Snake Modules, Extended T-Systems and Correlation Functions for higher rank

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February 6, 2023
Overview

An inhomogeneous XXX-Model
  Correlation functions in the $\mathfrak{sl}_2$ case
  The Snail Construction for $\mathfrak{sl}_2$
  T-Systems

Generalization to higher rank
  Snake modules and extended t-systems

Conclusion
1. 2006 Boos, Jimbo, Miwa, Smirnov and Takeyama
   'A RECUSION FORMULA FOR THE CORRELATION FUNCTIONS
   OF AN INHOMOGENEOUS XXX-MODEL'

2. 2012 E. Moukhin · C. A. S. Young
   'Extended T-Systems'

3. 2018 Boos, Hutsalyuk and Nirov
   'ON THE CALCULATION OF THE CORRELATION FUNCTIONS OF
   THE $\mathfrak{sl}_3$-MODEL BY MEANS OF THE REDUCED QKZ EQUATION'

4. 2019 Klümper, Nirov and Razumov
   'REDUCED QKZ EQUATION: GENERAL CASE'
An inhomogeneous XXX-Model

- \( H_{\text{XXX}} = \frac{1}{2} \sum_j \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z \right) \)
  gapless case \( q \to 1, \Delta = \frac{q+q^{-1}}{2} \)

- 'Inhomogeneous chain' generated by
  \[
  \text{tr} \left( R_{a,-N} (\lambda) \cdots R_{a,0} (\lambda) R_{a,1} (\lambda - \lambda_1) \cdots R_{a,m} (\lambda - \lambda_m) R_{a,m+1} (\lambda) \cdots R_{a,N} (\lambda) \right)
  \]

- Still exactly solvable but the interaction is nonlocal.

- Vertex operator approach (XXZ gapped regime)
  - multiple integrals which can be analytically continued for \( q \to 1 \)

- Factorisation (Boos, Korepin and Smirnov)
Conjecture 1.1.

\[
[D_{1,\ldots,m}(\lambda_1, \ldots, \lambda_m)]_{\bar{\varepsilon}_1 \ldots \bar{\varepsilon}_m} := \langle \text{vac} \mid (E_{\bar{\varepsilon}_1}^{\varepsilon_1})_1 \cdots (E_{\bar{\varepsilon}_m}^{\varepsilon_m})_m \mid \text{vac} \rangle \quad (1.1)
\]

\[
= \sum \prod \omega(\lambda_i - \lambda_j) f(\lambda_1, \ldots, \lambda_m),
\]

where \( \omega(\lambda) \) is a single transcendental function and the functions \( f(\lambda_1, \ldots, \lambda_n) \) are rational.

- Odd integer values of \( \zeta \) appear in the Taylor series of \( \omega \) for the homogeneous limit.
- Proof of (1.1) via the 'Snail Construction'.
- Result can be written in terms of a transfer matrix over an auxiliary space of 'fractional dimension'.
- Generalization to XXZ
- Fermionic structure \( \rightarrow \) HGS papers.
Correlation functions in the $\mathfrak{sl}_2$ case

- Define the rational R-matrix by

\[ R(\lambda) = \frac{\rho(\lambda)}{\lambda + 1} (\lambda + P), \quad \rho(\lambda) = -\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2} - \frac{\lambda}{2})}{\Gamma(-\frac{\lambda}{2})\Gamma(\frac{1}{2} + \frac{\lambda}{2})} \]

Proposition 1.2. The reduced density matrix $D_{1,...,m}$ fulfills

1. The global $GL_2(\mathbb{C})$-invariance.
2. The R-matrix relations

\[ D_{1,...,i+1,i,...,m}(\lambda_1, ..., \lambda_{i+1}, \lambda_i, ..., \lambda_m) = R_{i+1,i}(\lambda_{i+1},i)D_{1,...,m}(\lambda_1, ..., \lambda_m)R_{i,i+1}(\lambda_i,i+1). \]

3. Left-right reduction relations

\[ tr_1(D_{1,...,m}(\lambda_1, ..., \lambda_m)) = D_{2,...,m}(\lambda_2, ..., \lambda_m) \]
\[ tr_n(D_{1,...,m}(\lambda_1, ..., \lambda_m)) = D_{1,...,m-1}(\lambda_1, ..., \lambda_{m-1}). \]
4. The rqKZ-equation

\[ D_{1,\ldots,m}(\lambda_1 - 1, \lambda_2, \ldots, \lambda_m) = A_{1,1}(\lambda_1, \ldots, \lambda_m)(D_{1,2,\ldots,m}(\lambda_1, \lambda_2, \ldots, \lambda_m)). \]

Note:

• Due to \( \rho(\lambda)\rho(-\lambda) = 1 \) and \( \rho(\lambda - 1)\rho(\lambda) = -\frac{\lambda}{\lambda - 1} \) the coefficients in 2. and 4. are rational.

