Hilbert space fragmentation in integrable systems

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Thermalization and integrability

• The observable O thermalizes if after some relaxation time, the average expectation value of this observable agrees with the microcanonical expectation value
For the initial state $|\psi_0\rangle$ and (an arbitrary) operator O evolution is provided by

$$|\psi(t)\rangle = \sum_m C_m e^{-iE_m t}|m\rangle, \qquad C_m = \langle m|\psi_0\rangle$$

$$\langle\psi(t)|\hat{O}|\psi(t)\rangle = \sum_{m,n} C_m^* C_n e^{i(E_m-E_n)t} O_{mn} = \sum_m |C_m|^2 O_{mm} + \sum_{n\neq m} C_m^* C_n e^{i(E_m-E_n)t} O_{mn}$$

i. How to proceed to a microcanonical ensemble in first term (time independent!)
ii. What to do with states that are exponentially close to each other in the second term

ETH

- Description of a *thermalization* relies on the eigenstate thermalization hypothesis (ETH) J. M. Deutsch (1991) *Phys. Rev. A* **43** 2046; M. Srednicki (1994) *Phys. Rev. E* **50** 888
- ETH could be considered as an ansatz for matrix elements

$$O_{mn} = \delta_{mn} O(\bar{E}) + e^{-S(\bar{E})} f_O(\bar{E}, \omega) R_{mn}, \qquad \omega = E_n - E_m, \qquad \bar{E} = \frac{E_m + E_n}{2}$$

- $S(E)$ is an entropy
- f_O is a smooth function (operator dependent); $\overline{R_{mn}^2} = 1$, $\overline{|R_{mn}|^2} = 1$
- $O(\bar{E})$ coincides with an expectation value of the *microcanonical ensemble*

Using ETH it is possible to proceed to averaging with a *diagonal ensemble* ρ_{DE}

$$\langle O \rangle = \lim_{t_0 \to \infty} \frac{1}{t_0} \int_0^{t_0} dt \; O(t) = \sum_m |C_m|^2 O_{mm} = \mathrm{Tr}\left[\rho_{\mathrm{DE}} \mathrm{O}\right]$$

Integrable system, degeneracies and ETH

- In the integrable systems they are replaced by Generalized Gibbs ensemble (GGE)

$$\rho = \exp(-\beta H) \longrightarrow \exp\left(-\sum \beta_k H_k\right)$$

M. Rigol *et. al.* (2007) *Phys. Rev. Lett.* **98** 050405; P. Calabrese *et. al.* (2011) *Phys. Rev. Lett.* **106**, 227203; (2012) *J. Stat. Mech.* P07022

- Very successful for a lot of applications
- Note: only *weak ETH* is satisfied: an (exponentially!) small fraction of eigenstates does not obey the ETH, they have expectation values significantly different from the microcanonical ensemble
- However the complete breakdown of ETH could be expected in case of degeneracy!

Folded-XXZ via crystal limit

$$H = Q_4 = -\frac{1}{4}\sum_{j=1}^{L}\left(1+\sigma_j^z\sigma_{j+3}^z\right)\left(\sigma_{j+1}^+\sigma_{j+2}^- + \sigma_{j+1}^-\sigma_{j+2}^+\right)$$

Could be derived from XXZ chain using the crystal limit (note: Q_2 is nothing but Ising model)

$$Q_k = \lim_{\Delta\to\infty}\frac{\tilde{Q}_k}{\Delta^{[k/2]}}$$

where $\tilde{Q}_k$ is kth integral of motion of XXZ chain Earlier this model was discovered (at least...)
Z.-C. Yang, F. Liu, A. V. Gorshkov, T. Iadecola, *Phys. Rev. Lett.* **124**, 207602 (2020)

L. Zadnik, M. Fagotti, *SciPost Phys. Core* **4**, 10, (2021); L. Zadnik, K. Bidzhiev, M. Fagotti, *SciPost Phys.* **10**, 99 (2021)

