Constant mean curvature embeddings and the Fateev model IQFT

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Integrability in Condensed Matter Physics and IQFT

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The ODE/IQFT (ODE/IM,...) correspondence

[Voros'92; Dorey-Tateo'99; BLZ'98,03; LZ'10]

Classically integrable system (non-linear PDE ) $\Longleftrightarrow$ any $\hbar$ Spectral problem in integrable QFT

Special case explored in [Bazhanov, GK, Lukyanov '13,'14]:

– Modified sinh-Gordon PDE and integrable QFT formulated by Fateev'96 (first formulated in [Lukyanov'13])

Plan:

- Constant mean curvature embedding of punctured Riemann sphere into AdS_3 and its numerical solution
- Relation to the Fateev model

Why Fateev model?

- More transparent mathematical structures

Fateev model	$\sim$	Hypergeometric equation (Fuchsian ODE with 3 regular singular points)
KdV/sin(h)-Gordon model	$\sim$	Confluent hypergeometric equation (irregular singularity at ∞)

- Fateev model in certain parameter domain dual to integrable deformation of $\mathrm{SU}(2)$ Principal Chiral Field

ODE/IQFT approach to integrable non-linear sigma models

- First principles quantization of integrable non-linear sigma models initiated in [Polyakov, Wiegmann'83; Faddeev, Reshetikhin'86]

  $\approx$ 30 year pause due to non-ultralocality problem [Maillet'86]

- Fateev model = starting point for developing a new approach to sigma models based on ODE/IQFT correspondence
  [Bazhanov, GK, Lukyanov'14 '17'18; GK, Lacroix, Teschner '22]

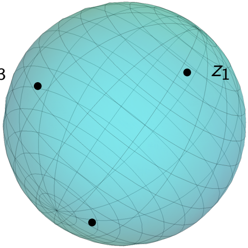
- Potential applications to:

  (i) Mathematical physics: asymptotic freedom, large N expansion, dualities, instanton counting and resurgence (e.g., Bajnok's talk at this conference) ...

  (ii) Condensed Matter theory: main theoretical approach to disordered electronic systems [Efetov'96]

  (iii) High Energy Theory: AdS/CFT correspondence, e.g., complement existing techniques [Minahan,Zarembo '02; Gromov,Kazakov,Leurent,Volin'14]

Metric for the punctured sphere



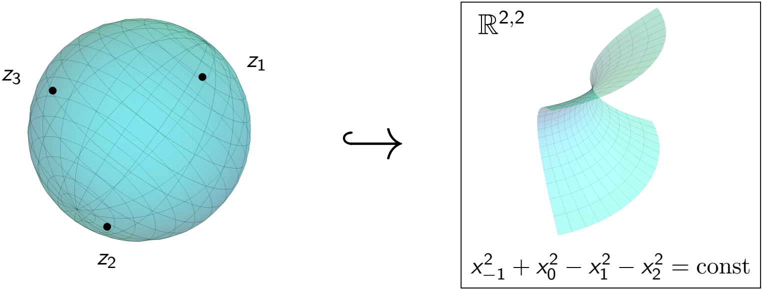
- Flat metric

$$(\mathrm{d}s)^2 = |p(z)|\,\mathrm{d}z\mathrm{d}\bar{z}\,, \qquad p(z) = \rho^2\,(z-z_1)^{a_1-2}\,(z-z_2)^{a_2-2}(z-z_3)^{a_3-2}$$

$z = z_j$: conical singularities with angle deficit $0 < (2-a_j)\,\pi < 2\pi$

- If $a_1 + a_2 + a_3 = 2$ then $p(z)\,(\mathrm{d}z)^2$ is quadratic differential under Möbius transformations