• \( D_{1,\ldots,m}(\lambda_1, \ldots, \lambda_m) \) is translationally invariant.

\[ D_{1,\ldots,m}(\lambda_1 + u, \ldots, \lambda_m + u) = D_{1,\ldots,m}(\lambda_1, \ldots, \lambda_m) \]

• \( D_{1,\ldots,m}(\lambda_1, \ldots, \lambda_m) \) fullfills the spin-conservation rule.

\[ [D_{1,\ldots,m}(\lambda_1, \ldots, \lambda_m)]_{\epsilon_1,\ldots,\epsilon_m} = 0 \quad \text{if} \quad n_1(\epsilon) \neq n_1(\bar{\epsilon}). \]
Proposition 1.3.

5. $D_{1,\ldots,m}(\lambda_1,\ldots,\lambda_m)$ is meromorphic in $\lambda_1,\ldots,\lambda_m$ with at most simple poles at $\lambda_i - \lambda_j \in \mathbb{Z}\setminus\{0, \pm 1\}$.

6. $\forall 0 < \delta < \pi$

$$\lim_{\lambda_1 \to \infty, \lambda_1 \in S_\delta} D_{1,\ldots,m}(\lambda_1,\ldots,\lambda_m) = \frac{1}{2} 1_{1}D_{2,\ldots,m}(\lambda_2,\ldots,\lambda_m),$$

where $S_\delta := \{\lambda \in \mathbb{C}|\delta < |\arg(\lambda)| < \pi - \delta\}$.

- 1. - 6. determine $D_m$ completely.

Remark 1.4.

7. From 2., 3., 4. and the analyticity of $D_m$ at $\lambda_1 = \lambda_2$ and $\lambda_1 = \lambda_2 - 1$

$$\Rightarrow P_{1,2}^- D_{1,2,\ldots,m}(\lambda - 1, \lambda, \ldots, \lambda_m) = P_{1,2}^- D_{3,\ldots,m}(\lambda_3, \ldots, \lambda_m).$$
The Snail Construction for $\mathfrak{sl}_2$

- **5.** Since $D_m$ is **meromorphic** in $\lambda_1$ with at most **simple poles**, it is enough to calculate the **residues** and consider the **asymptotic behaviour**.

- **Claim:** We have the relation

$$
\text{res}_{\lambda_1,j=k} \ D_{1,\ldots,m}(\lambda_1,\ldots,\lambda_m) = \quad (1.2)
$$

$$
\text{res}_{\lambda_1,j=k} \left\{ \frac{\omega(\lambda_{1,j})}{1 - \lambda_{1,j}^2} \tilde{X}[1,j](\lambda_1,\ldots,\lambda_m) \right\} (D_{m-2}(\lambda_2,\ldots,\hat{\lambda}_j,\ldots,\lambda_m))
$$

for the residues of $D_{1,\ldots,m}(\lambda_1,\ldots,\lambda_m)$, where $\frac{\omega(\lambda_{1,j})}{1 - \lambda_{1,j}^2} \tilde{X}[1,j](\lambda_1,\ldots,\lambda_m)$ is a single meromorphic function, the 'Snail Operator'.

- At integer values $\lambda_{1,j} = k$, the **Snail Operator** is completely determined by the **Kirillov Reshetikhin modules** $W^{(k)}$.

- Looking at the **asymptotics** w.r. to $\lambda_1$ after substracting the poles, it was possible to prove a **recursion relation** for $D_{1,\ldots,m}(\lambda_1,\ldots,\lambda_m)$ using Liouville’s theorem.
Taking the residue at $\mu_1 = \mu_2$, $R_{12}(\mu_1 - \mu_2 - 1)$ in the red circle (figure 1) reduces to $2P_{12}^-$ up to a scalar prefactor. As a consequence, we can apply the relation 7. to obtain the result in figure 2.
Figure 2 has to be understood as \( \lim_{\mu_1 \to \mu_2 - 1} (\mu_1 - \mu_2 + 1) \times \) Figure 1, where we split the operator \( 2P_{12}^- \) (a cross) into the tensor product of a singlet and its dual. The operator in the red box is the Snail Operator with \( k = 2 \) loops.
Drawing the Snail Operator in a slightly less compact way by not splitting up the projector $P^-$, we can see that it has $k$ closed loops (figure 3).

![Figure 3: The Snail Operator with $k$ (closed) loops.](image)

Note again that the picture is defined via the residuum at $\mu = \mu_2$ of the meromorphic function defined through the same picture, but with general parameter $\mu_2 \neq \mu$ of the second line.
T-Systems

Let us try to understand $\star$ algebraically:

- For the residuum at $\lambda_{1,2} = -k - 1$, we have to consider the Snail Operator with $k$ loops.

- Since every line can be regarded a fundamental representation of the Yangian $Y(\mathfrak{sl}_2)$, we have to deal with the tensor product of fundamental representations $W^{(1)}(\mu - k) \otimes W^{(1)}(\mu - k + 1) \otimes \cdots W^{(1)}(\mu)$.

- Note that the spectral parameters are in special position w.r. to their respective neighbours, i.e. we have a **short exact sequence**:

\[
W^{(0)} \hookrightarrow W^{(1)}(\mu) \otimes W^{(1)}(\mu + 1) \rightarrow W^{(2)}(\mu)
\]

- Considering a partition of unity with the respective projectors in the Snail Operator, the projector onto $W^{(0)}$ cancels out.
• Writing only the **irreducible composition factors** of the possible **short exact sequences**, we get equations in the **Grothendieck ring**, the **$t$-systems**. For instance

$$[W^{(k)}(\mu)][W^{(k)}(\mu + 1)] = [W^{(k+1)}(\mu)][W^{(k-1)}(\mu + 1)] + 1.$$ 

• These are sometimes written in terms of **transfer matrices** with the respective representations in the auxiliary space.

• Using the $t$-system above, one can derive the **$t$-systems**

$$[W^{(p)}(\mu - p)][W^{(1)}(\mu)] = [W^{(p+1)}(\mu - p)] + [W^{(p-1)}(\mu - p)]$$

which **appear in the Snail Operator successively**.

• The second component cancels out as above. Thus, only the **Kirillov Reshetikhin module** $W^{(k)}$ remains.
Generalization to higher rank

- Properties 1., 2., and 3. are straightforward to generalize.
- To write a rqKZ equation for rank $n \geq 3$, we need to introduce an additional density matrix $D^{(1)}$. Then we have two rqKZ equations between $D$ and $D^{(1)}$.

![ rqKZ equation combined ]

**Figure 4:** rqKZ equation combined
• 5. and 6. are thought to have a straightforward generalization as well, but it has to be proven (limit from the massive Vertex Operator approach).

• Finally, a **generalization of the identity 7**. for the projector $P^-$ on the singlet **exists**, but it only applies for the combined rqKZ equation when calculating the residue.

  ➤ **Snail Operator decouples** only when taking the **residue at**
  \[ \lambda_{12} = \pm(n + 1)k, \; k \in \mathbb{N}\backslash\{0\}. \]

• The other residues at \[ \lambda_{12} = \pm(n + 1)(k + \frac{1}{2}), \; k \in \mathbb{N}\backslash\{0\}, \] have to be considered separately.

  ➤ There is a way to get a relation with the **projector onto** the **antifundamental** representation for $\mathfrak{sl}_3$, **generally** it is **still a problem**.

  ➤ Possible to **calculate the residues** of $D_4$ in terms of $D_3$ and $D_2$ for $\mathfrak{sl}_3$ in mathematica using the result of Boos et al. 2018.
However, the Snail Operator can still be defined in the same way as for $sl_2$.

*Figure 5: The Snail Operator with three loops.*

- What are the **composition factors** in the tensor product of fundamental and antifundamental representations of the Yangian?
Snake modules and extended t-systems

- Category of fin. dim. repr. of $\mathcal{Y}(\mathfrak{sl}_{n+1})$ is far from semisimple, but there is a **multiplicative** notion of **dominant highest (loop-)weights**.

- The **fundamental evaluation modules** have (multiplicative!) highest $l$-weights $Y_{i,a}$, $i \in \{1, \ldots, n\} =: I$, $a \in \mathbb{C}$.

- Any **irreducible module** is contained in the **tensor product** of fundamental modules.

- For any **monomial** $m = \prod_{i \in I, a \in \mathbb{C}} Y_{i,a}$ in the fundamental $l$-weights, there is a **unique** fin. dim. simple module $L(m)$.

- W.l.o.g. we can assume all **loop parameters** to be **half integer valued** since
  - any **fusion** is happening when the loop parameters differ by half integers.
  - we can **recover any simple module** by adjoining new parameters $a \in \mathbb{C}$ via the hopf algebra automorphism $\tau_a$ and considering **tensor products**.

- We can visualize these modules $L(m)$ on the lattice $I \times \mathbb{Z} =: \mathcal{X}$!
Snake position and snakes

- Let $(i, k) \in \mathcal{X}$. A point $(i', k')$ is said to be in \textit{snake position} with respect to $(i, k)$ iff $k' - k \geq |i' - i| + 2$.
- The point $(i', k')$ is in \textit{minimal snake position} to $(i, k)$ iff $k' - k$ is equal to the \textit{lower bound}.
- We say that $(i', k') \in \mathcal{X}$ is in \textit{prime snake position} with respect to $(i, k)$ iff $i' + i \geq k' - k \geq |i' - i| + 2$.
- A finite sequence $(i_t, k_t) \ (1 \leq t \leq M \in \mathbb{N})$ of points in $\mathcal{X}$ is a \textit{snake} iff for all $2 \leq t \leq M$, $(i_t, k_t)$ is in \textit{snake position} with respect to $(i_{t-1}, k_{t-1})$.
- It is a \textit{minimal} (resp. \textit{prime}) \textit{snake} iff any two successive points are in \textit{prime snake position} to each other.
Snake modules

- The simple module $L(m)$ is a (minimal/prime) snake module iff $m = \prod_{t=1}^{M} Y_{i_t,k_t}$ for some (minimal/prime) snake $(i_t, k_t)_{1 \leq t \leq M}$.
- A snake module is prime iff its snake is prime.
- Prime snake modules are real.
• For any two successive points define the \textit{neighbouring points} by
\[
\begin{align*}
X_{i,k}^{i',k'} &:= \begin{cases} 
((\frac{1}{2}(i+k+i'-k'), \frac{1}{2}(i+k-i'+k'))) & k+i > k'-i' \\
\emptyset & k+i = k'-i'
\end{cases} \\
Y_{i,k}^{i',k'} &:= \begin{cases} 
((\frac{1}{2}(i' + k' + i - k), \frac{1}{2}(i' + k' - i + k))) & k+N+1 - i > k'-(N+1-1') \\
\emptyset & k+N+1 - i = k' - N-1 + i'.
\end{cases}
\end{align*}
\]

• For any \textbf{prime snake} \((i_t, k_t)_{1 \leq k \leq M}\) we define its \textit{neighbouring snakes}
\[
\mathcal{X} := \mathcal{X}_{(i_t,k_t)_{1 \leq k \leq M}} \quad \text{and} \quad \mathcal{Y} := \mathcal{Y}_{(i_t,k_t)_{1 \leq k \leq M}}
\]
by concatenating its neighbouring points.

\[\Rightarrow\] \textit{We have the \textit{extended t-system}}
\[
\begin{align*}
&\left[ L \left( \prod_{t=1}^{M-1} Y_{i_t,k_t} \right) \right] \left[ L \left( \prod_{t=2}^{M} Y_{i_t,k_t} \right) \right] = \\
&\left[ L \left( \prod_{t=2}^{M-1} Y_{i_t,k_t} \right) \right] \left[ L \left( \prod_{t=1}^{M} Y_{i_t,k_t} \right) \right] \\
&+ \left[ L \left( \prod_{(i,k) \in \mathcal{X}} Y_{i_t,k_t} \right) \right] \left[ L \left( \prod_{(i,k) \in \mathcal{Y}} Y_{i_t,k_t} \right) \right],
\end{align*}
\]
(2.1)

where the summands on the right hand side are classes of simple modules.
• Let’s draw an example for $A_4$.

\[ A_4 \]

0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots

• The extended $t$-system includes the usual $t$-system.

