# Snake Modules, Extended T-Systems and Correlation Functions for higher rank

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# Overview

## An inhomogeneous XXX-Model

### Correlation functions in the $\mathfrak{sl}_2$ case

### The Snail Construction for $\mathfrak{sl}_2$

### T-Systems

## Generalization to higher rank

### Snake modules and extended t-systems

## Conclusion

1. 2006 Boos, Jimbo, Miwa, Smirnov and Takeyama
   'A RECUSION FORMULA FOR THE CORRELATION FUNCTIONS OF AN INHOMOGENEOUS XXX-MODEL'
2. 2012 E. Moukhin · C. A. S. Young
   'Extended T-Systems'
3. 2018 Boos, Hutsalyuk and Nirov
   'ON THE CALCULATION OF THE CORRELATION FUNCTIONS OF THE $\mathfrak{sl}_3$-MODEL BY MEANS OF THE REDUCED QKZ EQUATION'
4. 2019 Klümper, Nirov and Razumov
   'REDUCED QKZ EQUATION: GENERAL CASE'

# An inhomogeneous XXX-Model

- $H_{XXX} = \frac{1}{2}\sum_j \left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \sigma_j^z\sigma_{j+1}^z\right)$
  gapless case $q \to 1$, $\Delta = \frac{q+q^{-1}}{2}$
- 'Inhomogeneous chain' generated by
  $tr\,(R_{a,-N}(\lambda)\cdots R_{a,0}(\lambda)R_{a,1}(\lambda-\lambda_1)\cdots R_{a,m}(\lambda-\lambda_m)R_{a,m+1}(\lambda)\cdots R_{a,N}(\lambda))$
- Still exactly solvable but the interaction is nonlocal.
- Vertex operator approach (XXZ gapped regime)
  - multiple integrals which can be analytically continued for $q \to 1$
- Factorisation (Boos, Korepin and Smirnov)

Conjecture 1.1.

$$[D_{1,..,m}(\lambda_1,...,\lambda_m)]^{\epsilon_1\cdots\epsilon_m}_{\bar{\epsilon_1}\cdots\bar{\epsilon_m}} := \langle vac|\,(E^{\epsilon_1}{}_{\bar{\epsilon_1}})_1\cdots(E^{\epsilon_m}{}_{\bar{\epsilon_m}})_m\,|vac\rangle \quad (1.1)$$
$$= \sum\prod\omega(\lambda_i-\lambda_j)f(\lambda_1,...,\lambda_m),$$

where $\omega(\lambda)$ is a single transzendental function and the functions $f(\lambda_1,...,\lambda_n)$ are rational.

- Odd integer values of $\zeta$ appear in the Taylor series of $\omega$ for the homogeneous limit.
- Proof of (1.1) via the 'Snail Construction'.
- Result can be written in terms of a transfer matrix over an auxilliary space of 'fractional dimension'.
- Generalization to XXZ
- Fermionic structure $\longrightarrow$ HGS papers.

# Correlation functions in the $\mathfrak{sl}_2$ case

- Define the rational R-matrix by

$$R(\lambda) = \frac{\rho(\lambda)}{\lambda + 1} (\lambda + P), \quad \rho(\lambda) = -\frac{\Gamma(\frac{\lambda}{2})\Gamma(\frac{1}{2} - \frac{\lambda}{2})}{\Gamma(-\frac{\lambda}{2})\Gamma(\frac{1}{2} + \frac{\lambda}{2})}$$

Proposition 1.2. The reduced density matrix $D_{1,\ldots,m}$ fulfills

1. The global $GL_2(\mathbb{C})$-invariance .
2. The R-matrix relations

$$D_{1,\ldots,i+1,i,\ldots,m}(\lambda_1, \ldots, \lambda_{i+1}, \lambda_i, \ldots, \lambda_m) =$$
$$R_{i+1,i}(\lambda_{i+1,i}) D_{1,\ldots,m}(\lambda_1, \ldots, \lambda_m) R_{i,i+1}(\lambda_{i,i+1}).$$

