

# ***Tracy-Widom distributions in supersymmetric gauge theories***

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*Integrability in Condensed Matter Physics and Quantum Field Theory*  
*February 10, 2023*

# Outline

Ultimate goal is to solve four-dimensional super Yang–Mills theories

$$\text{SYM theory} = [\text{gauge fields}] + [\text{fermions}] + [\text{scalars}]$$

for arbitrary 't Hooft coupling  $\lambda = g_{\text{YM}}^2 N_c$ , to any order in  $1/N_c^2$

Integrability allows us to realize this program for a special class of observables

- ✓ Weak coupling expansion is easy, what we expect at strong coupling?
- ✓ Tracy-Widom distribution in random matrices
- ✓ Free energy in  $\mathcal{N} = 2$  SYM on sphere
- ✓ Correlation function of (infinitely) heavy half-BPS operators in  $\mathcal{N} = 4$  SYM
- ✓ Strong coupling expansion from Szegő–Akhiezer-Kac formula

## What we expect at strong coupling

Simple example: circular Wilson loop in planar  $\mathcal{N} = 4$  SYM

$$W = \frac{1}{N_c} \langle \text{tr} P e^{i g_{\text{YM}} \oint ds \dot{x} \cdot A(x(s)) + i \Phi(x(s)) |\dot{x}(s)|} \rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$$

\lambda = g\_{\text{YM}}^2 N\_c

Expansion at weak and strong coupling

$$W \stackrel{\lambda \ll 1}{\approx} 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \frac{\lambda^3}{9216} + \frac{\lambda^4}{737280} + \dots$$

$$W \stackrel{\lambda \gg 1}{\approx} \exp \left( \sqrt{\lambda} - \frac{1}{2} \log \left( \frac{\pi}{2} \lambda^{3/2} \right) - \frac{3}{8\sqrt{\lambda}} - \frac{3}{16\lambda} - \frac{21}{128\lambda^{3/2}} + \dots \right)$$

Semiclassical asymptotics in AdS/CFT

$$\log W = -\sqrt{\lambda} A_0 - A_1 \log(\sqrt{\lambda}) - B - \frac{A_2}{\sqrt{\lambda}} - \frac{A_3}{\lambda} - \frac{A_4}{\lambda^{3/2}} + \dots$$

- ✓  $A_0$  minimal area in  $\text{AdS}_5$
- ✓  $A_i$  and  $B$  come from fluctuations (very hard to compute in AdS/CFT)

# Tracy-Widom distribution

Describes statistics of the spacing of the eigenvalues of  $N \times N$  hermitian matrices for  $N \rightarrow \infty$

Gaussian Unitary Ensemble

$$Z_{\text{GUE}} = \int d^{N \times N} a e^{-\frac{1}{2} \text{tr} a^2} = \int_{-\infty}^{\infty} d\lambda_1 \dots d\lambda_N \prod_{i \neq j} (\lambda_i - \lambda_j)^2 e^{-\frac{1}{2} \sum_i \lambda_i^2}$$

Laguerre ensemble (Wishart matrix theory)

$$Z_{\text{Laguerre}} = \int_0^{\infty} d\lambda_1 \dots d\lambda_N \prod_{i \neq j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^N \lambda_i^{\ell} e^{-\lambda_i}$$

where  $\ell > -1$  and eigenvalues are located on semi-axis  $[0, \infty)$ .

The probability density for eigenvalues

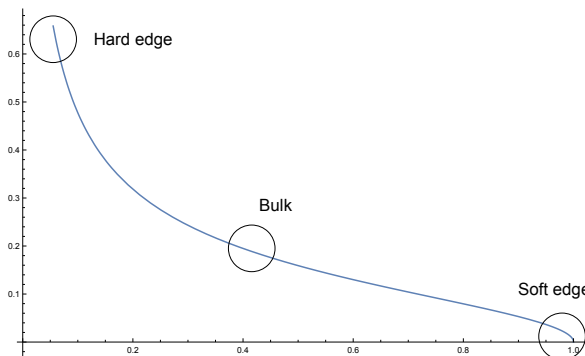
$$R_n(x_1, \dots, x_n) = \left\langle \prod_{i=1}^n \delta(\lambda_i - x_i) \right\rangle = \det K_N(x_i, x_j) \Big|_{i,j=1, \dots, n}$$

