrm{d}z$$

Quadratic density

$$\Gamma^{(\mathrm{qt})}(z,x) = \frac{\mathrm{J}^{(1)}(x)}{z} + \frac{\mathrm{J}^{(2)}(x)}{z-1} \qquad \text{and} \qquad \mathcal{P}(z) = z^{k_1}(z-1)^{k_2}$$

- Quadratic density:

$$\mathrm{W}_{2,\gamma}(x) = \eta^{ab} \oint_\gamma \mathcal{P}(z)^{-1/h^\vee} \; :\Gamma_a^{(\mathrm{qt})}(z,x)\,\Gamma_b^{(\mathrm{qt})}(z,x):\; \mathrm{d}z$$

- Explicitely computed using Euler B-function:

$$\mathrm{W}_{2,\gamma}(x) \propto \frac{\eta^{ab}}{2}\left(\frac{:\mathrm{J}_a^{(1)}(x)\mathrm{J}_b^{(1)}(x):}{k_1+h^\vee} + \frac{:\mathrm{J}_a^{(2)}(x)\mathrm{J}_b^{(2)}(x):}{k_2+h^\vee} - \frac{:\mathrm{J}_a^{\mathrm{diag}}(x)\mathrm{J}_b^{\mathrm{diag}}(x):}{k_1+k_2+h^\vee} \right)$$

- Recover the GKO coset energy-momentum tensor (quantum corrections by $h^\vee$ come from Pochhammer integrals)

Quartic density for $\mathfrak{g} = \mathfrak{sl}_2$

- Further specialise to $\mathfrak{g} = \mathfrak{sl}_2$
- $\widehat{\mathfrak{sl}}(2)_{k_1} \oplus \widehat{\mathfrak{sl}}(2)_{k_2} / \widehat{\mathfrak{sl}}(2)^{\text{diag}}_{k_1+k_2}$ coset: corner-brane $\mathcal{W}$-algebra [Fateev '96, Feigin Semikhatov '01, Lukyanov Zamolodchikov '12]:
  - one spin-2 field: energy-momentum tensor T
  - one spin-3 field: descendant $\partial\mathrm{T}$
  - three spin-4 fields: descendants $\partial^2\mathrm{T}$, $:\mathrm{T}^2:$ and primary $\mathrm{W}_{4,P}$

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  - three spin-4 fields: descendants ∂^2T, $:\mathrm{T}^2:$ and primary $\mathrm{W}_{4,P}$
- Quartic density from AGM construction: [Kotousov SL Teschner '22]

$$\mathrm{W}_{4,\gamma}(x) \propto \mathrm{W}_{4,P}(x) + \delta_1 :\mathrm{T}^2(x): + \delta_2 \, \partial^2 \mathrm{T}(x)$$

$$\delta_1 = \frac{15(k_1+2)(3k_1+4)(k_2+2)(3k_2+4)(k_1+k_2+2)(3k_1+3k_2+8)}{176 - 44(k_1+2)^2 - 44(k_2+2)^2 - k_1 k_2 (37k_1 + 37k_2 + 192)}$$

- Agrees with screening-charge computation in Fateev integrable structure [Fateev '96, Feigin Semikhatov '01, Lukyanov Zamolodchikov '12]