3. Left-right reduction relations

$$tr_1(D_{1,\ldots,m}(\lambda_1, \ldots, \lambda_m)) = D_{2,\ldots,m}(\lambda_2, \ldots, \lambda_m)$$
$$tr_n(D_{1,\ldots,m}(\lambda_1, \ldots, \lambda_m)) = D_{1,\ldots,m-1}(\lambda_1, \ldots, \lambda_{m-1}).$$

4. The rqKZ-equation

$$D_{1,\ldots,m}(\lambda_1 - 1, \lambda_2, \ldots, \lambda_m) = A_{\bar{1},1}(\lambda_1, \ldots, \lambda_m)(D_{\bar{1},2,\ldots,m}(\lambda_1, \lambda_2, \ldots, \lambda_m)).$$

Note:

- Due to $\rho(\lambda)\rho(-\lambda) = 1$ and $\rho(\lambda - 1)\rho(\lambda) = -\frac{\lambda}{\lambda-1}$ the coefficients in 2. and 4. are rational.
- $D_{1,..,m}(\lambda_1, \ldots, \lambda_m)$ is translationally invariant.

$$D_{1,..,m}(\lambda_1 + u, \ldots, \lambda_m + u) = D_{1,..,m}(\lambda_1, \ldots, \lambda_m)$$

- $D_{1,..,m}(\lambda_1, \ldots, \lambda_m)$ fullfills the spin-conservation rule.

$$[D_{1,..,m}(\lambda_1, \ldots, \lambda_m)]^{\epsilon_{\mathbf{1}} \cdots \epsilon_m}_{\bar{\epsilon_{\mathbf{1}}} \cdots \bar{\epsilon_m}} = 0 \quad \text{if} \quad n_1(\epsilon) \neq n_1(\bar{\epsilon}).$$

Proposition 1.3.

5. $D_{1,..,m}(\lambda_1, ..., \lambda_m)$ is meromorphic in $\lambda_1, ..., \lambda_m$ with at most simple poles at $\lambda_i - \lambda_j \in \mathbb{Z} \backslash \{0, \pm 1\}$.
6. $\forall\, 0 < \delta < \pi$

$$\lim_{\substack{\lambda_{\mathbf{1}} \to \infty \\ \lambda_{\mathbf{1}} \in S_\delta}} D_{1,..,m}(\lambda_1, ..., \lambda_m) = \frac{1}{2} \mathbf{1}_1 D_{2,...,m}(\lambda_2, ..., \lambda_m),$$

where $S_\delta := \{\lambda \in \mathbb{C} | \delta < |\arg(\lambda)| < \pi - \delta\}$.

- 1. - 6. determine $D_m$ completely.

Remark 1.4.

7. From 2., 3., 4. and the analyticity of $D_m$ at $\lambda_1 = \lambda_2$ and $\lambda_1 = \lambda_2 - 1$

$$\Rightarrow \quad P^{-}_{1,2} D_{1,2,..,m}(\lambda - 1, \lambda, ..., \lambda_m) = P^{-}_{1,2} D_{3,..,m}(\lambda_3, ..., \lambda_m).$$

# The Snail Construction for $\mathfrak{sl}_2$

- 5.: Since $D_m$ is **meromorphic** in $\lambda_1$ with at most **simple poles**, it is enough to calculate the **residues** and consider the **asymptotic behaviour**.
- Claim: We have the relation

$$\operatorname*{res}_{\lambda_{1,j}=k} D_{1,..,m}(\lambda_1, ..., \lambda_m) = \tag{1.2}$$

$$\operatorname*{res}_{\lambda_{1,j}=k} \left\{ \frac{\omega(\lambda_{1,j})}{1-\lambda_{1,j}^2} \tilde{X}^{[1,j]}(\lambda_1, ..., \lambda_m) \right\} (D_{m-2}(\lambda_2, ..., \hat{\lambda}_j, ..., \lambda_m))$$

for the residues of $D_{1,..,m}(\lambda_1, ..., \lambda_m)$, where $\frac{\omega(\lambda_{1,j})}{1-\lambda_{1,j}^2}\tilde{X}^{[1,j]}(\lambda_1, ..., \lambda_m)$ is a single meromorphic function, the 'Snail Operator'.