$$K_N(x, y) = \sum_{k=0}^{N-1} \phi_k(x) \phi_k(y)$$

where  $\phi_k(x)$  are orthonormal functions  $x^k e^{-x^2/2} + \dots$  (GUE) and  $x^k x^{\ell/2} e^{-x/2} + \dots$  (Laguerre)

## Tracy-Widom distribution II

The distribution of the eigenvalues in the Laguerre ensemble in the limit  $N \rightarrow \infty$



$$R_1(4Nx) \sim \frac{1}{2\pi} \sqrt{\frac{1-x}{x}}$$

Scaling behaviour of  $K_N(x, y)$  around  $x = 0$  (hard edge),  $x = 1$  (soft edge) and  $0 < x < 1$  (bulk)

bulk :	$\frac{\sin \pi(x - y)}{\pi(x - y)}$
soft edge :	$\frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}(y)\text{Ai}'(x)}{x - y}$
hard edge :	$\frac{J_\ell(\sqrt{x})\sqrt{y}J'_\ell(\sqrt{y}) - \sqrt{x}J'_\ell(\sqrt{x})J_\ell(\sqrt{y})}{2(x - y)}$

The probability that there are eigenvalues on the interval  $[0, s]$

$$E(0; s) = \det(1 - K)_{[0, s]} = 1 + \sum_{n \geq 1} \frac{(-1)^n}{n!} \int_0^s dx_1 \dots dx_n \det \|K(x_i, x_j)\|_{1 \leq i, j \leq n}$$

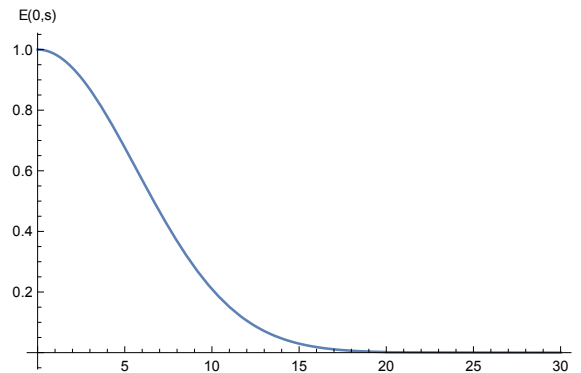
# Bessel kernel

Tracy-Widom distribution close to the hard edge

$$E(0, s) = \det(1 - K_{\text{Bessel}})_{[0, s]} = \exp\left(-\frac{1}{4} \int_0^s dx \log(s/x) Q^2(x)\right)$$

$Q(s)$  satisfies Painlevé V differential equation

Dependence of the probability  $E(0, s)$  on the interval length  $s$



Asymptotics of  $E(0, s)$  at small and large  $s$

$$E(0, s) \stackrel{s \ll 1}{\approx} 1 - \frac{(s/4)^{\ell+1}}{\Gamma^2(\ell+2)} + \dots$$

$$E(0, s) \stackrel{s \gg 1}{\approx} \exp\left(-s/4 - \frac{\ell^2}{4} \log s + \frac{\ell}{8} s^{-1/2} + \dots\right)$$

Remarkably similar to weak/strong coupling expansion in gauge theory for  $s \sim \sqrt{\lambda}$

## Bessel kernel at finite temperature

$$K_\ell(x, y) = \sum_{n \geq 1} \phi_n(x) \phi_n(y), \quad \phi_n(x) = \sqrt{2n + \ell - 1} \frac{J_{2n + \ell - 1}(\sqrt{x})}{\sqrt{x}}$$