• Note that the **loop parameters** of successive lines in the **Snail Operator** are in **minimal snake position**.
Conclusion

One can now prove the assertions

- The tensor product $V_{N(m),k} \otimes V_{N(m+1),k+n+1} \otimes \cdots \otimes V_{N(m+l),k+l(n+1)}$ of $l + 1$ many antifundamental and fundamental modules has Fibonacci$(l + 1)$ many composition factors, one of which is the minimal snake module $S_{m}^{(l+1)} := L(\prod_{t=0}^{l} Y_{N(t+m),k+t(n+1)})$, where $N(t) := \begin{cases} 1, & t \text{ even} \\ n, & t \text{ odd} \end{cases}$.

- Using the ext. $t$-system above, one can derive the ext. $t$-systems

$$[S_{m}^{(p)}(\mu - p \frac{n+1}{2})][S_{m+1}^{(1)}(\mu)] = [S_{m+1}^{(p+1)}(\mu - p \frac{n+1}{2})] + [S_{m+1}^{(p-1)}(\mu - p \frac{n+1}{2})]$$

which appear in the Snail Operator successively.

- Conjecture 2.1. Only the minimal snake module $S^{(k)}$ remains in the Snail Operator - completely analogue to $sl_2$. 

Let’s draw the snake in the snail for $\mathfrak{sl}_3$ and $(k = 4)$:
Let the action of the density-Matrices $D$ and $D^{(1)}$ be defined as

$$D_{1,...,m}(\lambda_1, \ldots, \lambda_n) (X_{1,...,m}) := tr_{1,...,m} (D_{1,...,m}(\lambda_1, \ldots, \lambda_m)X_{1,...,m}),$$

$$D^{(1)}_{\bar{1},2,...,m}(\lambda_1, \ldots, \lambda_m) (X_{\bar{1},2,...,m}) := tr_{\bar{1},2,...,m} \left( D^{(1)}_{\bar{1},2,...,m}(\lambda_1, \ldots, \lambda_m)X_{\bar{1},2,...,m} \right),$$

Then one can write the two reduced qKZ-equations as

$$D^{(1)}_{\bar{1},2,...,m}(\lambda_1 - \frac{n+1}{2}, \lambda_2, \ldots, \lambda_m) = A^{(1)}_{1,\bar{1}|2,...,m}(\lambda_1|\lambda_2, \ldots, \lambda_m) (D_{1,...,m}(\lambda_1, \lambda_2, \ldots, \lambda_m))$$
$$:= tr_1 \left( R_{1m}(\lambda_1 - \lambda_m) \cdots R_{12}(\lambda_1 - \lambda_2) D_{1,...,m}(\lambda_1, \lambda_2, \ldots, \lambda_m)(n+1)P^\perp_{1\bar{1}} \right) \tag{3.1}$$

$$R_{21}(\lambda_2 - \lambda_1) \cdots R_{m1}(\lambda_m - \lambda_1), \tag{3.2}$$

$$D_{1,...,m}(\lambda_1 - \frac{n+1}{2}, \lambda_2, \ldots, \lambda_m) = A^{(2)}_{\bar{1},1|2,...,m}(\lambda_1|\lambda_2, \ldots, \lambda_m) \left( D^{(1)}_{\bar{1},2,...,m}(\lambda_1, \lambda_2, \ldots, \lambda_m) \right)$$
$$:= tr_{\bar{1}} \left( \bar{R}_{\bar{1}m}(\lambda_1 - \lambda_m) \cdots \bar{R}_{\bar{1}2}(\lambda_1 - \lambda_2) D^{(1)}_{\bar{1},2,...,m}(\lambda_1, \lambda_2, \ldots, \lambda_m)(n+1)P^\perp_{1\bar{1}} \right) \tag{3.3}$$

$$\bar{R}_{2\bar{1}}(\lambda_2 - \lambda_1) \cdots \bar{R}_{m\bar{1}}(\lambda_m - \lambda_1), \tag{3.4}$$

where $(P^-_{1\bar{1}})^2 = P^-_{1\bar{1}}$ is the projector onto the singlet in the tensor product $V \otimes \bar{V}$ of the fundamental and antifundamental representation of $\mathfrak{sl}_n$. 

25 / 26
Proof of the projection relation 7. for \( \mathfrak{sl}_2 \):