Constant mean curvature embedding into AdS_3



Embedding problem leads to Modified sinh-Gordon equation (MshG) [Pohlmeyer'76; Bobenko'91; de Vega and Sanchez'93; Alday, Maldacena'09; Dorey, Dunning, Negro, Tateo'20]

$$\partial_z \partial_{\bar{z}} \eta - \mathrm{e}^{2\eta} + |p(z)|^2 \, \mathrm{e}^{-2\eta} = 0$$

for induced metric $\propto \mathrm{e}^{2\eta} \, \mathrm{d}z \mathrm{d}\bar{z}$

Sinh-Gordon equation

$$\partial_z \partial_{\bar{z}} \eta - \mathrm{e}^{2\eta} + |p(z)|^2 \, \mathrm{e}^{-2\eta} = 0 \, ,$$

Redefine

$$\hat{\eta} = \eta - \tfrac{1}{2} \log |p(z)|$$

Apply Schwarz-Christoffel map

$$z \to w \qquad \text{with} \qquad w = \int \mathrm{d}z \sqrt{p(z)}$$

PDE becomes ordinary sinh-Gordon equation

$$\partial_w \partial_{\bar{w}} \hat{\eta} = 2 \sinh(\hat{\eta})$$

but on different domain

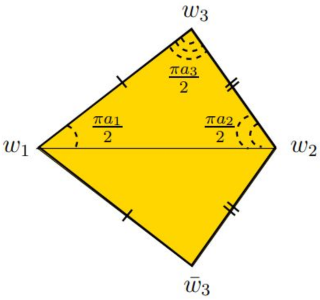
Domain

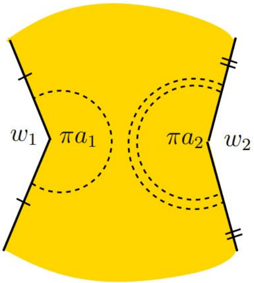
– (I) Compact case:

$$0 < a_1, a_2, a_3 < 2$$

– (II) Non-compact case:

$$0 < a_1, a_2 \text{ and } a_3 < 0$$





Formulation of the problem

$$\partial_z \partial_{\bar{z}} \eta = e^{2\eta} - |p(z)|^2 e^{-2\eta}\,, \qquad p(z) = \rho^2 \prod_{j=1}^{3} (z - z_j)^{a_j - 2}$$

Supplement with:

(I) Compact case: $0 < a_1, a_2, a_3 \leq 2$

$$\eta = 2m_j \log|z - z_j| + O(1) \quad \text{as} \quad z \to z_j \quad (j = 1, 2, 3)$$

$$\eta = -2 \log|z| + O(1) \quad \text{as} \quad z \to \infty$$

m_j real such that $m_j \in \big(-\frac{1}{2}, -\frac{1}{4}(2 - a_j)\big]$

(II) Non-compact case: $a_1, a_2 > 0$ and $a_3 < 0$

Send $z_3 \to \infty$ and specify:

$$\eta = 2m_j \log|z - z_j| + O(1) \quad \text{as} \quad z \to z_j \quad (j = 1, 2)$$

$$\eta = \frac{1}{2} \log|p(z)| + o(1) \quad \text{as} \quad z \to \infty$$

(Regularized) surface area

All geometric properties of embedding can be computed from η, e.g.,

Surface area of embedded surface

$$\mathit{Vol} = \int_\Sigma \mathrm{d}^2 z \, \sqrt{-\det g} \propto \int_\Sigma \mathrm{d}^2 z \, \mathrm{e}^{2\eta} = \int_\Sigma \mathrm{d}^2 z \, |p(z)|^2 \mathrm{e}^{-2\eta} - \pi \, (1 + m_1 + m_2 + m_3)$$

$$\text{(!!!)} \qquad \mathrm{e}^{2\eta} = |p(z)| \big(1 + o(1)\big) \sim |z|^{-2-a_3} \qquad \text{as} \qquad z \to \infty$$

$\Longrightarrow$ For non-compact case *Vol* diverges with $a_3 < 0$

Solution: define *regularized volume* by subtracting $|p(z)|$ from integrand

$$\mathfrak{F} = -\frac{2}{\pi} \int \mathrm{d}^2 z \, \left(\mathrm{e}^{\eta} - \sqrt{|p(z)|} \, \mathrm{e}^{-\eta}\right)^2 + \text{constant}$$