  - ⋆ At integer values $\lambda_{1,j} = k$, the **Snail Operator** is completely determined by the **Kirillov Reshetikhin modules** $W^{(k)}$.
  - ▶ Looking at the **asymptotics** w.r. to $\lambda_1$ after substracting the poles, it was possible to prove a **recursion relation** for $D_{1,..,m}(\lambda_1, ..., \lambda_m)$ using Liouvilles theorem.

## In Pictures:

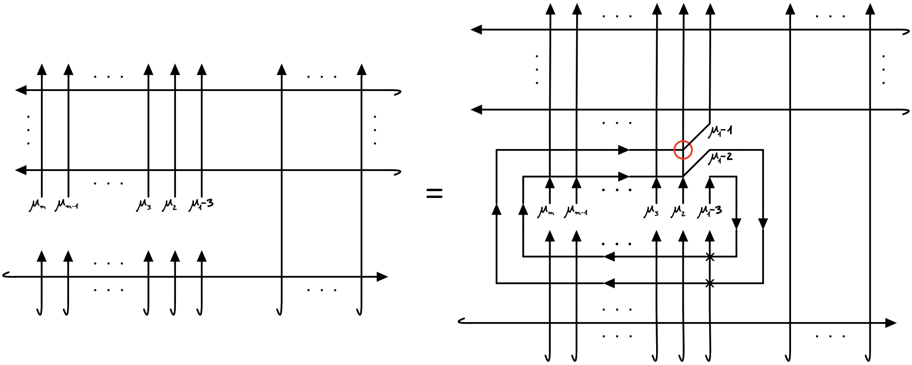



Figure 1: rqKZ-equation two times.

Taking the residue at $\mu_1 = \mu_2$, $R_{12}(\mu_1 - \mu_2 - 1)$ in the red circle (figure 1) reduces to $2P^{-}_{12}$ up to a scalar prefactor. As a consequence, we can apply the relation 7. to obtain the result in figure 2.

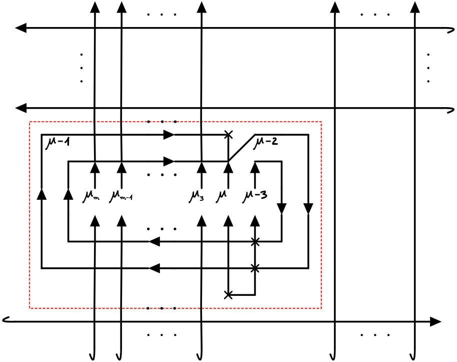



Figure 2: The Snail with two loops ($k = 2$).

Figure 2 has to be understood as $\lim_{\mu_1 \to \mu_2 - 1} (\mu_1 - \mu_2 + 1) \times$Figure 1, where we split the operator $2P^-_{12}$ (a cross) into the tensor product of a singlet and its dual. The operator in the red box is the Snail Operator with $k = 2$ loops.

Drawing the Snail Operator in a slightly less compact way by not splitting up the projector $P^-$, we can see that it has $k$ closed loops (figure 3).

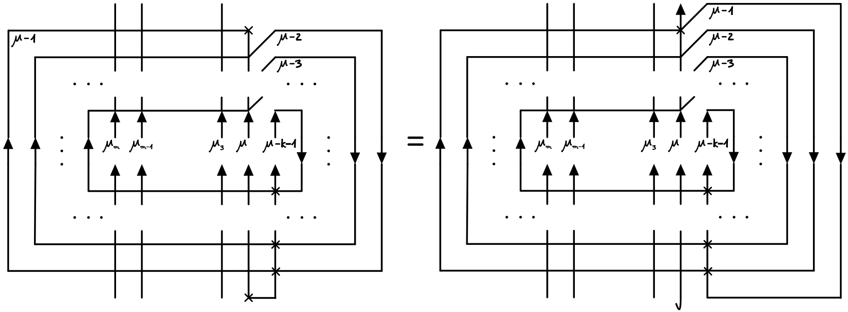



Figure 3: The Snail Operator with $k$ (closed) loops.