Can be represented by a semi-infinite matrix

$$\int_0^\infty dy K_\ell(x, y) \phi_n(x) = K_{nm} \phi_m(x)$$

$$K_{nm} = 2(-1)^{n+m} \sqrt{(2n + \ell - 1)(2m + \ell - 1)} \int_0^\infty \frac{dx}{x} J_{2n + \ell - 1}(x) J_{2m + \ell - 1}(x) \chi\left(\frac{x}{2g}\right)$$

$\chi(x)$  is the *symbol* of the Bessel operator

$$\det(1 - \mathbf{K}_\chi) = \det(\delta_{nm} - K_{nm}) \Big|_{n, m \geq 1}$$

- ✓ For  $\chi(x) = \theta(1 - x)$  coincides with the Tracy-Widom distribution  $E(0, s)$  for  $s = (2g)^2$
- ✓ Finite-temperature generalization:  $\chi(x) = 1/(1 + e^{\frac{x-\mu}{T}})$
- ✓ In supersymmetric gauge theories we encounter symbol of the form

$$\chi_{\text{loc}}(x) = -\frac{1}{\sinh^2(x/2)}, \quad \chi_{\text{oct}}(x) = \frac{\cosh y + \cosh \xi}{\cosh y + \cosh \sqrt{x^2 + \xi^2}}$$

$y$  and  $\xi$  are kinematical variables

## Free energy in $\mathcal{N} = 2$ super Yang-Mills theory

- ✓  $\mathcal{N} = 2$  supersymmetric Yang-Mills theory with gauge group  $SU(N)$  coupled to matter multiplets in the symmetric ( $N_S = 1$ ) and anti-symmetric ( $N_A = 1$ ) representations  
The beta function vanishes  $\beta_0 = 2N - N_S(N + 2) - N_A(N - 2) = 0$ ,

- ✓ The partition function on sphere  $S^4$  is given by a matrix integral [Pestun]

$$Z_{S^4} = e^{-F} = \int da e^{-\frac{8\pi^2 N}{\lambda} \text{tr } a^2} |Z_{1\text{-loop}}(a) Z_{\text{inst}}(a)|^2$$

Non-perturbative instanton contribution  $Z_{\text{inst}}(a)$  is exponentially small at large  $N$

- ✓ Perturbative corrections  $Z_{1\text{-loop}}(a) = \exp(-S_{\text{int}}(a))$  only come from one loop

$$S_{\text{int}}(a) = \sum_{i,j} \left[ \log H(\lambda_i + \lambda_j) - \log H(\lambda_i - \lambda_j) \right] \quad (\lambda_i \text{ are eigenvalues of } a)$$

$$= 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} \zeta_{2n+1} \sum_{p=0}^n \binom{2n+2}{2p+1} \text{tr } a^{2p+1} \text{tr } a^{2(n-p)+1}$$

$$H(x) = \prod_{n=1}^{\infty} \left( 1 + \frac{x^2}{n^2} \right)^n e^{-\frac{x^2}{n}} = \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} \zeta_{2n+1} x^{2n+2} \right)$$



# Large $N$ expansion

$$e^{-F} = \left( \frac{8\pi^2}{\lambda} \right)^{-(N^2-1)/2} \int da \exp \left( -N \operatorname{tr} a^2 + \frac{1}{2} \sum_{k,n} C_{kn} O_{2k+1} O_{2n+1} \right)$$

The interaction term is a sum over double traces  $O_k = \operatorname{tr} a^k$  with the couplings

$$C_{kn} = 4 \frac{(-1)^{k+n+1}}{k+n+1} \zeta_{2(k+n)+1} \binom{2(k+n+1)}{2k+1} \left( \frac{\lambda}{8\pi^2} \right)^{k+n+1}$$

Large  $N$  expansion

$$F = N^2 F_0(\lambda) + F_1(\lambda) + F_2(\lambda)/N^2 + \dots$$

The interaction term does not contribute to  $F_0$

$$F_1 = \sum_{n \geq 1} \left( \text{Diagram of a torus with two handles labeled } 1 \text{ and } 2 \text{, and } n \text{ other handles} \right) = - \sum_{n \geq 1} \frac{1}{2n} \operatorname{tr} [(QC)^n] = \frac{1}{2} \log \det(1 - QC)$$