'constant' fixed by some renormalization condition

Numerical solution (focus on compact case)

$$\underbrace{\partial_z\partial_{\bar z}\eta}_{\text{Laplacian}} = \mathrm{e}^{2\eta} - |p(z)|^2\,\mathrm{e}^{-2\eta} \qquad \sim \qquad \text{Poisson's equation}$$

Green's function for 2D Laplacian:

$$G(z,z') = \frac{2}{\pi}\,\log|z-z'|\,, \qquad\qquad \partial_z\partial_{\bar z}G(z,z') = \delta(z-z')$$

one can re-write differential equation as non-linear integral equation:

$$\eta(z) = \int_\Sigma \mathrm{d}^2z\,G(z-z')\left[\mathrm{e}^{2\eta(z')} - |p(z')|^2\,\mathrm{e}^{-2\eta(z')}\right] + \eta_\infty + \sum_{j=1}^{3} 2m_j\,\log|z-z_j|$$

- Σ – thrice punctured Riemann sphere
- η_∞ - constant in subleading asymptotics

$$\eta(z) = -2\log|z| + \eta_\infty + o(1) \qquad \text{as} \qquad z\to\infty$$

Numerical solution

Define $\psi(z) = \eta(z) - \eta_\infty$ and re-write PDE:

$$\partial_z \partial_{\bar{z}} \psi(z) = \mathrm{e}^{2\psi(z) + 2\eta_\infty} - |p(z)|^2 \, \mathrm{e}^{-2\eta - 2\eta_\infty}$$

To determine η_∞ integrate both sides over punctured sphere

$$\underbrace{\int_\Sigma \mathrm{d}^2 z \, \partial_z \partial_{\bar{z}} \, \psi}_{= -\pi(1+m_1+m_2+m_3) \equiv A} = \mathrm{e}^{2\eta_\infty} \underbrace{\int_\Sigma \mathrm{d}^2 z \, \mathrm{e}^{2\psi(z)}}_{\equiv B} - \mathrm{e}^{-2\eta_\infty} \underbrace{\int_\Sigma \mathrm{d}^2 z |p(z')|^2 \, \mathrm{e}^{-2\psi(z')}}_{\equiv C}$$

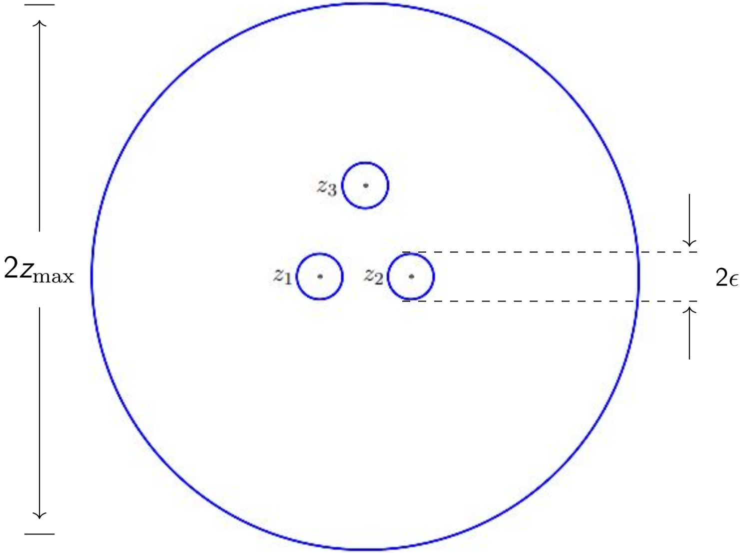
$\Longrightarrow$ quadratic equation for $\mathrm{e}^{2\eta_\infty}$

Solve and choose positive real root:

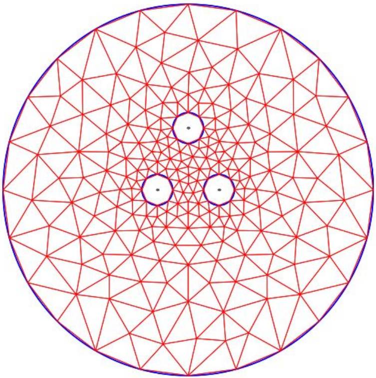
$$\eta_\infty = \frac{1}{2} \log \left(\frac{A + \sqrt{A^2 + 4BC}}{2B} \right) .$$

On this basis develop iteration procedure to solve non-linear integral eqs

Domain for numerical integration: $\Sigma \mapsto \Sigma_{\epsilon, z_{\max}}$



Domain for numerical integration



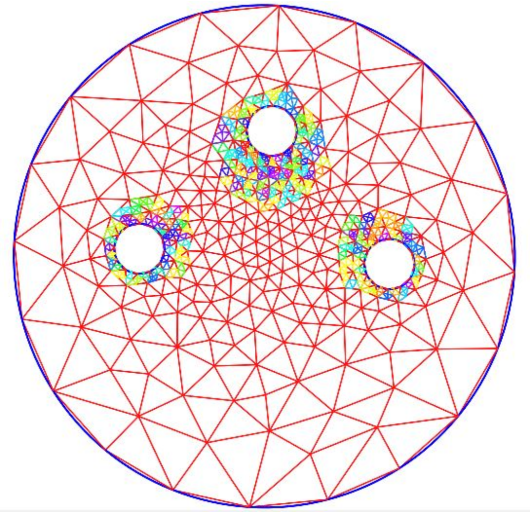
$$\int_{\Sigma_{\epsilon,R}} f(x) = \frac{1}{3} \sum_{\text{triangles}} \text{Area}\left(j^{\text{th}} \text{ triangle}\right) \times \left[f(v_{j,1}) + f(v_{j,2}) + f(v_{j,3})\right]$$

$v_{j,1}$ = vertex of triangle j

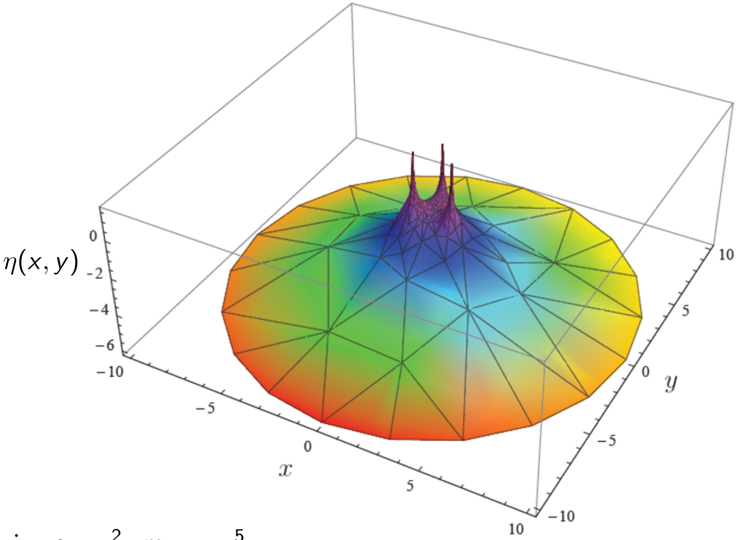
Triangulation done by GMSH finite mesh generator [Geuzaine, Remacle'09]

Adaptive meshing

At n^{th} step refine mesh based on growth of function:

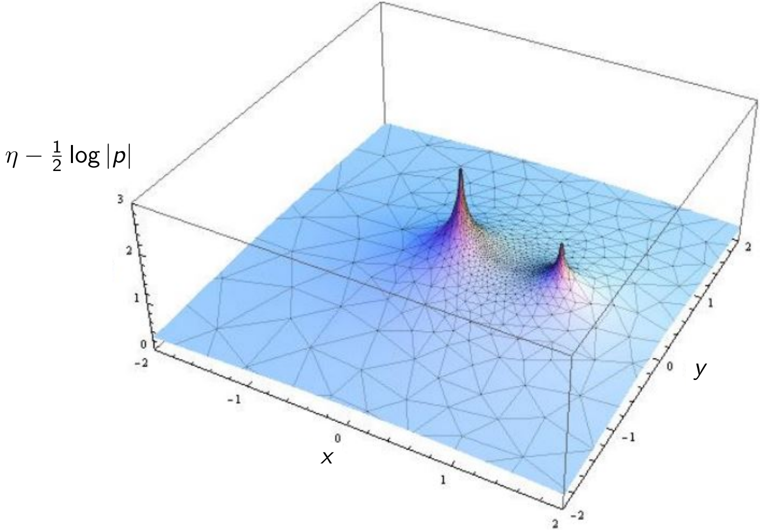


Solution for compact case



$z = x + \mathrm{i}y,\ a_j = \frac{2}{3},\ m_j = -\frac{5}{12}$

Solution for non-compact case



$z = x + \mathrm{i}y,\ a_1 = 1.7,\ a_2 = 1.5,\ m_1 = -0.3,\ m_2 = -0.25$

(2000 CPU hours total)

Up till now:

Constant mean curvature embedding problem

Next:

The Fateev model

The Fateev model [Fateev'95]

Integrable affine Toda type theory in $1+1$D with 3 bosonic fields

$$\mathcal{L} = \frac{1}{16\pi}\sum_{j=1}^{3} \partial_\mu \varphi_j \, \partial^\mu \varphi_j + 2\mu \left(e^{i\alpha_3\varphi_3} \cos(\alpha_1\varphi_1 + \alpha_2\varphi_2) + e^{-i\alpha_3\varphi_3} \cos(\alpha_1\varphi_1 - \alpha_2\varphi_2) \right)$$

μ – mass, α_j – couplings

$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \frac{1}{2}$: infinite number of conservation laws labeled by Lorentz spin

$$\partial_- T_{2n} = \partial_+ \Theta_{2n-2}\,, \qquad \partial_+ T_{-2n} = \partial_- \Theta_{2-2n}\,, \qquad \left(\partial_\pm = \tfrac{1}{2}\left(\partial_t \pm \partial_x\right)\right)$$

Finite volume $x \sim x + R$ (spacetime cylinder)

$$\mathbb{I}_{2n-1} = \int_0^R dx \left(T_{2n} - \Theta_{2n-2} \right), \qquad \bar{\mathbb{I}}_{2n-1} = \int_0^R dx \left(T_{-2n} - \Theta_{2-2n} \right)$$

IMs conserved and commute with each other

$$[\mathbb{I}_{2n-1}, \mathbb{I}_{2m-1}] = [\bar{\mathbb{I}}_{2n-1}, \mathbb{I}_{2m-1}] = [\bar{\mathbb{I}}_{2n-1}, \bar{\mathbb{I}}_{2m-1}] = 0$$

Regimes

- Compact case: $\alpha_j \in \mathbb{R}$

  Periodic potential

  $$= 2\mu \left(e^{i\alpha_3\varphi_3} \cos(\alpha_1\varphi_1 + \alpha_2\varphi_2) + e^{-i\alpha_3\varphi_3} \cos(\alpha_1\varphi_1 - \alpha_2\varphi_2)\right)$$

  Space of states splits onto twisted sectors labeled by quasi-momenta:

  $$\varphi_j \mapsto \varphi_j + 2\pi/\alpha_j \ : \qquad \Psi_{\mathbf{k}}[\varphi_1, \varphi_2, \varphi_3] \mapsto e^{2\pi i k_j}\, \Psi_{\mathbf{k}}[\varphi_1, \varphi_2, \varphi_3]\,.$$

  with $|k_j| < \frac{1}{2}$.

  Fields φ_j taken to belong to compact segment $[0, 2\pi)$.

- Non-compact case: $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\alpha_3 = ib$ pure imaginary

  Model can be reformulated as a 1+1 dimensional sigma model with target space two parameter deformation of $\mathbb{S}^3$

Non linear sigma model

$$\mathcal{L} = G_{ab}(X)\,\partial_\mu X^a \partial^\mu X^b$$

$X^a = X^a(t, x)$ – coordinates on $\mathbb{S}^3$

G_{ab} – asymetric 2 parameter deformation of three sphere metric

Special cases:

- 2D Sausage (deformed $\mathrm{O}(3)$ model) [Fateev,Onofri,Zamolodchikov'93]

  Target space:

- Anisotropic $\mathrm{SU}(2)$ principal chiral field (squashed sphere or def. $\mathrm{O}(4)$ model)

  Target space:

. . .  . . .

Quantum/classical cor. [Bazhanov, GK, Lukyanov '13;'14]

- Quantum problem: computation of vacuum energy in twisted sectors

$$\mathbb{H}\,|vac_{\mathbf{k}}\rangle = E_{\mathbf{k}}^{(\mathrm{vac})}(\mu, R)\,|vac_{\mathbf{k}}\rangle$$

R - compactification radius

$E_{\mathbf{k}}^{(\mathrm{vac})} = \frac{1}{R}\times$ function of dimensionless combination (μR)

- Classical problem: Area of embedded surface in AdS_3

$$\mathfrak{F} = -\frac{2}{\pi}\int \mathrm{d}^2 z\, \left(\mathrm{e}^{\eta} - \sqrt{|p(z)|}\,\mathrm{e}^{-\eta}\right)^2 + \sum_{j=1}^{3} 2\,(m_j + \tfrac{1}{2}) - \tfrac{a_j}{2}$$

Renormalization condition $\lim_{\rho\to\infty}\mathfrak{F} = 0$

Quantum/classical cor. [Bazhanov, GK, Lukyanov '13;'14]

- Quantum/classical correspondence:

$$\frac{R}{\pi}\, E_{\mathbf{k}}^{(\mathrm{vac})} = \mathfrak{F} - 4\rho^2 \prod_{j=1}^{3} \frac{\Gamma(a_j/2)}{\Gamma(1-a_j/2)}$$

Identification of parameters:

$$\rho = \mu R\,, \qquad a_j = 4\alpha_j^2\,, \qquad |k_j| = \tfrac{2}{a_j}\,(m_j + \tfrac{1}{2})$$

- Non-compact case:

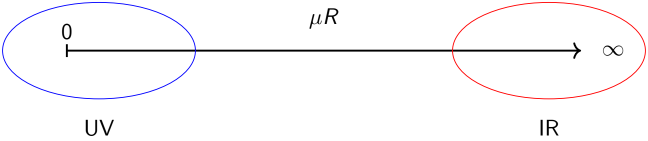
$$a_1, a_2 > 0,\ a_3 < 0 \quad \Longleftrightarrow \quad \alpha_1, \alpha_2 \text{ real and } \alpha_3 \text{ pure imaginary}$$

Solution to PDE has two punctures in finite plane $\eta \sim 2m_j \log|z - z_j|$ $\Longleftrightarrow$

Twisted sectors labeled by two quasi-momenta $\mathbf{k} = (k_1, k_2)$

Checks

$E_{\mathbf{k}}^{(\mathrm{vac})}$ may be computed in two important limits:



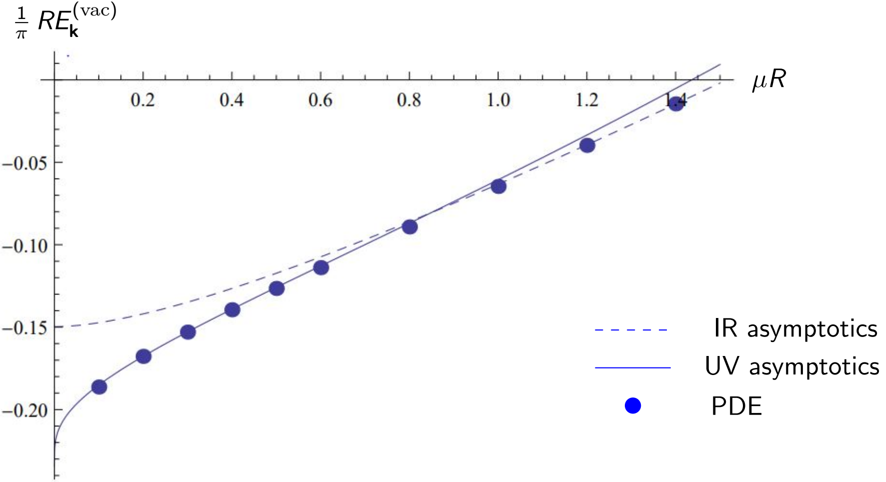
- IR: Lüscher corrections [Lüscher '86] coming from exact S-matrix
- UV: conformal perturbation theory
- Special cases with $\mathbf{k} = 0$:

  – $4\alpha_1^2, 4\alpha_2^2 = 2, 3, \ldots$ (more generally rational) TBA derived in [Fateev'96]

  – Any $\alpha_1^2, \alpha_2^2 \geq 0$ NLIE obtained in [Hegedus'03] along lines of [Klumper, Batchelor, Pearce '91; Destri, de Vega '95]

  – $\mathrm{O}(4)$ model studied in [Balog, Hegedus '04; Gromov, Kazakov, Viera '09; Caetano '10]

Results ($a_1 = 1.7$, $a_2 = 1.5$, $k_1 = \frac{4}{17} = 0.235\ldots$, $k_2 = \frac{1}{3}$)



Future directions – excited states

Zero-curvature representation for the Modified sinh-Gordon equation:

$$\partial_z \partial_{\bar{z}} \eta - \mathrm{e}^{2\eta} + |p(z)|^2 \, \mathrm{e}^{-2\eta} = 0 \qquad \Longleftrightarrow \qquad \big[\partial_z - \boldsymbol{A}_z\{\eta\}\,,\, \partial_{\bar{z}} - \boldsymbol{A}_{\bar{z}}\{\eta\}\big] = 0$$

Find all η such that monodromy properties of linear system are unchanged

Apart from $\eta \propto \log|z - z_j|$ at $z = z_j$, additional "vortex" singularities allowed:

$$\eta = \log\left(\frac{z - x_a}{\bar{z} - \bar{x}_a}\right) + O(1)\,, \qquad \eta = \log\left(\frac{\bar{z} - \bar{y}_b}{z - y_b}\right) + O(1)$$

where $\{x_1, \ldots, x_{\mathrm{L}}\}$, $\{y_1, \ldots, y_{\bar{\mathrm{L}}}\}$ obey (transcendental) system with discrete solution set

Correspondence

$$\text{states } |\boldsymbol{v}\rangle \quad \leftrightarrow \quad \eta^{(\boldsymbol{v})} \qquad \text{such that} \qquad \frac{R}{\pi}\, E_{\mathbf{k}}^{(\boldsymbol{v})} = \mathfrak{F}\{\eta\} + \ldots$$

Conclusion

In this talk:

- Minimal surface embedding problem $\Sigma \mapsto \mathrm{AdS}_3$
- Modified sinh-Gordon equation and its numerical solution
- (Regularized) area of embedded surface $\leftrightarrow$ vacuum energy of Fateev model

Future directions:

- Excited states: classification of all solutions η subject to certain 'monodromy free' condition
- Geometric interpretation
- Problem may be studied in context of ODE/IQFT correspondence for sinh-Gordon QFT using excited states TBA derived in [Teschner'07]