Note again that the picture is defined via the residuum at $\mu = \mu_2$ of the meromorphic function defined through the same picture, but with general parameter $\mu_2 \neq \mu$ of the second line.

# T-Systems

Let us try to understand $\star$ algebraically:

- For the residuum at $\lambda_{1,2} = -k - 1$, we have to consider the Snail Operator with $k$ loops.
- Since every line can be regarded a fundamental representation of the Yangian $Y(\mathfrak{sl}_2)$, we have to deal with the tensor product of fundamental representations $W^{(1)}(\mu - k) \otimes W^{(1)}(\mu - k + 1) \otimes \cdots W^{(1)}(\mu)$.
- Note that the spectral parameters are in special position w.r. to their respecive neighbours, i.e. we have a ***short exact sequence***:

$$W^{(0)} \hookrightarrow W^{(1)}(\mu) \otimes W^{(1)}(\mu + 1) \twoheadrightarrow W^{(2)}(\mu)$$

- Considering a partition of unity with the repective projectors in the Snail Operator, the projector onto $W^{(0)}$ cancels out.

- Writing only the **irreducible composition factors** of the possible **short exact sequences**, we get equations in the **Grothendieck ring**, the ***t*-systems**. For instance

$$[W^{(k)}(\mu)][W^{(k)}(\mu+1)] = [W^{(k+1)}(\mu)][W^{(k-1)}(\mu+1)] + 1.$$

- These are somtimes written in terms of ***transfer matrices*** with the respective representations in the auxiliary space.
- Using the $t$-system above, one can derive the ***t*-systems**

$$[W^{(p)}(\mu-p)][W^{(1)}(\mu)] = [W^{(p+1)}(\mu-p)] + [W^{(p-1)}(\mu-p)]$$

  which **appear in the Snail Operator successively**.
- The second component cancels out as above. Thus, only the **Kirillov Reshetikhin module** $W^{(k)}$ remains.

# Generalization to higher rank

- Properties 1.,2. and 3. are straightforward to generalize.
- To write a rqKZ equation for rank $n \geq 3$, we need to introduce an additional density matrix $D^{(1)}$. Then we have **two rqKZ equations** between $D$ and $D^{(1)}$.

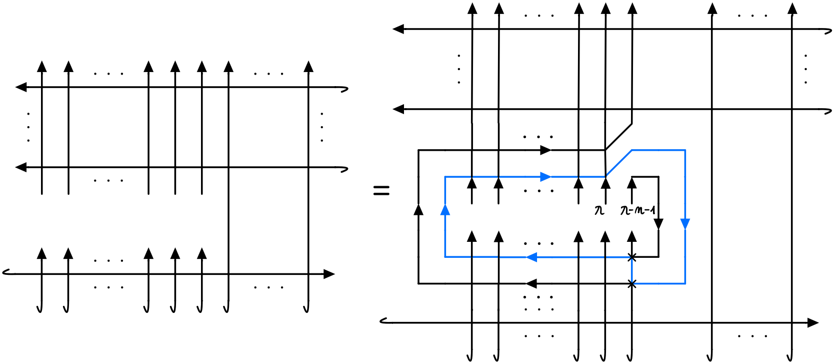



Figure 4: rqKZ equation combined

- 5. and 6. are thought to have a straightforward generalization aswell, but it has to be proven (limit from the massive Vertex Operator approach).
- Finally, a **generalization of the identity 7.** for the projector $P^-$ on the singlet **exists**, but it only applies for the combined rqKZ equation when calculating the residue.

- **Snail Operator decouples** only when taking the **residue at $\boldsymbol{\lambda_{12} = \pm(n+1)k}$**, $k \in \mathbb{N}\backslash\{0\}$.