Cylinders  $Q_{kn} = \left( \text{Diagram of a cylinder with two faces labeled } k \text{ and } n \right) = \langle \operatorname{tr} a^{2k+1} \operatorname{tr} a^{2n+1} \rangle_{\text{GUE}}$  are glued together with the weight  $C_{kn}$

## Relation to Bessel kernel

Explicit expressions for semi-infinite matrices

$$Q_{kn} = \frac{2\beta_k\beta_n}{k+n+1} + O(1/N^2), \quad \beta_n = \frac{2^n n \Gamma(n + \frac{3}{2})}{\sqrt{\pi} \Gamma(n + 2)}$$

$$C_{kn} = 4 \frac{(-1)^{k+n+1}}{k+n+1} \zeta_{2(k+n)+1} \binom{2(k+n+1)}{2k+1} \left( \frac{\lambda}{8\pi^2} \right)^{k+n+1}$$

The matrix  $(QC)$  is related to the Bessel kernel by a similarity transformation

[Beccaria et al]

$$\begin{aligned} K_{nm} &= (U^{-1}QC U)_{nm} \\ &= 2(-1)^{n+m} \sqrt{2n+1} \sqrt{2m+1} \int_0^\infty \frac{dt}{t} J_{2n+1}(t) J_{2m+1}(t) \chi\left(\frac{x}{2g}\right) \end{aligned}$$

Special form of the symbol

$$\chi(x) = -\frac{1}{\sinh^2(x/2)}, \quad g = \frac{\sqrt{\lambda}}{4\pi}$$

The free energy coincides with the Tracy-Widom distribution at the hard edge for  $\ell = 2$

$$F_1 = \frac{1}{2} \log \det(1 - QC) = \frac{1}{2} \text{tr} \log(1 - \mathbf{K}_\chi)$$

# Correlation functions in $\mathcal{N} = 4$ SYM

- ✓ Half-BPS operators

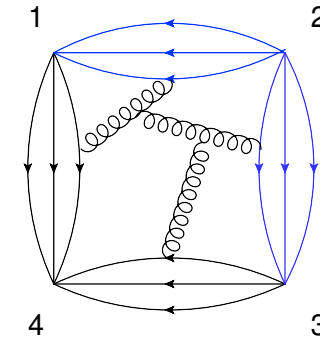
$$O_1 = \text{tr}(Z^{K/2} \bar{X}^{K/2}) + \text{permutations}, \quad O_2 = \text{tr}(X^K), \quad O_3 = \text{tr}(\bar{Z}^K)$$

Exact scaling dimension (R-charge)  $\Delta = K$

Two- and three-point functions are protected

- ✓ “Simplest” four-point function

$$\langle O_1(x_1) O_2(x_2) O_1(x_3) O_3(x_4) \rangle = \frac{\mathcal{G}_K(z, \bar{z})}{(x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2)^{K/2}}$$



Depends on two cross ratios and 't Hooft coupling  $g^2 = g_{\text{YM}}^2 N_c / (4\pi)^2$

$$\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}, \quad \frac{x_{23}^2 x_{41}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z})$$

- ✓ Examine  $\mathcal{G}_K(z, \bar{z})$  in the limit  $K \rightarrow \infty$  (infinitely heavy operators) with  $g^2$  kept fixed

# Weak coupling expansion

$$\lim_{K \rightarrow \infty} \mathcal{G}_K = [\mathbb{O}(z, \bar{z})]^2$$

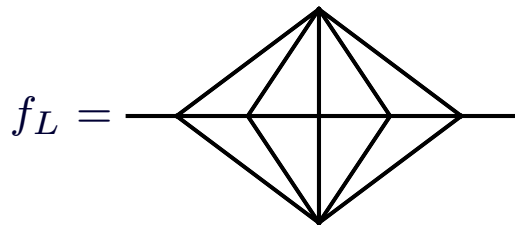
$\mathbb{O}(z, \bar{z})$  is a multilinear combination of ladder integrals

