Light-cone and double-scaling limits of rectangular fishnets

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Integrability in Condensed Matter Physics and Quantum Field Theory

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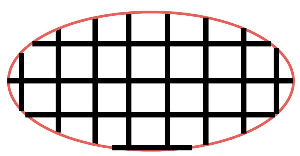
Basso-Dixon integral for rectangular fishnets

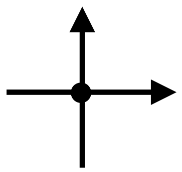
Fishnet Feynman graphs in 4d, or "fishnets", are integrable



[**Alexander Zamolodchikov 1980**, Gürdogan-Kazakov 2015,

Basso, Caetano, Derkachov, Dixon, Fleury, Gromov, Kazakov, Korchemsky, Negro, Olivucci, Preti, Sever, Sizov, Zhong, …]

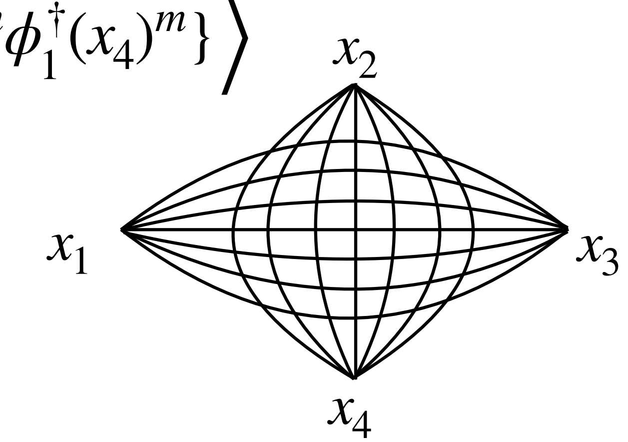
Fishnet QFT: - 4d planar massless QFT of two complex matrix fields $\phi_1(x), \phi_2(x)$ with non-unitary interaction $\mathrm{Tr}\{\phi_1(x)\phi_2(x)\phi_1^\dagger(x_3)\phi_2^\dagger(x_4)\}$



Rectangular fishnets - particular case of open fishnets, special 4-point correlators in the fishnet CFT:

$$G_{m,n}(x_1, x_2, x_3, x_4) = \left\langle \mathrm{Tr}\{\phi_2(x_1)^n\phi_1(x_2)^m\phi_2^\dagger(x_3)^n\phi_1^\dagger(x_4)^m\} \right\rangle$$

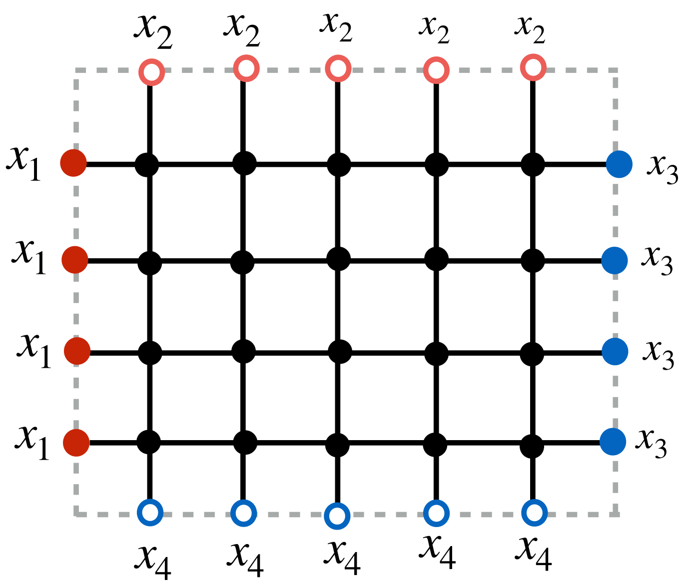
= single planar graph



computed by Basso and Dixon from integrability

[B. Basso, L. Dixon 1705.03545]

Basso-Dixon integral for rectangular fishnets



Can be viewed as a lattice model defined on a rectangle with four different Dirichlet b.c. on the edges

- Fluctuation variable $x \in \mathbb{R}^4$,
- nearest-neighbour interaction $|x-y|^{-2}$

$$G_{m,n}(x_1,x_2,x_3,x_4) = \int_{\mathbb{R}^4} \prod_{r\bullet \in \text{bulk}} d^4x(r) \prod_{r\bullet\!\!-\!\!\!-\!\!\bullet r'} \frac{1}{|x(r)-x(r')|^2}$$

Exactly solvable open spin chain with $SO(1,5)$ symmetry
[Derkachov-Olivucci, 2020], using the techniques in [Derkachov-Korchemsky-Manashov,2001].