- The other residues at $\lambda_{12} = \pm(n+1)(k+\frac{1}{2})$, $k \in \mathbb{N}\backslash\{0\}$, have to be considered seperately.

- There is a way to get a relation with the **projector onto** the **antifundamental** representation for $\mathfrak{sl}_3$, **generally** it is **still a problem**.

- Possible to **calculate the residues** of $D_4$ in terms of $D_3$ and $D_2$ for $\mathfrak{sl}_3$ in mathematica using the result of Boos et al. 2018.

However, the Snail Operator can still be defined in the same way as for $\mathfrak{sl}_2$.

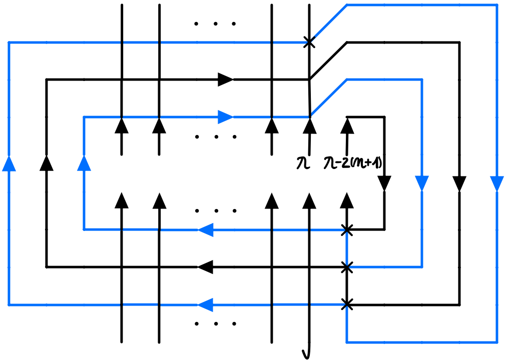



Figure 5: The Snail Operator with three loops.

- What are the **composition factors** in the tensor product of fundamental and antifundamental representations of the Yangian?

## Snake modules and extended t-systems

- Category of fin. dim. repr. of $Y(\mathfrak{sl}_{n+1})$ is far from semisimple, but there is a **multiplicative** notion of **dominant highest (loop-)weights**.
- The **fundamental evaluation modules** have (multiplicative!) highest **l-weights** $Y_{i,a}$, $i \in \{1, \dots, n\} =: I$, $a \in \mathbb{C}$.
- Any **irreducible module** is contained in the **tensor product** of fundamental modules.
- For any **monomial** $m = \Pi_{i \in I, a \in \mathbb{C}} Y_{i,a}$ in the **fundamental l-weights**, there is a **unique** fin. dim. simple module $\boldsymbol{L(m)}$.
- W.l.o.g. we can assume all **loop parameters** to be **half integer valued** since
  - any **fusion** is happening when the **loop parameters** differ by **half integers**.
  - we can **recover any simple module** by **adjoining new parameters** $a \in \mathbb{C}$ via the hopf algebra automorphism $\tau_a$ and considering **tensor products**.

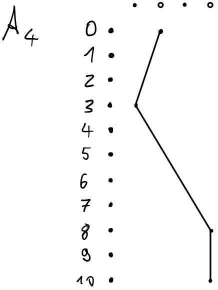
► We can visualize these modules $L(m)$ on the lattice $I \times \mathbb{Z} =: \mathcal{X}$!

## Snake position and snakes

- Let $(i, k) \in \mathcal{X}$. A point $(i', k')$ is said to be in ***snake position*** with respect to $(i, k)$ iff $k' - k \geq |i' - i| + 2$.
- The point $(i', k')$ is in ***minimal snake position*** to $(i, k)$ iff $k' - k$ is equal to the **lower bound**.
- We say that $(i', k') \in \mathcal{X}$ is in **prime snake position** with respect to $(i, k)$ iff $i' + i \geq k' - k \geq |i' - i| + 2$.
- A finite sequence $(i_t, k_t)$ $(1 \leq t \leq M \in \mathbb{N})$ of points in $\mathcal{X}$ is a ***snake*** iff for all $2 \leq t \leq M$, $(i_t, k_t)$ is in *snake position* with respect to $(i_{t-1}, k_{t-1})$.
- It is a ***minimal*** (resp. ***prime***) ***snake*** iff any two successive points are in *prime snake position* to each other.