[Coronado]

$$\begin{aligned} \mathbb{O}(z, \bar{z}) &= 1 + g^2 f_1 - 2g^4 f_2 + 6g^6 f_3 + g^8 (-20f_4 - \frac{1}{2}f_2^2 + f_1 f_3) + \dots \\ &= 1 + \sum_{\ell \geq 1} (g^2)^\ell \times \sum_{i_1 + \dots + i_n = \ell} d_{i_1 \dots i_n} f_{i_1} \dots f_{i_n} \end{aligned}$$

The expansion coefficients  $d_{i_1 \dots i_n}$  can be found to all loops from different OPE limits of  $\mathcal{G}_K$

Ladder integrals



$$f_L = \frac{1}{z - \bar{z}} \sum_{m=0}^L \frac{(-1)^m (2L - m)!}{L! (L - m)! m!} \ln^m(z\bar{z}) \underbrace{\left[ \text{Li}_{2L-m}(z) - \text{Li}_{2L-m}(\bar{z}) \right]}_{\text{polylog}}$$

The weak coupling expansion can be resummed to all orders in the coupling [Kostov, Petkova, Serban]

## Relation to Bessel kernel

$$\mathbb{O}(z, \bar{z}) = \exp \left[ -\frac{1}{2} \sum_{n \geq 1} \text{tr}(CH)^n \right] = \sqrt{\det(1 - CH)}$$

Semi-infinite matrices

$$H_{nm} = \frac{g}{2i} \int_{|\xi|}^{\infty} dt \frac{\left(i\sqrt{\frac{t+\xi}{t-\xi}}\right)^{m-n} - \left(i\sqrt{\frac{t+\xi}{t-\xi}}\right)^{n-m}}{\cosh y + \cosh t} \underbrace{J_m(2g\sqrt{t^2 - \xi^2}) J_n(2g\sqrt{t^2 - \xi^2})}_{\text{Bessel function}}$$

$$C_{nm} = 2(\cosh y + 1)(\delta_{n+1,m} - \delta_{n,m+1}),$$

Kinematical variables :  $z = -e^{-y-\xi}$ ,  $\bar{z} = -e^{+y-\xi}$

Similarity transformation

[Belitsky,GK],[Kostov,Petkova]

$$\begin{aligned} K_{nm} &= (\Omega^{-1}CH\Omega)_{nm} \\ &= 2(-1)^{n+m} \sqrt{2n+1} \sqrt{2m+1} \int_0^{\infty} \frac{dt}{t} J_{2n+1}(t) J_{2m+1}(t) \chi_{\text{oct}} \left( \frac{x}{2g} \right) \end{aligned}$$

$\mathbb{O}(z, \bar{z})$  coincides with the Tracy-Widom distribution for  $\ell = 0$  and the symbol

$$\chi_{\text{oct}} \left( \frac{x}{2g} \right) = \frac{\cosh y + \cosh \xi}{\cosh y + \cosh(\sqrt{x/(2g)^2 + \xi^2})} \quad \text{depends on } g, y, \xi$$

# Tracy-Widom distribution in super Yang-Mills theories

Different observables in SYM theories are given by the Tracy-Widom distribution  $\det(1 - \mathbf{K}_\chi)$

Choice of the observable fixes the form of the symbol:

- ✓ Free energy of  $\mathcal{N} = 2$  SYM

$$\chi(x) = -\frac{1}{\sinh^2(x/2)}$$

- ✓ Four-point correlator in  $\mathcal{N} = 4$  SYM

$$\chi(x) = \frac{\cosh y + \cosh \xi}{\cosh y + \cosh(\sqrt{x + \xi^2})}$$

- ✓ Circular Wilson loop

$$\chi(x) = -\frac{4}{x^2}$$

The coupling constant defines the interval length in the TW distribution  $s \sim g^2$