Continuum limit, if exists, is different from that for cylindrical fishnets
[Basso-Zhong, Gromov-Sever]

Conformal symmetry

$$G_{m,n}(x_1, x_2, x_3, x_4) = \langle \mathrm{Tr}\{\phi_2^n(x_1)\phi_1^m(x_2)\phi_2^{\dagger n}(x_3)\phi_1^{\dagger m}(x_4)\}\rangle$$

is a correlation function of spinless fields with dimensions $\Delta_2 = \Delta_4 = m,\ \Delta_1 = \Delta_3 = n$

By the conformal invariance, the correlator depends, up to a standard factor, on the positions x_1, x_2, x_3, x_4 through the two conformal invariant cross ratios (cf Volker's talk)

$$U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = \frac{z\bar{z}}{(1-z)(1-\bar{z})}, \qquad V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = \frac{1}{(1-z)(1-\bar{z})}$$

By conformal transformation $x_1 = (0,0),\ x_2 = (z, \bar{z}),\ x_3 = (\infty, \infty),\ x_4 = (1,1)$

Parametrisation by hyperbolic angles: $z = -e^{-\sigma-\varphi}, \qquad \bar{z} = -e^{-\sigma+\varphi}$

(in Minkowski kinematics $\sigma, \varphi \in \mathbb{R}$)

$$G_{m,n}(x_1, x_2, x_3, x_4) = \frac{g^{2mn}}{(x_{13}^2)^n (x_{24}^2)^m} \times I_{m,n}^{\mathrm{BD}}(z, \bar{z})$$

Basso-Dixon integral

“BMN integral representation”

Conjectured in [Basso-Dixon, 2017], proved in [Derkachov-Olivucci, 2020]

$$I^{\text{BD}}_{m,n}(z,\bar{z}) = (2\cosh\sigma + 2\cosh\varphi)^m \sum_{a_1,\ldots,a_m=1}^{\infty} \prod_{j=1}^{m} \frac{\sinh(a_i\varphi)}{\sinh\varphi}(-1)^{a_j-1} \int \prod_{j=1}^{m} \frac{du_j}{2\pi}\, e^{2i\sigma u_j}$$

$$\times \frac{\prod_{i=1}^{m} a_i \prod_{i<j}\left[(u_i-u_j)^2 + \frac{(a_i+a_j)^2}{4}\right]\left[(u_i-u_j)^2 + \frac{(a_i-a_j)^2}{4}\right]}{\left(u_j^2 + \frac{a_j^2}{4}\right)^{m+n}}$$

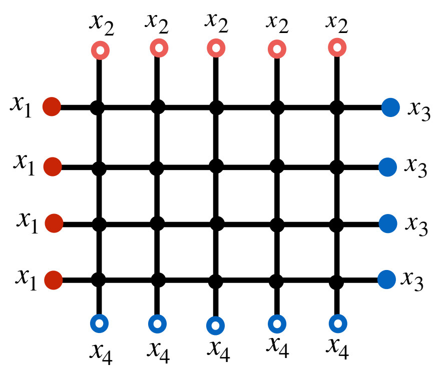
“Dual integral representation” [Basso-Dixon-Kosower-Krajenbrink-Zhong, 2021]

By Fourier transformation $u \to i\partial/\partial t,\ \partial/\partial u \to -it$, the discrete sum can be done explicitly

$$I^{\text{BD}}_{m,n}(z,\bar{z}) = \frac{1}{\mathcal{N}}\frac{1}{m!}\int_{|\sigma|}^{\infty}\prod_{j=1}^{m} dt_j\ t_j^{(n-m)^2-\sigma^2}\ \frac{\cosh\sigma + \cosh\varphi}{\cosh t_j + \cosh\varphi}\ \prod_{j,k=1}^{m}(t_j+t_k)\prod_{j<k}(t_j-t_k)^2$$

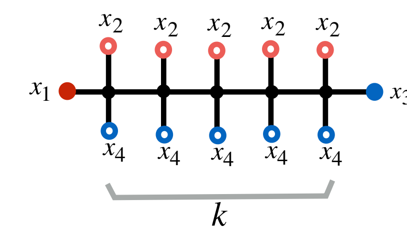
— Generalises the integral for the ladder diagrams ($m=1$) Broadhurst-Davydychev, 2010

Determinant representation (ladders glued into a fishnet) B. Basso, L. Dixon 1705.03545,



$$I^{\rm BD}_{m,m+\ell} = \frac{1}{\mathcal{N}} \det\left(\left[f_{j+k+\ell-1}\right]_{j,k=1,\dots,m}\right)$$

$$f_k(z,\bar z) = \int_{|\sigma|}^{\infty} \frac{\cosh\sigma + \cosh\varphi}{\cosh t + \cosh\varphi} (t^2-\sigma^2)^{n-1}\ 2t dt$$



k-ladder diagram

O(-2) matrix model: The B-D integral takes the form of the partition function of a certain $m \times m$ matrix model studied in the past:

$$I^{\rm BD}_{m,n} = \mathcal{Z}_m(\ell,\sigma,\varphi), \qquad \ell \equiv n-m \quad \text{- "bridge"}$$

$$\mathcal{Z}_m(\ell,\sigma,\varphi) = \frac{1}{\mathcal{N}} \frac{1}{m!} \int_{|\sigma|}^{\infty} \prod_{j=1}^{m} dt_j\ e^{-V(t_j)} \prod_{j,k=1}^{m} (t_j+t_k) \prod_{j<k}^{m} (t_j-t_k)^2$$

$$V(t) = \log \frac{\cosh t + \cosh\varphi}{\cosh\sigma + \cosh\varphi} - \ell \log(t^2-\sigma^2)$$

- unusual confining potential:

— grows slowly (linearly) at $t \to \pm\infty$;

— $V'(t)$ has an infinite array of simple poles on the imaginary axis.