Properties

- Crystal limit models are (at least supposedly) simpler than the original one
- Crystal limit automatically provides us with Hamiltonians that are integrable
- Limit is known for the Hamiltonian only. It is not clear how to generalize it for the **Lax operator and R-matrix** (?)
- It is naturally to expect a large degree of degeneracy in the spectrum since the crystal limit also affects Bethe equations

Bethe ansatz solution

Bethe equations naturally follows from the crystal limit of ordinary XXZ equations

$$e^{ip_{j,n}L}\prod_{\substack{m,k\\(j,n)\neq(k,m)}} S_{n,m}(p_{j,n}-p_{k,m})=1 \quad\longrightarrow e^{ip_{j,1}(L-N_1-2N_s)}=(-1)^{N_1-1}e^{-iP_1-2iP_s}$$

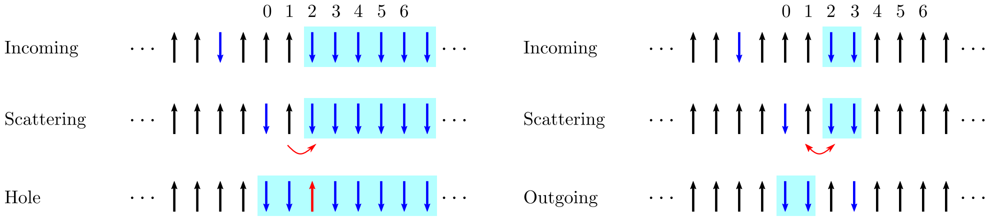
s is a number of strings, N_1 – number of particles
There are "particles" (and correspondingly scattering matrices) of two types

$$S_{1,1}(p,k)=-e^{-i(p-k)}, \qquad S_{1,n}(p,k)=e^{-2i(p-k)}$$ the last does not depend on n!

In this picture strings could be understood as "domain walls". The last are not dynamical on their own: **in the absence of particles** they lead to frozen configuration and exponentially degenerate Hilbert states

Domain walls (DW) dynamic

Scattering of a particle with a DW. As the result DW gets replaced by 2 sites to the left, and the trajectory of the particle receives a displacement of 1 site to the right



Dual transformation

Bond-site transformation

$$
\begin{aligned}
|\circ\rangle_j &\quad \text{if} \quad \sigma_j^z\sigma_{j+1}^z = 1 \\
|\bullet\rangle_j &\quad \text{if} \quad \sigma_j^z\sigma_{j+1}^z = -1
\end{aligned}
$$

We can interpret $|\circ\rangle$ as the up spin, and $|\bullet\rangle$ as the down spin
Note : transformation is good defined in boundary case. Some problems in a periodic case (we neglect it since TDL is object of interest at the very end)
Note: Q_1 becomes non-local

$$
Q_1 = \frac{1}{2}\sum_{j=1}^{L-1}\left[1 - \prod_{k=1}^{j}\sigma_j^z\right]
$$

Charges transformation (note: Q_4 is 3-site now)

$$Q_2 \to \sum_{j=1}^{L-1} \sigma_j^z$$

$$Q_3 \to i \sum_j \sigma_j^- P_{j+1}^\bullet \sigma_{j+2}^+ - \sigma_j^+ P_{j+1}^\bullet \sigma_{j+2}^-$$

$$Q_4 \to \sum_j \sigma_j^- P_{j+1}^\bullet \sigma_{j+2}^+ + \sigma_j^+ P_{j+1}^\bullet \sigma_{j+2}^-$$

here $P^\bullet = (1 - \sigma^z)/2$. The only non-zero elements of Qs are

$$|\circ \bullet \bullet\rangle \to |\bullet \bullet \circ\rangle, \qquad |\bullet \bullet \circ\rangle \to |\circ \bullet \bullet\rangle$$

Thus Q_3, Q_4, etc, move only double $\bullet\bullet$. Single $\bullet$ remains invariant!