- ✓ Weak coupling expansion is easy

$$\log \det(1 - \mathbf{K}_\chi) = -\text{tr } \mathbf{K}_\chi - \frac{1}{2} \text{tr}(\mathbf{K}_\chi^2) + \dots = c_1 g^2 + c_2 g^4 + \dots$$

- ✓ Strong coupling expansion is hard

## Szegő-Akhiezer-Kac formula

- ✓ Asymptotic behaviour for sufficiently smooth symbol  $\chi(z)$

$$\det(1 - \mathbf{K}_\chi) = e^{-gA_0 + B + O(1/g)}$$

SAK formula (1915-1966)

$$A_0 = -2\tilde{\psi}(0), \quad B = \frac{1}{2} \int_0^\infty dk k (\tilde{\psi}(k))^2,$$

$$\tilde{\psi}(k) = \int_0^\infty \frac{dz}{\pi} \cos(kz) \log(1 - \chi(z))$$

$B$  diverges for  $\chi(z) \sim 1 - z^{2\beta}$  or  $\tilde{\psi}(k) \sim -\beta/k$  at large  $k$

*Fisher-Hartwig singularity*

- ✓ The SAK formula for the Bessel kernel with Fisher-Hartwig singularity has not been derived yet

- ✓ Our conjecture

[Belitsky,GK]

$$\det(1 - \mathbf{K}_\chi) = e^{-gA_0 + A_1 \log g + B' + O(1/g)}$$

$$A_1 = \frac{1}{2}\beta^2,$$

$$B' = \frac{1}{2} \int_0^\infty dk \left[ k (\tilde{\psi}(k))^2 - \beta^2 \frac{1 - e^{-k}}{k} \right] + \frac{\beta}{2} \log(2\pi) - \log G(1 + \beta),$$

Power suppressed  $O(1/g)$  corrections are determined using the *method of differential equations*

# Method of differential equations

A powerful method for computing correlators in integrable models

[Its,Izergin,Korepin,Slavnov]

Logarithmic derivative of the octagon

$$U(g, y, \xi) = -2g\partial_g \log \det(1 - \mathbf{K}_\chi)$$

Satisfies the system of *exact* integro-differential equations

[Belitsky,GK]

$$\partial_y U = \int_0^\infty dx Q^2(x) \partial_y \chi(x),$$

$$g\partial_g U = -2 \int_0^\infty dx Q^2(x) x \partial_x \chi(x),$$

$$\partial_\xi U = 8g^2 \xi \int_0^\infty dx Q^2(x) \partial_x \chi(x) + \frac{\sinh \xi}{\cosh y + \cosh \xi} \int_0^\infty dx Q^2(x) \chi(x)$$

$$\chi = \frac{\cosh y + \cosh \xi}{\cosh y + \cosh(\sqrt{x/(2g)^2 + \xi^2})}$$

Auxiliary function  $Q(x)$  obeys a PDE

$$(g\partial_g + 2x\partial_x)^2 Q(x) + (x - g\partial_g U + U) Q(x) = 0$$

For  $\chi(x) = \theta(1 - x)$  reduces to Painleve V equation



# Tracy-Widom distribution at strong coupling

✓ Strong coupling expansion:

$$\log \det(1 - \mathbf{K}_\chi) = \underbrace{-gA_0 + A_1 \log g + B}_{\text{SAK formula}} + \frac{A_2}{4g} + \frac{A_3}{12g^2} + \frac{A_4}{24g^3} + \dots$$

✓ Exact expressions for the expansion coefficients

$$\begin{aligned} A_0 &= 2I_0, & A_1 &= \frac{1}{2}, \\ A_2 &= -\frac{3I_1}{4}, & A_3 &= -\frac{9I_1^2}{16}, \\ A_4 &= -\frac{3I_1^3}{8} + \frac{15I_2}{128}, & A_5 &= -\frac{15I_1^4}{64} + \frac{75I_1I_2}{256}, \quad \dots \end{aligned}$$

Dependence on symbol (=choice of observable) enters through a *profile function*

$$I_n(y, \xi) = \int_0^\infty \frac{dz}{\pi} \frac{(z^{-1} \partial_z)^n}{(2n-1)!!} z \partial_z \log(1 - \chi(z))$$

$A_1$  is universal, generated by the Fisher-Hartwig singularity

$B$  is the Dyson-Widom constant

## Application to free energy in $\mathcal{N} = 2$ SYM

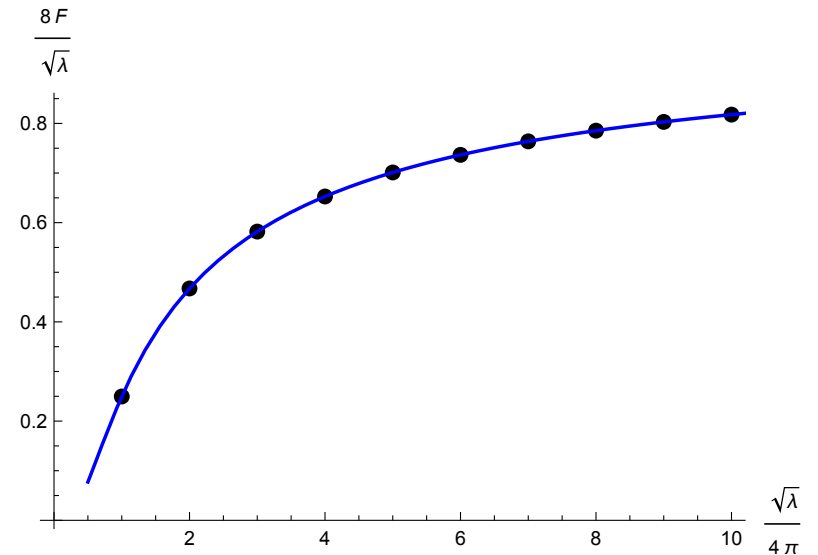
$$F = \frac{1}{2} \log \det(1 - \mathbf{K}_\chi), \quad \chi(x) = -\frac{1}{\sinh^2(x/2)}$$

✓ Weak coupling expansion in  $\hat{\lambda} = \lambda/(8\pi^2)$

$$F = 5\zeta_5 \hat{\lambda}^3 - \frac{105}{2} \zeta_7 \hat{\lambda}^4 + 441\zeta_9 \hat{\lambda}^5 - (25\zeta_5^2 + 3465\zeta_{11}) \hat{\lambda}^6 + \left( 525\zeta_5\zeta_7 + \frac{212355\zeta_{13}}{8} \right) \hat{\lambda}^7 + \dots$$

✓ Strong coupling expansion in  $1/\sqrt{\lambda}$

$$F = \frac{1}{8} \lambda^{1/2} - \frac{3}{8} \log \lambda - 3 \log A + \frac{1}{4} - \frac{11}{12} \log 2 + \frac{3}{4} \log(4\pi) \\ + \frac{3}{32} \log(\lambda'/\lambda) - \frac{15\zeta_3}{64 \lambda'^{3/2}} - \frac{945\zeta_5}{512 \lambda'^{5/2}} - \frac{765\zeta_3^2}{128 \lambda'^3} + \dots \\ - \frac{i}{4} \lambda^{1/2} e^{-\sqrt{\lambda}} (1 + O(\lambda^{-1/2})), \quad \lambda'^{1/2} = \lambda^{1/2} - 4 \log 2.$$



Series in  $1/\sqrt{\lambda}$  has factorially growing coefficients

Borel singularities are in one-to-one correspondence with nonperturbative corrections

## Conclusions and open questions

*Various* quantities (free energy, correlation functions, Wilson loop) in *different* 4d super Yang-Mills theories are expressed in terms of the *same* (temperature dependent) Tracy-Widom distribution

This relation is powerful enough to predict the dependence on 't Hooft coupling

- ✓ Who ordered this universality?
- ✓ What is the reason why the Bessel kernel appears in all cases?
- ✓ Nonplanar corrections?

*Thank you for your attention!*