Thermodynamical limit $(m \to \infty)$

• In the thermodynamical limit the "free energy" $\mathscr{F} = \log I^{\mathrm{BD}}_{m,n}$ grows as the "area" $mn = m(m+\ell)$.

In the "bulk" thermodynamical limit, $m, n \to \infty$ with fixed hyperbolic angles σ and φ , the free-energy density $\hat{\mathscr{F}} = \mathscr{F}/(mn)$ depends only on the aspect ratio m/n and **not** on the "boundary conditions" determined by σ and φ.

[Basso-Dixon-Kosower-Krajenbrink-Zhong, 2021]

• In the **scaling limit** $m, \sigma \to \infty$ with $\hat{\sigma} \sim \sigma/m$ finite, the saddle-point equation for the spectral density is equivalent to the Bethe equations for the Frolov-Tseytlin folded string rotating in $\mathrm{AdS}_3 \times S^1$ with $\{S, J\} = \{2m, \ell\}$

[Basso-Dixon-Kosower-Krajenbrink-Zhong, 2021]

THIS TALK:

• I will consider the **double scaling limit** $m, \sigma, \varphi \to \infty$ with $\hat{\sigma} \sim \sigma/m$, $\hat{\varphi} \sim \varphi/m$ finite. The interpretation in terms of Bethe equations still exists, but with unphysical choice of the mode numbers.

Saddle-point equations

At large argument, the derivative of the potential is approximated by a piecewise linear function:

$$V'(t) \underset{t\to\infty}{\to} \mathrm{sgn}(t)\,\theta\left(\,|\,t\,| - |\,\varphi\,|\,\right) \quad \Rightarrow \quad e^{2\pi i V'(t)} \to 1$$

Hence the saddle-point equations are equivalent modulo Bethe numbers to the BAE for a <u>symmetric</u> configuration of $2m$ magnons in a $\mathrm{XXX}_{-1/2}$ closed spin chain of length $J = \ell$

$$\left(\frac{t_j - \sigma^2/t_j + i\pi}{t_j - \sigma^2/t_j - i\pi}\right)^{\ell} \prod_{k\neq j}^{2m} \frac{t_j - t_k + 2\pi i}{t_j - t_k - 2\pi i} = 1, \qquad (j = 1,\ldots,2m)$$

The logarithmic form of the BAE is

$$\frac{2\ell\, t_j}{t_j^2 - \sigma^2} + \sum_{k\neq j}^{m} \frac{2}{t_j - t_k} + \sum_{k=1}^{m} \frac{2}{t_j + t_k} = n_j \qquad (j = 1,\ldots,m) \qquad \{t_j\} = \{-t_j\}$$

Bethe numbers $n_j = -\, n_{2m-j+1}$

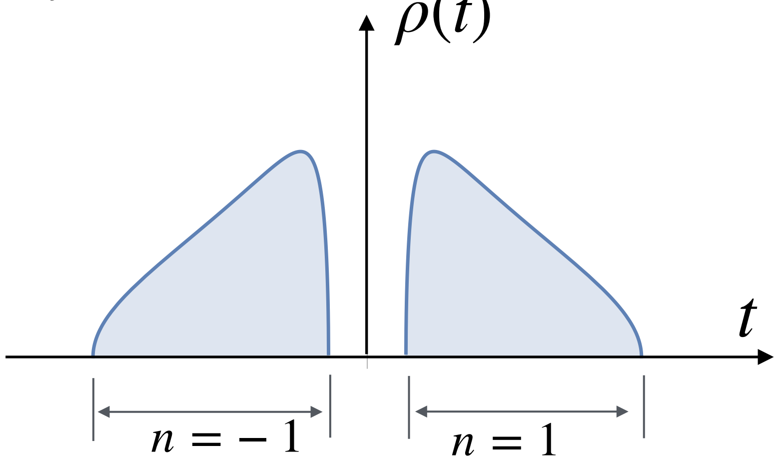
The BAE depend on the angle φ only through the Bethe numbers.

Two choices for the Bethe numbers

$$\frac{2\ell\, t_j}{t_j^2 - \sigma^2} + \sum_{k\neq j}^{m} \frac{2}{t_j - t_k} + \sum_{k=1}^{m} \frac{2}{t_j + t_k} = n_j \qquad (j = 1,\ldots,m) \qquad \{t_j\} = \{-t_j\}$$

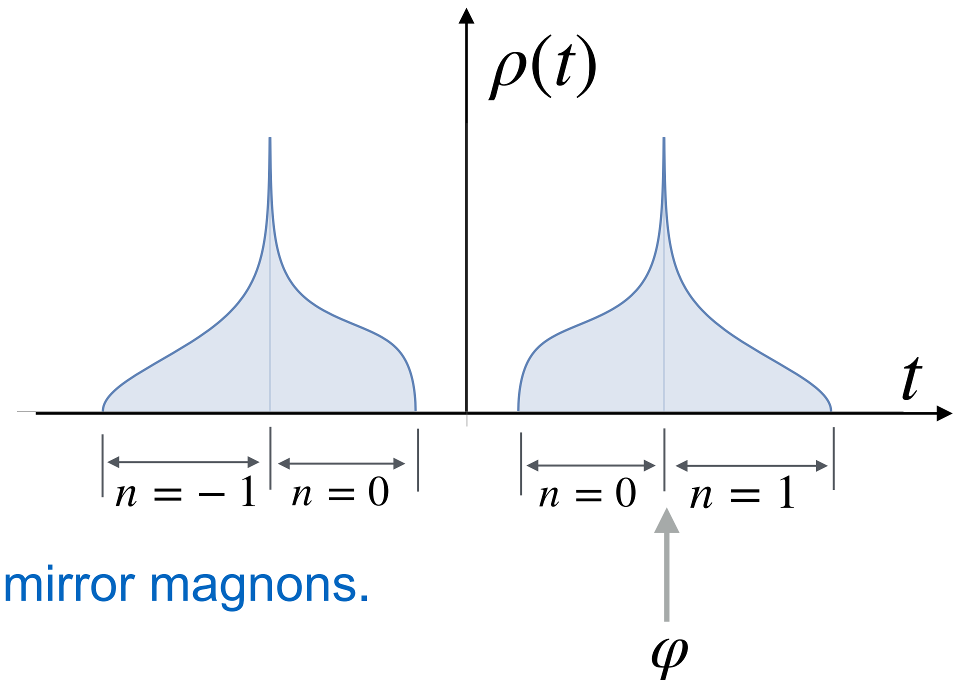
Regime I. If $|\varphi| \leq |\sigma|$, then $n_j = \mathrm{sign}(t_j), \quad j = 1,\ldots,2m$

For large number of magnons this is the finite-zone solution for the the Frolov-Tseytlin folded string rotating in $\mathrm{AdS}_3 \times S^1$ with $\{S, J\} = \{2m, \ell\}$ [Basso et al, 2021]



Regime II. If $|\varphi| > |\sigma|$, then $n_j = \mathrm{sign}(t_j)$ if $|t_j| > |\varphi|$ and $n_j = 0$ if $|t_j| < |\varphi|$, $j = 1,\ldots,2m$.

Not a finite-gap solution: the two groups of roots (with mode numbers 1 and 0 respectively) do not repel but attract. Logarithmic cusp of the spectral density observed at the collision point.



- Note that these fictive magnons have nothing to do with the original mirror magnons.

Saddle-point solution in the double scaling limit (regime II)

$\ell, m, \sigma, \varphi \to \infty$ with $\hat{\sigma} = \frac{\sigma}{m}, \hat{\varphi} = \frac{\varphi}{m}, \hat{\ell} = \frac{\ell}{m}$ fixed

"Free energy" $\mathcal{F}_m(\ell, \sigma, \varphi) \equiv \log \mathcal{Z}_m(\ell, \sigma, \varphi)$ grows as "area" $mn = m(m + \ell)$

$\hat{\mathcal{F}}(\hat{\sigma}, \hat{\varphi}, \hat{\ell}) = \mathrm{Lim}_{m\to\infty} \frac{\mathcal{F}_m(\ell, \sigma, \varphi)}{m(m+\ell)}$ — free energy per unit area (finite)

- spectral density $\rho(t)$ encoded in the resolvent

$$G(t) = \sum_{k=1}^{m} \frac{1}{t - t_k} = \int_b^a \frac{dt' \rho(t')}{t - t'}$$

- saddle-point equations reformulated as a Riemann-Hilbert problem for

$$H(t) \equiv -\frac{1}{2} V'(t) + G(t) - G(-t) \; = -2 \int_b^a \frac{dt_1}{2\pi} \, \frac{y(t)}{y(t_1)} \, \frac{tV'(t) - t_1 V'(t_1)}{t^2 - t_1^2}$$

$y(t)$ = projection of the elliptic curve $a^2 y^2 = (a^2 - t^2)(t^2 - b^2)$

- support of density $[-a, -b] \cup [b, a]$ determined by

$$\int_b^a \frac{dt}{y(t)} V'(t) = 0, \quad \int_b^a \frac{dt}{y(t)} t^2 V'(t) = 2\pi m a$$

Ivan Kostov, IPhT Saclay, 09/02/2023

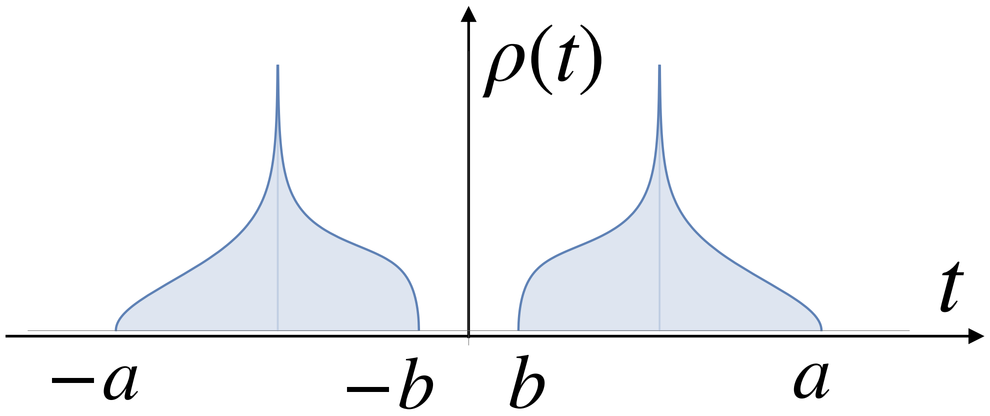
Explicit expression of spectral density in regime II $\ell, \sigma, \varphi \sim m, \quad |\varphi| > |\sigma|$

- Density:

$$\rho(t) = \frac{1}{\pi}\frac{\ell\, t}{t^2-\sigma^2}\sqrt{\frac{(a^2-t^2)(t^2-b^2)}{(a^2-\sigma^2)(b^2-\sigma^2)}} + \frac{1}{\pi^2}\frac{t}{a}\sqrt{\frac{t^2-b^2}{a^2-t^2}}\;\Pi\left(\frac{a^2-b^2}{a^2-t^2};\psi\middle|k^2\right)$$

with $k^2 = 1-\dfrac{b^2}{a^2}, \qquad \psi = \arcsin\dfrac{\sqrt{a^2-\varphi^2}}{\sqrt{a^2-b^2}}$

$$\left.\begin{aligned} a^2E\left(\psi\middle|k^2\right) - \sigma^2 F\left(\psi\middle|k^2\right) &= \pi(2m+\ell)a \\ F\left(\psi\middle|k^2\right)\sqrt{\left(a^2-\sigma^2\right)\left(b^2-\sigma^2\right)} &= \pi\ell a \end{aligned}\right\} \Rightarrow a, b$$



- Free energy:

$$\partial_m\mathcal{F} = (2m+\ell)\log\frac{(a^2-b^2)}{4(2m+\ell)^2} + \frac{2\varphi}{\pi}\arctan\frac{\sqrt{a^2-\varphi^2}}{\sqrt{\varphi^2-b^2}} + 2\ell\,\mathrm{arctanh}\frac{\sqrt{b^2-\sigma^2}}{\sqrt{a^2-\sigma^2}} - \frac{2\ell\,\sigma^2}{\sqrt{\left(a^2-\sigma^2\right)\left(b^2-\sigma^2\right)}}$$

$$F(\psi\,|\,k^2) = \int_0^{\psi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \quad E(\psi\,|\,k^2) = \int_0^{\psi} d\theta\,\sqrt{1-k^2\sin^2\theta}$$

incomplete elliptic integrals of first and second kind

$$\Pi(\alpha^2;\psi\,|\,k^2) = \int_0^{\psi}\frac{d\theta}{(1-\alpha^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}$$

incomplete elliptic integral of third kind

Explicit expression of spectral density in regime I

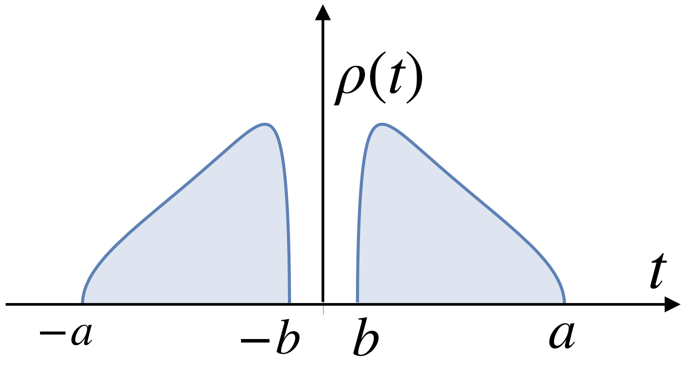
— obtained by setting $\varphi = b \quad (\psi = \pi/2)$ in the solution in regime II

- Density:

$$\rho(t) = \frac{1}{\pi}\frac{\ell\, t}{t^2-\sigma^2}\sqrt{\frac{(a^2-t^2)(t^2-b^2)}{(a^2-\sigma^2)(b^2-\sigma^2)}} + \frac{1}{\pi^2}\frac{t}{a}\sqrt{\frac{t^2-b^2}{a^2-t^2}}\ \Pi\left(\frac{a^2-b^2}{a^2-t^2}\,\middle|\, 1-\frac{b^2}{a^2}\right)$$

$$k^2 = 1-(k')^2, \quad k' = \frac{b}{a}$$

$$\left.\begin{aligned} a^2\mathbb{E} - \sigma^2\mathbb{K} &= \pi(2m+\ell)a \\ \sqrt{(a^2-\sigma^2)(b^2-\sigma^2)}\,\mathbb{K} &= \pi\ell\, a \end{aligned}\right\} \Rightarrow \ a, b$$



This is the density of the Bethe roots that correspond to the Frolov-Tseytlin folded string.

[Basso-Dixon-Kosower-Krajenbrink-Zhong, 2021]

- Free energy:

$$\partial_m \mathcal{F} = (2m+\ell)\log\frac{(a^2-b^2)}{4(2m+\ell)^2} + 2\ell\,\text{arctanh}\frac{\sqrt{b^2-\sigma^2}}{\sqrt{a^2-\sigma^2}} - \frac{2\ell\,\sigma^2}{\sqrt{(a^2-\sigma^2)\,(b^2-\sigma^2)}} + \max(|\varphi|, |\sigma)$$

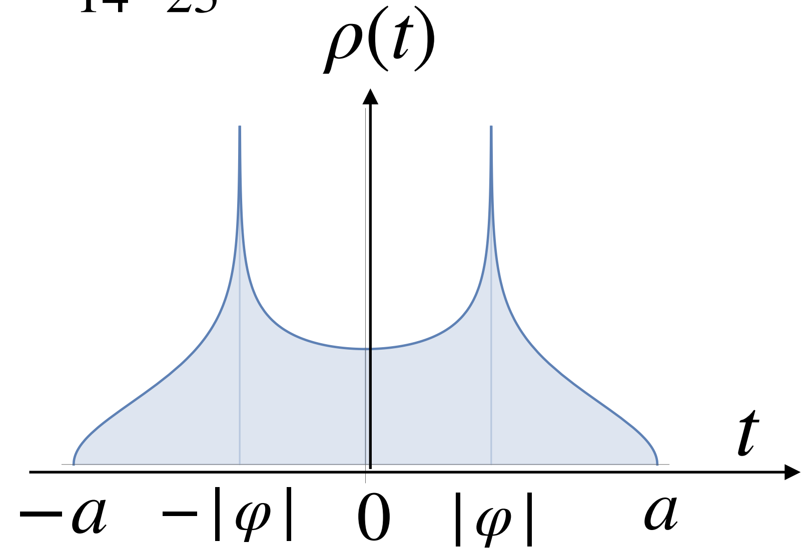
$$\mathbb{E} = E(k^2) = \int_0^{\pi/2} d\theta\,\sqrt{1-k^2\sin^2\theta}, \quad \mathbb{K} = K(k^2) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \quad \Pi(\alpha^2|\,k^2) = \int_0^{\pi/2}\frac{d\theta}{(1-\alpha^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}$$

Explicit solution for square fishnet $(\ell = 0)$ with $\sigma = 0 \Leftrightarrow x_{12}^2 x_{34}^2 = x_{14}^2 x_{23}^2$

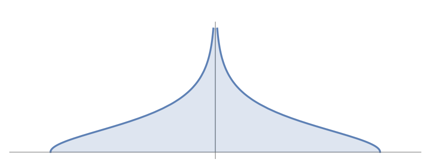
$\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = m$ and special kinematics $x_{12}^2 x_{34}^2 = x_{14}^2 x_{23}^2$

- Density: $b = 0, \quad a = \sqrt{\varphi^2 + 4\pi^2 m^2}$

$$\rho(t) = \frac{1}{2\pi^2} \log \left| \frac{\sqrt{\varphi^2 + 4\pi^2 m^2 - t^2} + 2\pi m}{\sqrt{\varphi^2 + 4\pi^2 m^2 - t^2} - 2\pi m} \right|$$



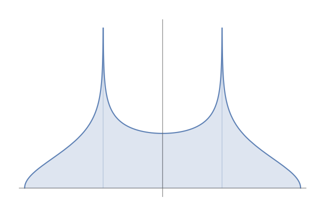
- Free energy: $\mathscr{F} = m^2 \log\left(\frac{\varphi^2 + 4\pi^2 m^2}{16 m^2}\right) - \frac{\varphi^2}{4\pi^2} \log\left(\frac{\varphi^2 + 4\pi^2 m^2}{\varphi^2}\right) + \frac{2\varphi m}{\pi} \operatorname{arccot}\left(\frac{\varphi}{2\pi m}\right)$



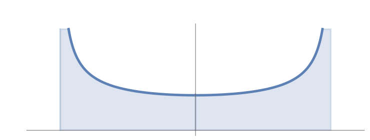
Bulk thermodynamical limit

[Basso-Dixon-Kosower-Krajenbrink-Zhong, 2021]

$\longleftarrow$ $\varphi \to 0$



$\longrightarrow$ $\varphi \to \infty$



"Double light-like limit"

Euclidean OPE and light-like limits

- **Euclidean short-distance (OPE) limit** $\left(\hat{\sigma} \to \infty \text{ with } \hat{\varphi} \text{ fixed}\right)$

  [Basso-Dixon-Kosower-Krajenbrink-Zhong, 2021]

$$\sigma \to \infty \;\; \Rightarrow U \to 0,\; V \to 1$$

$$|x_{12}|^2, |x_{34}|^2 \sim \sqrt{U}\; |x_{13}|^2 \qquad (U \to 0,\; V \to 1)$$

i.e. $x_1 \sim x_2, \;\; x_3 \sim x_4$ => OPE limit in the U-channel.

- **Double light-cone, or nul, limit** $\left(\hat{\varphi} \to \infty \;\; \text{with} \;\; \hat{\sigma} \text{ fixed}\right)$

$$\varphi \to \infty \quad \Rightarrow \quad U \to 0,\; V \to 0$$

$$x_{12}^2,\, x_{34}^2 \sim \sqrt{U}\; |x_{13}||x_{24}|; \quad x_{14}^2,\, x_{23}^2 \sim \sqrt{V}\; |x_{13}||x_{24}|$$

i.e. Minkowski intervals $x_{12}^2,\, x_{23}^2, x_{34}^2, x_{41}^2$ become simultaneously light-like

Exact solutions in the Euclidean OPE and in the double light-like limits

- **Euclidean short-distance limit** $(\hat{\sigma} \to \infty \text{ with } \hat{\varphi} \text{ fixed})$:

Ladders: $f_k(z,\bar{z}) \to \int_0^\infty (2|\sigma|)^k\, t^{k-1} e^{-t} dt = (2|\sigma|)^k\, (k-1)!$

$$I^{\mathrm{BD}}_{m,n} \to \frac{(2|\sigma|)^{mn}}{\mathcal{N}} \det_{j,k} \left[(j+k+\ell-2)! \right] = \left(\log \frac{1}{U} \right)^{mn} \; C_{m,n}$$

[Basso-Dixon-Kosower-Krajenbrink-Zhong, 2021]

$$C_{m,n} = \frac{G(m+1)G(n+1)}{G(m+n+1)}, \quad G(m) = 1!2!\ldots(m-2)!$$

Barnes' G-function

- **Double light-cone, or nul, limit** $(\hat{\varphi} \to \infty \quad \text{with} \quad \sigma \text{ fixed}):$

Ladders: $f_k(z,\bar{z}) \underset{\varphi \gg k}{\to} 2 \int_0^\varphi t^{2k-1} dt = \frac{\varphi^{2k}}{k}$

$$I^{\mathrm{BD}}_{m,n} \to \frac{\varphi^{2m(m+\ell)}}{\mathcal{N}} \times \det \left[\frac{1}{i+j-1+n-m} \right]_{i,j=1,\ldots,m} = \frac{\varphi^{2mn}}{\mathcal{N}} \times \mathcal{N}\,(C_{m,n})^2$$

$$= C_{m,n} \left(\log \frac{1}{U} \right)^{mn} \times C_{m,n} \left(\log \frac{1}{V} \right)^{mn}$$

Comparison with the solution in the double scaling limit

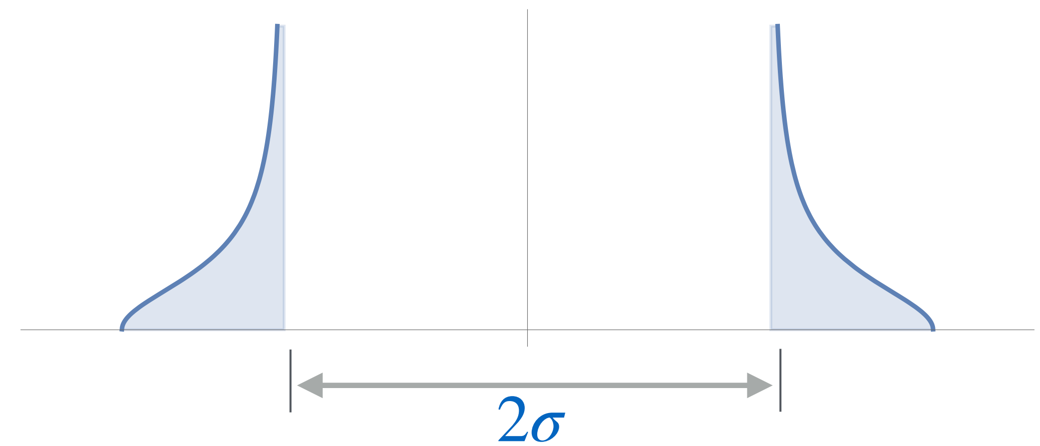
- $m \to \infty$ asymptotics of exact solution in Euclidean OPE and double light-cone limits matches $\hat{\sigma} \to \infty$ and $\hat{\varphi} \to \infty$ limits of the saddle-point solution

- **Euclidean short-distance limit** $(\hat{\sigma} \to \infty \text{ with } \hat{\varphi} \text{ fixed}):$

[Basso-Dixon-Kosower-Krajenbrink-Zhong, 2021]

$$a \approx \sigma + (\sqrt{m} + \sqrt{n})^2, \; b \approx \sigma + (\sqrt{m} - \sqrt{n})^2$$

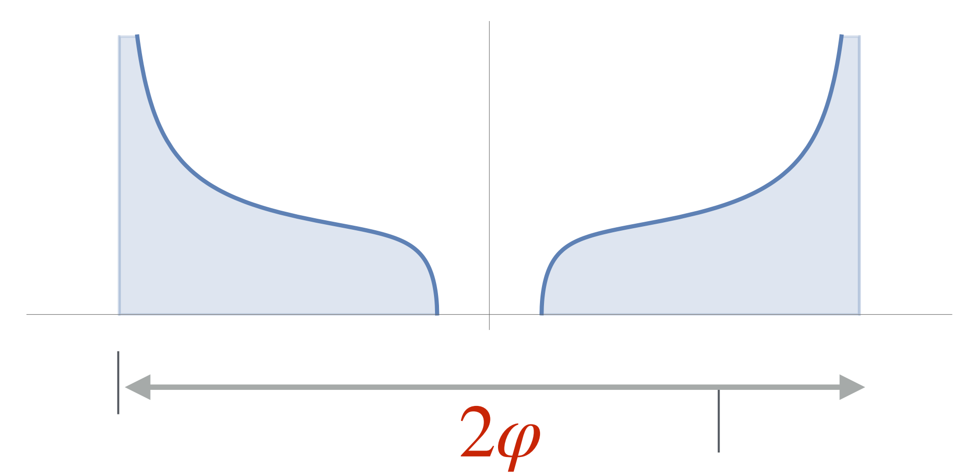
$$\mathcal{F} = mn\log(2\sigma) + \frac{3}{2}mn + \frac{1}{2}m^2\log(m) + \frac{1}{2}n^2\log n - \frac{1}{2}(m+n)^2\log(m+n)$$



- **Double light-cone, or nul, limit** $(\hat{\varphi} \to \infty \text{ with } \sigma \text{ fixed}):$