Overlaps in folded-XXZ

Define set of states

$$|x_1, \dots, x_N\rangle = \sigma^-_{x_1} \dots \sigma^-_{x_N} |\emptyset\rangle,$$

restriction $1 \leq x_1 < \cdots < x_N \leq L$ is imposed to avoid double counting
Specific Neél state with N particles in a volume $L = 3N$

$$|\Psi'_0\rangle = |3, 6, 9, \dots\rangle$$

From the limit of Slavnov's determinant (overlap of XXZ Bethe vectors)

$$\langle \Psi'_0 | \boldsymbol{p} \rangle = \det_{jk} e^{ip_j(2k+1)}$$

On-shell overlap with initial state

We consider the initial state

$$|\Psi_0\rangle = \frac{1+U+U^2}{\sqrt{3}}|\Psi_0'\rangle,$$

where U is the one-site cyclic shift operator

In the zero momentum sector the Bethe equations are

$$e^{i2Np_j} = -1, \qquad j = 1, \ldots, N,$$

$L = 3N$, N assumed to be even for simplicity

The overlaps can be expressed simply as

$$|\langle\Psi_0|\boldsymbol{p}\rangle|^2 = \prod_{j<k} |e^{i2p_j} - e^{i2p_k}|^{-2} \to N^N$$

Solvable quench dynamics

Form factor sum could be written as

$$\langle \mathcal{O}(t) \rangle = \sum_{\boldsymbol{p},\boldsymbol{k}} \frac{\langle \Psi_0 | \boldsymbol{p} \rangle \langle \boldsymbol{p} | \mathcal{O} | \boldsymbol{k} \rangle \langle \boldsymbol{k} | \Psi_0 \rangle}{\langle \boldsymbol{p} | \boldsymbol{p} \rangle \langle \boldsymbol{k} | \boldsymbol{k} \rangle} e^{-i(E_{\mathbf{k}} - E_{\mathbf{p}})t}$$

Emptiness formation probability (EFP)

$$\mathbb{E}_\ell(x) = \prod_{j=1}^{\ell} \frac{1 + \sigma^z_{x-1+j}}{2}.$$

- Form factors should be computed $\langle \boldsymbol{p} | \mathcal{O} | \boldsymbol{k} \rangle$
- Summation over $\boldsymbol{p}$, $\boldsymbol{k}$ should be performed

Form factors

N. M. Bogoliubov, C. L. Malyshev, *Theor. Math. Phys.* **169**, 1517 (2011)

$$\langle \boldsymbol{p}|\mathbb{E}_\ell(x)|\boldsymbol{k}\rangle = \prod_{j\le k} \frac{1}{\left(e^{iu_j^C} - e^{iu_k^C}\right)\left(e^{iu_j^B} - e^{iu_k^B}\right)} \det \mathcal{T},$$

$$\mathcal{T}_{jj} = (N - L + \ell - 1)e^{-2ik_j}, \quad u_j^C = u_k^B,$$
$$\mathcal{T}_{jk} = e^{i(\ell-1)(u_j^B+u_k^C)} \frac{\sin\left((\ell-1)(u_j^B - u_k^C)\right)}{\sin(u_j^B - u_k^C)}, \quad u_j^C \neq u_k^B.$$

For diagonal part of $\mathcal{T}$ the rank is given by the number of coinciding elements in the sets $\bar{u}^C$, $\bar{u}^B$! This put very heavy restrictions on the summation

After simple summation over form factors the exact quench for $\mathbb{E}_3$ is given by

$$\langle\psi(t)|\mathbb{E}_3(x)|\psi(t)\rangle = \frac{1}{6} - \frac{1}{6}\left(\frac{1}{N}\sum_a \cos(2\cos(c_a)t)\right)^2 - \frac{1}{6}\left|\frac{1}{N}\sum_a \sin(2\cos(c_a)t)e^{ic_a}\right|^2$$

$c_a = \pi(2a-1)/(2N)$, $a = 1,\dots,N$. In the thermodynamic limit the last expression is

$$\langle\psi(t)|\mathbb{E}_3(x)|\psi(t)\rangle = \frac{1}{6}\left[1 - (J_0(2t))^2 - (J_1(2t))^2\right]$$