## Snake modules

- The simple module ***L*(*m*)** is a (***minimal/prime***) ***snake module*** iff $m = \prod_{t=1}^{M} Y_{i_t,k_t}$ for some (*minimal/prime*) *snake* $(i_t, k_t)_{1 \le t \le M}$.
- A ***snake module*** is **prime** iff its *snake* is ***prime***.
- ***Prime snake modules*** are **real**.



- For any two successive points define the **neighbouring points** by

$$\mathbb{X}_{i,k}^{i',k'} := \begin{cases} ((\frac{1}{2}(i+k+i'-k'), \frac{1}{2}(i+k-i'+k'))) & {\scriptstyle k+i>k'-i'} \\ \emptyset & {\scriptstyle k+i=k'-i'} \end{cases}$$

$$\mathbb{Y}_{i,k}^{i',k'} := \begin{cases} ((\frac{1}{2}(i'+k'+i-k), \frac{1}{2}(i'+k'-i+k))) & {\scriptstyle k+N+1-i>k'-(N+1-i')} \\ \emptyset & {\scriptstyle k+N+1-i=k'-N-1+i'.} \end{cases}$$

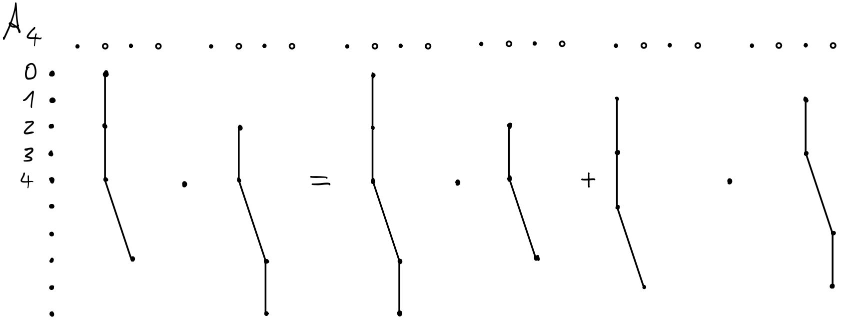
- For any **prime snake** $(i_t, k_t)_{1\leq k\leq M}$ we define its neighbouring snakes $\mathbb{X} := \mathbb{X}_{(i_t,k_t)_{1\leq k\leq M}}$ and $\mathbb{Y} := \mathbb{Y}_{(i_t,k_t)_{1\leq k\leq M}}$ by concatenating its neighbouring points.

$\Rightarrow$ We have the ***extended t-system***

$$\left[L\left(\prod_{t=1}^{M-1} Y_{i_t,k_t}\right)\right]\left[L\left(\prod_{t=2}^{M} Y_{i_t,k_t}\right)\right] = \left[L\left(\prod_{t=2}^{M-1} Y_{i_t,k_t}\right)\right]\left[L\left(\prod_{t=1}^{M} Y_{i_t,k_t}\right)\right] + \left[L\left(\prod_{(i,k)\in\mathbb{X}} Y_{i_t,k_t}\right)\right]\left[L\left(\prod_{(i,k)\in\mathbb{Y}} Y_{i_t,k_t}\right)\right], \quad (2.1)$$

where the summands on the right hand side are classes of simple modules.

- Let's draw an example for $A_4$.




- The extended $t$-system includes the usual $t$-system.
- Note that the **loop parameters** of successive lines in the **Snail Operator** are in ***minimal snake position***.

# Conclusion

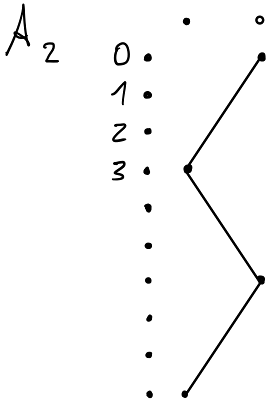
One can now proof the assertions

- The tensor product $V_{N(m),k} \otimes V_{N(m+1),k+n+1} \otimes \cdots \otimes V_{N(m+l),k+l(n+1)}$ of $l+1$ many **antifundamental** and **fundamental** modules has **Fibonacci**$(l+1)$ many **composition factors**, one of which is the ***minimal snake module*** $S_m^{(l+1)} := L(\prod_{t=0}^{l} Y_{N(t+m),k+t(n+1)})$, where $N(t) := \begin{cases} 1, t \text{ even} \\ n, t \text{ odd} \end{cases}$.
- Using the ext. $t$-system above, one can derive the ext. $t$-systems