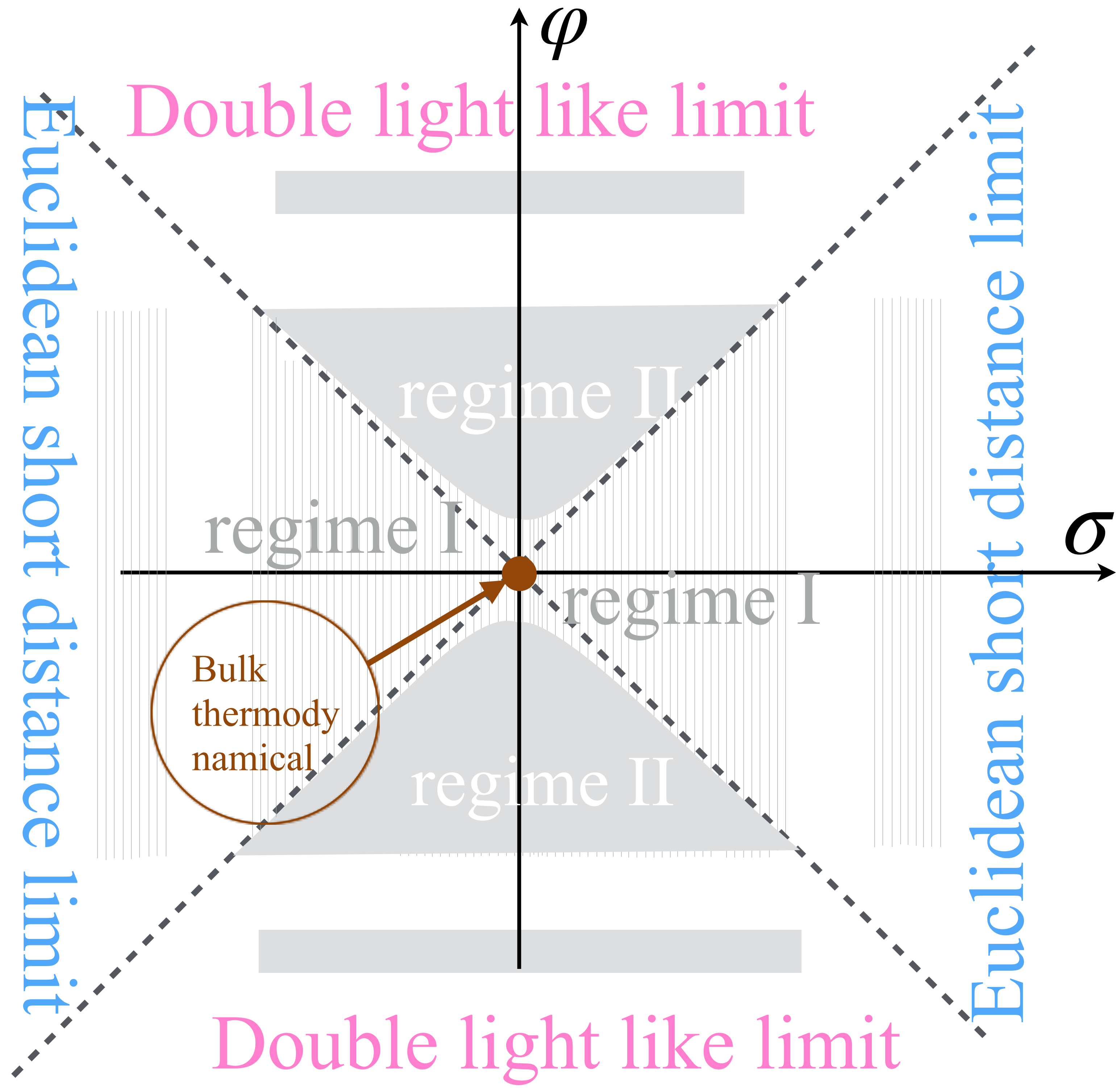
$$a \approx \varphi, \; b \approx \frac{n-m}{n+m}\varphi$$

$$\mathcal{F} = 2mn\log(\varphi) + 3mn + m^2\log(m) + n^2\log(n) - (m+n)^2\log(m+n)$$



Summary

- The bulk thermodynamical limit $\hat{\sigma} = \hat{\varphi} = 0$, the Euclidean OPE limit $\hat{\sigma} \to \infty$ and the double light-cone limit $\hat{\varphi} \to \infty$ are analytically related.



- HOLOGRAPHIC DUAL OF OPEN FISHNETS?

Results compatible with existence of holographic dual.
Saddle-point equations = Bethe equations for some magnons in t-space.

However, not clear how to interpret the “unphysical” mode numbers in regime II.

— Problem still open.

- Curious factorisation observed in the light-cone limit where the result is a product of two factors associated with the direct and with the cross channels

$$I^{\mathrm{BD}}_{m,n} = C_{m,n}\left(\log\frac{1}{U}\right)^{mn} \times C_{m,n}\left(\log\frac{1}{V}\right)^{mn}$$

— There is interpretation of the OPE limit in terms of hopping magnons (“stampedes”) [Olivucci-Vieira, 2022].
If it can be extended to the light-like limit, how the above factorisation appears?

— Possible arctic curve phenomenon.