Thermalization in a fragmented Hilbert space

Problem of thermalization from the initial state $|\psi_0\rangle$

$$\langle \mathcal{O}(t)\rangle = \sum \langle \Psi_0|a\rangle\langle a|\mathcal{O}|b\rangle\langle b|\Psi_0\rangle e^{-i(E_b-E_a)t}$$

We denote by E_a the energy eigenvalues, and by a further discrete index j the states $|a,j\rangle$ in the degenerate eigenspaces. Then

$$\lim_{t\to\infty} \langle \mathcal{O}(t)\rangle = \sum_a \sum_{j,k} \langle \Psi_0|a,j\rangle\langle a,j|\mathcal{O}|a,k\rangle\langle a,k|\Psi_0\rangle.$$

We choose the initial state and the operator as

$$|\Psi_0\rangle = \otimes_{j=1}^{L} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad D_k = \prod_{j=1}^{k} \sigma_j^-$$

Breakdown of GGE: Persistent oscillation

GE (GGE) predicts the relaxation to a stationary equilibrium state. Persistent oscillations that we observe for above operator clearly violate this hypothesis

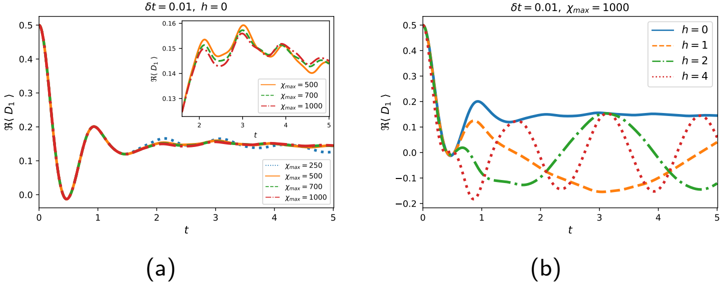


Figure: Simulation is done by time evolved block decimation, the maximal bond dimension is $\chi_{max} = 1000$, the Trotter time step is $\delta t = 0.01$. The oscillation of the expectation value with a frequency directly given by h.

$T\bar{T}$-deformations in lattice models

For any pair of extensive charges Q_α and Q_β a $T\bar{T}$-like deformation could be defined. To first order the deformation consists in modifying the Hamiltonian H as

$$H' = H + \kappa\left(J_\alpha(x)q_\beta(x) - q_\alpha(y)J_\beta(x+1)\right) + \mathcal{O}(\kappa^2),$$

here $Q_\alpha = \sum_x q_\alpha(x)$, $J_\alpha = \sum_x J_\alpha(x)$ satisfy the continuity equation

$$\partial_t q_\alpha(x,t) + \partial_x J_\alpha(x,t) = 0$$

Then the two-particle scattering matrix $S(p,k) = e^{i\delta(p,k)}$ gets deformed as

$$\delta'(p,k) = \delta(p,k) + \kappa(h_\alpha(p)h_\beta(k) - h_\alpha(k)h_\beta(p)) + \mathcal{O}(\kappa^2)$$

where $h_{\alpha,\beta}$ are the one-particle eigenvalue functions of the charges Q_α and Q_β

$T\bar{T}$-deformations in folded XXZ

Usual $T\bar{T}$: **energy** and **momentum**
We choose instead: **particle number** and **momentum**
This case is called **hard-rod deformation**

$$S(p,k) \to S(p,k)\exp(i\kappa(p-k))$$

Folded-XXZ case

$$S(p,k) = -e^{i(p-k)}$$

thus $S(p,k) = -1$ (XX model), $\kappa = 1$ and folded-XXZ is equal to the **hard-rod deformation of XX** model
Note: rigorously speaking $T\bar{T}$ is not defined on the lattice since there is no momentum operator on lattice which would be ab extensive local charge

Hard-rod XXZ

We can consider following projection operators

$$P^{\circ} = \frac{1+\sigma^z}{2}, \qquad P^{\bullet} = \frac{1-\sigma^z}{2}.$$

Then ordinary XXZ spin chain could be written as

$$h_{j,j+1} = \sigma_j^- \sigma_{j+1}^+ + \sigma_j^+ \sigma_{j+1}^- - \Delta(P_j^{\circ} P_{j+1}^{\bullet} + P_j^{\bullet} P_{j+1}^{\circ}), \qquad H = 2\sum_j h_{j,j+1}.$$

Hard-rod deformation of XXZ with core of length ℓ is given by

$$H = \sum_j \left[h_{j,j+\ell} \prod_{k=1}^{\ell-1} P_{j+k}^{\bullet} \right].$$

Properties

$$H=\sum_j \left[h_{j,j+\ell}\prod_{k=1}^{\ell-1}P^{\bullet}_{j+k}\right].$$

- $\ell=1$ is ordinary XXZ chain
- $\ell=2$ with $\Delta=0$ is a folded-XXZ in a "dual representation" (thus folded-XXZ could be called a hard-rod deformed XX)
- $\ell=2$ with $\Delta=0$ coincides with a Bariev model at $U=1$

$$H=\sum_j \left[\sigma_j^-\sigma_{j+2}^+ + \sigma_j^+\sigma_{j+2}^-\right]\frac{1-U\sigma^z_{j+1}}{2},$$

Wave function is given by

$$|\Psi\rangle = \sum_{x_1 \leq x_2 \leq \ldots x'_N} \sum_{\mathcal{P} \in S_{N'}} e^{i \sum_{j=1}^{N'} q_{\mathcal{P}_j} x_j} \prod_{j \leq k} S_{a_j, a_k}(q_j, q_k) \prod_{j=1}^{N'} A_{x_j}^{a_{\mathcal{P}_j}} |\Omega\rangle$$

$$A_j^a = \begin{cases} \sigma_j^- & \text{if } a = 1 \\ \sigma_j^- \sigma_{j+1}^- & \text{if } a = 2. \end{cases}$$

Here scattering matrices are

$$S_{1,1}(q_1, q_2) = -e^{-i(q_1 - q_2)},$$
$$S_{2,2}(q_1, q_2) = e^{-i(q_1 - q_2)} S_{XXZ}(q_1, q_2),$$
$$S_{1,2}(q_1, q_2) = e^{-i(q_1 - 2q_2)},$$

where S_{XXZ} is an ordinary scattering matrix of magnons in XXZ chains

Yang-Baxter integrability:

$$R_{12}(\lambda_1,\lambda_2)R_{23}(\lambda_2,\lambda_3)R_{13}(\lambda_1,\lambda_3)=R_{13}(\lambda_1,\lambda_3)R_{23}(\lambda_2,\lambda_3)R_{12}(\lambda_1,\lambda_2)$$

$$R_{B,A}(\nu,\mu)\mathcal{L}_{B,j}(\nu)\mathcal{L}_{A,j}(\mu)=\mathcal{L}_{A,j}(\mu)\mathcal{L}_{B,j}(\nu)R_{B,A}(\nu,\mu)$$

- **Lax operator** could be derived (rather guessed) from the set of condition (self-commutativity of the transfer matrix $[t(u),t(v)]=0$ could be checked on a final chain)

$$\check{\mathcal{L}}_{a,b,j}(u)=\check{\mathcal{L}}^{(XXZ)}_{a,j}(u)P^{\bullet}_{b}+P^{\circ}_{b} \qquad \mathcal{L}_{a,b,j}(u)=\mathcal{P}_{a,j}\mathcal{P}_{b,j}\check{\mathcal{L}}_{a,b,j}(u)$$

3rd index denotes quantums space, the 1st and the 2nd – auxiliary spaces (i.e. $A,B=a\otimes b$)

- R-matrix could be fixed then from the RLL-relation
- Yang-Baxter equation provides the final check (!) for the R-matrix

$$R(\lambda,\mu) = \begin{pmatrix} E_{11}+E_{44}\rho_1 & E_{21}+E_{43}\rho_2 & E_{31} & E_{41}\rho_5 \\ E_{12} & E_{22}+E_{44}\rho_6 & E_{32} & E_{42}\rho_5 \\ E_{13}+E_{24}\rho_2 & E_{23}\rho_3 & E_{33}+E_{44}\rho_6 & E_{21}\rho_4+E_{43}\rho_5 \\ E_{14}\rho_5 & E_{13}\rho_4+E_{24}\rho_5 & E_{34}\rho_5 & E_{11}\rho_7+(E_{22}+E_{33})\rho_6+E_{44} \end{pmatrix}$$

- Matrix has a non-difference form
- Diagonal elements (Cartans) are “degenerated” (Gauss decomposition issues?)
- Rational limit ($\Delta = 1$)

$$\check{R}_{12,34}(u,v) = 1 + \frac{u-v}{u-v+1}\left(h_{234}+h_{123}+\frac{u}{u+1}h_{234}h_{123}+\frac{v}{v-1}h_{123}h_{234}\right)$$

$$\check{\mathcal{L}}_{a,b,j}(u) = \check{\mathcal{L}}^{(XXX)}_{a,j}(u)P^{\bullet}_{b} + P^{\circ}_{b}, \qquad \check{R}_{ab,cd}(u) = \mathcal{P}_{a,c}\mathcal{P}_{b,d}R_{ab,cd}(u)$$

Conclusions

- Folded-XXZ is only one model in a big family of hard-rod models
- Moreover, each given R-matrix assume existence of (in general infinite) family of Lax operators (so, infinite family of models)
- Folded-XXZ Hamiltonian arises as a crystal limit of usual XXZ (algebra symmetry $\mathfrak{gl}(2)$). Immediate generalization for higher rank algebra related cases turns out to be possible (crystal limit in Perk-Schultz models)
- All of such models posses quite degenerate spectrum
- RLL expansion of these models in terms of Drinfeld currents is quite challenging task because of the "degeneracy of Cartans" but desirable task (algebraic Bethe ansatz)

$$\rho_1 = \frac{\sinh(\lambda-\mu)\sinh(\mu)}{\sinh(\lambda-\mu+\eta)\sinh(\mu-\eta)}, \qquad \rho_2 = -\frac{\sinh(\lambda-\mu)\sinh(\eta)}{\sinh(\lambda-\mu+\eta)\sinh(\mu-\eta)},$$

$$\rho_3 = \frac{1}{\sinh(\lambda-\mu+\eta)}\left(\frac{\sinh(\eta)\sinh(\eta+\mu)}{\sinh(\eta+\lambda)} + \frac{\sinh(\lambda-\mu)\sinh(\mu)}{\sinh(\mu-\eta)}\right),$$

$$\rho_4 = \frac{\sinh(\lambda-\mu)\sinh(\eta)}{\sinh(\lambda-\mu+\eta)\sinh(\lambda+\eta)}, \qquad \rho_5 = \frac{\sinh(\eta)}{\sinh(\lambda-\mu+\eta)},$$

$$\rho_6 = \frac{\sinh(\lambda-\mu)}{\sinh(\lambda-\mu+\eta)}, \qquad \rho_7 = \frac{\sinh(\lambda-\mu)\sinh(\lambda)}{\sinh(\lambda-\mu+\eta)\sinh(\lambda+\eta)}.$$

Rational limit $\lambda \to \eta\lambda$, $\mu \to \eta\mu$, $\eta \to 0$

$$e^{ip_j(L-N-M)}e^{iP}e^{2iK} = (-1)^{N-1},$$
$$e^{ik_\ell(L-2N-M)}e^{iK}e^{iP} = (-1)^{M-1},$$

$$P = \sum_{j=1}^{N} p_j, \qquad K = \sum_{j=1}^{M} k_j.$$

The energy is carried only by the particles (!), and the effect of the domain walls is only a change in the available volume