  $$[S_m^{(p)}(\mu - p\frac{n+1}{2})][S_{m+1}^{(1)}(\mu)] = [S_{m+1}^{(p+1)}(\mu - p\frac{n+1}{2})] + [S_{m+1}^{(p-1)}(\mu - p\frac{n+1}{2})]$$

  which **appear in the Snail Operator successively**.
- Conjecture 2.1. **Only the minimal snake module $S^{(k)}$ remains** in the Snail Operator - completely analogue to $\mathfrak{sl}_2$.

Let's draw the snake in the snail for $\mathfrak{sl}_3$ and $(k = 4)$:



Let the action of the density-Matrices $D$ and $D^{(1)}$ be defined as

$$D_{1,\dots,m}(\lambda_1,\dots,\lambda_n)(X_{1,\dots,m}) := tr_{1,\dots,m}\left(D_{1,\dots,m}(\lambda_1,\dots,\lambda_m)X_{1,\dots,m}\right),$$
$$D^{(1)}_{\bar{1},2,\dots,m}(\lambda_1,\dots,\lambda_m)(X_{\bar{1},2,\dots,m}) := tr_{\bar{1},2,\dots,m}\left(D^{(1)}_{\bar{1},2,\dots,m}(\lambda_1,\dots,\lambda_m)X_{\bar{1},2,\dots,m}\right),$$

Then one can write the two reduced qKZ-equations as

$$D^{(1)}_{\bar{1},2,\dots,m}(\lambda_1-\frac{n+1}{2},\lambda_2,\dots,\lambda_m) = A^{(1)}_{1,\bar{1}|2,\dots,m}(\lambda_1|\lambda_2,\dots,\lambda_m)\left(D_{1,\dots,m}(\lambda_1,\lambda_2,\dots,\lambda_m)\right)$$
$$:= tr_1\Big(R_{1m}(\lambda_1-\lambda_m)\cdots R_{12}(\lambda_1-\lambda_2)D_{1,\dots,m}(\lambda_1,\lambda_2,\dots,\lambda_m)(n+1)P^{-}_{1\bar{1}} \tag{3.1}$$
$$R_{21}(\lambda_2-\lambda_1)\cdots R_{m1}(\lambda_m-\lambda_1)\Big), \tag{3.2}$$
$$D_{1,\dots,m}(\lambda_1-\frac{n+1}{2},\lambda_2,\dots,\lambda_m) = A^{(2)}_{\bar{1},1|2,\dots,m}(\lambda_1|\lambda_2,\dots,\lambda_m)\left(D^{(1)}_{\bar{1},2,\dots,m}(\lambda_1,\lambda_2,\dots,\lambda_m)\right)$$
$$:= tr_{\bar{1}}\Big(\bar{\bar{R}}_{\bar{1}m}(\lambda_1-\lambda_m)\cdots\bar{\bar{R}}_{\bar{1}2}(\lambda_1-\lambda_2)D^{(1)}_{\bar{1},2,\dots,m}(\lambda_1,\lambda_2,\dots,\lambda_m)(n+1)P^{-}_{1\bar{1}} \tag{3.3}$$
$$\bar{R}_{2\bar{1}}(\lambda_2-\lambda_1)\cdots\bar{R}_{m\bar{1}}(\lambda_m-\lambda_1)\Big), \tag{3.4}$$

where $(P^{-}_{1\bar{1}})^2 = P^{-}_{1\bar{1}}$ is the projector onto the singlet in the tensor product $V\otimes\bar{V}$ of the fundamental and antifundamental representation of $\mathfrak{sl}_n$.

Proof of the projection relation 7. for $\mathfrak{sl}_2$: