Hilbert space fragmentation in integrable systems

A. Hutsalyuk
Eötvös Loránd University

Les Diablerets

February 6, 2023
• The observable $O$ thermalizes if after some relaxation time, the average expectation value of this observable agrees with the microcanonical expectation value.

For the initial state $|\psi_0\rangle$ and (an arbitrary) operator $O$ evolution is provided by

$$|\psi(t)\rangle = \sum_m C_m e^{-iE_m t} |m\rangle,$$

$C_m = \langle m|\psi_0\rangle$

$$\langle \psi(t)|\hat{O}|\psi(t)\rangle = \sum_{m,n} C^*_m C_n e^{i(E_m - E_n)t} O_{mn} = \sum_m |C_m|^2 O_{mm} + \sum_{n \neq m} C^*_m C_n e^{i(E_m - E_n)t} O_{mn}$$

i. How to proceed to a microcanonical ensemble in first term (time independent!)
ii. What to do with states that are exponentially close to each other in the second term
ETH

- ETH could be considered as an ansatz for matrix elements

\[
O_{mn} = \delta_{mn}O(\bar{E}) + e^{-S(\bar{E})}f_O(\bar{E}, \omega)R_{mn}, \quad \omega = E_n - E_m, \quad \bar{E} = \frac{E_m + E_n}{2}
\]

- \(S(E)\) is an entropy
- \(f_O\) is a smooth function (operator dependent); \(\overline{R^2_{mn}} = 1, \overline{|R_{mn}|^2} = 1\)
- \(O(\bar{E})\) coincides with an expectation value of the microcanonical ensemble

Using ETH it is possible to proceed to averaging with a diagonal ensemble \(\rho_{DE}\)

\[
\langle O \rangle = \lim_{t_0 \to \infty} \frac{1}{t_0} \int_0^{t_0} dt \ O(t) = \sum_m |C_m|^2 O_{mm} = \text{Tr} \ [\rho_{DE}O]
\]
Integrable system, degeneracies and ETH

• In the integrable systems they are replaced by Generalized Gibbs ensemble (GGE)

$$\rho = \exp(-\beta H) \longrightarrow \exp\left(-\sum \beta_k H_k\right)$$


• Very successful for a lot of applications

• Note: only weak ETH is satisfied: an (exponentially!) small fraction of eigenstates does not obey the ETH, they have expectation values significantly different from the microcanonical ensemble

• However the complete breakdown of ETH could be expected in case of degeneracy!
Folded-XXZ via crystal limit

\[ H = Q_4 = -\frac{1}{4} \sum_{j=1}^{L} (1 + \sigma_j^z \sigma_{j+3}^z) \left( \sigma_{j+1}^+ \sigma_{j+2}^- + \sigma_{j+1}^- \sigma_{j+2}^+ \right) \]

Could be derived from XXZ chain using the crystal limit (note: \( Q_2 \) is nothing but Ising model)

\[ Q_k = \lim_{\Delta \to \infty} \tilde{Q}_k / \Delta^{[k/2]} \]

where \( \tilde{Q}_k \) is \( k \)th integral of motion of XXZ chain Earlier this model was discovered (at least...) Z.-C. Yang, F. Liu, A. V. Gorshkov, T. Iadecola, *Phys. Rev. Lett.* 124, 207602 (2020)

Properties

• Crystal limit models are (at least supposedly) simpler than the original one
• Crystal limit automatically provides us with Hamiltonians that are integrable
• Limit is known for the Hamiltonian only. It is not clear how to generalize it for the Lax operator and $R$-matrix (?)
• It is naturally to expect a large degree of degeneracy in the spectrum since the crystal limit also affects Bethe equations
Bethe ansatz solution

Bethe equations naturally follows from the crystal limit of ordinary XXZ equations

\[ e^{ip_{j,n}L} \prod_{m,k} S_{n,m}(p_{j,n} - p_{k,m}) = 1 \rightarrow e^{ip_{j,1}(L-N_1-2N_s)} = (-1)^{N_1-1} e^{-iP_1-2iP_s} \]

\( s \) is a number of strings, \( N_1 \) – number of particles

There are “particles” (and correspondingly scattering matrices) of two types

\[ S_{1,1}(p, k) = -e^{-i(p-k)}, \quad S_{1,n}(p, k) = e^{-2i(p-k)} \]  the last does not depend on \( n \)!

In this picture strings could be understood as “domain walls”. The last are not dynamical on their own: **in the absence of particles** they lead to frozen configuration and exponentially degenerate Hilbert states
Scattering of a particle with a DW. As the result DW gets replaced by 2 sites to the left, and the trajectory of the particle receives a displacement of 1 site to the right.
Dual transformation

Bond-site transformation

\[ |\circ\rangle_j \quad \text{if} \quad \sigma^z_j \sigma^z_{j+1} = 1 \]
\[ |\bullet\rangle_j \quad \text{if} \quad \sigma^z_j \sigma^z_{j+1} = -1 \]

We can interpret \(|\circ\rangle\) as the up spin, and \(|\bullet\rangle\) as the down spin.

Note: transformation is good defined in boundary case. Some problems in a periodic case (we neglect it since TDL is object of interest at the very end).

Note: \(Q_1\) becomes non-local

\[
Q_1 = \frac{1}{2} \sum_{j=1}^{L-1} \left[ 1 - \prod_{k=1}^{j} \sigma^z_j \right]
\]
Charges transformation (note: $Q_4$ is 3-site now)

\[
Q_2 \rightarrow \sum_{j=1}^{L-1} \sigma_j^z
\]

\[
Q_3 \rightarrow i \sum_j \sigma_j^- P_{j+1}^\bullet \sigma_{j+2}^+ - \sigma_j^+ P_{j+1}^\bullet \sigma_{j+2}^-
\]

\[
Q_4 \rightarrow \sum_j \sigma_j^- P_{j+1}^\bullet \sigma_{j+2}^+ + \sigma_j^+ P_{j+1}^\bullet \sigma_{j+2}^-
\]

here $P^\bullet = (1 - \sigma_z^z)/2$. The only non-zero elements of $Q$s are

\[
|\circ \bullet \bullet\rangle \rightarrow |\bullet \circ \circ\rangle, \quad |\bullet \circ \circ\rangle \rightarrow |\circ \bullet \bullet\rangle
\]

Thus $Q_3, Q_4$, etc, move only double $\bullet \bullet$. Single $\bullet$ remains invariant!
Overlaps in folded-XXZ

Define set of states

$$|x_1, \ldots, x_N \rangle = \sigma_{x_1}^{-} \cdots \sigma_{x_N}^{-} |\emptyset \rangle,$$

restriction $1 \leq x_1 < \cdots < x_N \leq L$ is imposed to avoid double counting

Specific Neél state with $N$ particles in a volume $L = 3N$

$$|\Psi'_0 \rangle = |3, 6, 9, \ldots \rangle$$

From the limit of Slavnov's determinant (overlap of XXZ Bethe vectors)

$$\langle \Psi'_0 | p \rangle = \det_{jk} e^{ip_j(2k+1)}$$
On-shell overlap with initial state

We consider the initial state

$$|\psi_0\rangle = \frac{1 + U + U^2}{\sqrt{3}} |\psi'_0\rangle,$$

where $U$ is the one-site cyclic shift operator.

In the zero momentum sector the Bethe equations are

$$e^{i2Np_j} = -1, \quad j = 1, \ldots, N, \quad L = 3N, \quad N \text{ assumed to be even for simplicity.}$$

The overlaps can be expressed simply as

$$|\langle \psi_0 | \mathbf{p} \rangle|^2 = \prod_{j < k} |e^{i2p_j} - e^{i2p_k}|^{-2} \to N^N.$$
Solvable quench dynamics

Form factor sum could be written as

\[ \langle O(t) \rangle = \sum_{p,k} \frac{\langle \Psi_0 | p \rangle \langle p | O | k \rangle \langle k | \Psi_0 \rangle}{\langle p | p \rangle \langle k | k \rangle} e^{-i(E_k - E_p)t} \]

Emptiness formation probability (EFP)

\[ E_\ell(x) = \prod_{j=1}^{\ell} \frac{1 + \sigma_z^x - 1 + j}{2} \]

- Form factors should be computed \( \langle p | O | k \rangle \)
- Summation over \( p, k \) should be performed
Form factors


\[ \langle p | \Xi_\ell(x) | k \rangle = \prod_{j \leq k} \frac{1}{(e^{iu_j^C} - e^{iu_k^C})(e^{iu_j^B} - e^{iu_k^B})} \det \mathcal{T}, \]

\[ \mathcal{T}_{jj} = (N - L + \ell - 1)e^{-2ikj}, \quad u_j^c = u_k^B, \]

\[ \mathcal{T}_{jk} = e^{i(\ell - 1)(u_j^B + u_k^C)} \frac{\sin ((\ell - 1)(u_j^B - u_k^C))}{\sin(u_j^B - u_k^C)}, \quad u_j^c \neq u_k^B. \]

For diagonal part of $\mathcal{T}$ the rank is given by the number of coinciding elements in the sets $\bar{u}_j^C$, $\bar{u}_j^B$! This put very heavy restrictions on the summation.
After simple summation over form factors the exact quench for $E_3$ is given by

$$
\langle \psi(t)|E_3(x)|\psi(t) \rangle = \frac{1}{6} - \frac{1}{6} \left( \frac{1}{N} \sum_a \cos(2 \cos(c_a)t) \right)^2 - \frac{1}{6} \left| \frac{1}{N} \sum_a \sin(2 \cos(c_a)t) e^{ic_a} \right|^2
$$

$c_a = \pi(2a - 1)/(2N)$, $a = 1, \ldots, N$. In the thermodynamic limit the last expression is

$$
\langle \psi(t)|E_3(x)|\psi(t) \rangle = \frac{1}{6} \left[ 1 - (J_0(2t))^2 - (J_1(2t))^2 \right]
$$
Thermalization in a fragmented Hilbert space

Problem of thermalization from the initial state $|\psi_0\rangle$

$$\langle O(t) \rangle = \sum \langle \Psi_0 | a \rangle \langle a | O | b \rangle \langle b | \Psi_0 \rangle e^{-i(E_b - E_a)t}$$

We denote by $E_a$ the energy eigenvalues, and by a further discrete index $j$ the states $|a, j\rangle$ in the degenerate eigenspaces. Then

$$\lim_{t \to \infty} \langle O(t) \rangle = \sum_a \sum_{j, k} \langle \Psi_0 | a, j \rangle \langle a, j | O | a, k \rangle \langle a, k | \Psi_0 \rangle.$$ 

We choose the initial state and the operator as

$$|\psi_0\rangle = \bigotimes_{j=1}^{L} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad D_k = \prod_{j=1}^{k} \sigma_j^-$$
Breakdown of GGE: Persistent oscillation

GE (GGE) predicts the relaxation to a stationary equilibrium state. Persistent oscillations that we observe for above operator **clearly violate this hypothesis**

![Graph](a)

**Figure:** Simulation is done by time evolved block decimation, the maximal bond dimension is $\chi_{\text{max}} = 1000$, the Trotter time step is $\delta t = 0.01$. The oscillation of the expectation value with a frequency directly given by $h$. 
For any pair of extensive charges $Q_\alpha$ and $Q_\beta$ a $T \bar{T}$-like deformation could be defined. To first order the deformation consists in modifying the Hamiltonian $H$ as

$$H' = H + \kappa \left( J_\alpha(x)q_\beta(x) - q_\alpha(y)J_\beta(x + 1) \right) + \mathcal{O}(\kappa^2),$$

here $Q_\alpha = \sum_x q_\alpha(x)$, $J_\alpha = \sum_x J_\alpha(x)$ satisfy the continuity equation

$$\partial_t q_\alpha(x, t) + \partial_x J_\alpha(x, t) = 0$$

Then the two-particle scattering matrix $S(p, k) = e^{i\delta(p, k)}$ gets deformed as

$$\delta'(p, k) = \delta(p, k) + \kappa(h_\alpha(p)h_\beta(k) - h_\alpha(k)h_\beta(p)) + \mathcal{O}(\kappa^2)$$

where $h_{\alpha, \beta}$ are the one-particle eigenvalue functions of the charges $Q_\alpha$ and $Q_\beta$. 
Usual $T \bar{T}$: energy and momentum
We choose instead: particle number and momentum
This case is called hard-rod deformation

$$S(p, k) \rightarrow S(p, k) \exp(i\kappa(p - k))$$

Folded-XXZ case

$$S(p, k) = -e^{i(p - k)}$$

thus $S(p, k) = -1$ (XX model), $\kappa = 1$ and folded-XXZ is equal to the hard-rod deformation of XX model

Note: rigorously speaking $T \bar{T}$ is not defined on the lattice since there is no momentum operator on lattice which would be an extensive local charge
Hard-rod XXZ

We can consider following projection operators

\[ P^\circ = \frac{1 + \sigma^z}{2}, \quad P^\bullet = \frac{1 - \sigma^z}{2}. \]

Then ordinary XXZ spin chain could be written as

\[ h_{j,j+1} = \sigma_j^- \sigma_{j+1}^+ + \sigma_j^+ \sigma_{j+1}^- - \Delta (P^\circ_j P^\bullet_{j+1} + P^\bullet_j P^\circ_{j+1}), \quad H = 2 \sum_j h_{j,j+1}. \]

Hard-rod deformation of XXZ with core of length \( \ell \) is given by

\[ H = \sum_j \left[ h_{j,j+\ell} \prod_{k=1}^{\ell-1} P^\bullet_{j+k} \right]. \]
Properties

\[ H = \sum_j \left[ h_{j,j+\ell} \prod_{k=1}^{\ell-1} P_{j+k} \right]. \]

- \( \ell = 1 \) is ordinary XXZ chain
- \( \ell = 2 \) with \( \Delta = 0 \) is a folded-XXZ in a “dual representation” (thus folded-XXZ could be called a hard-rod deformed XX)
- \( \ell = 2 \) with \( \Delta = 0 \) coincides with a Bariev model at \( U = 1 \)

\[ H = \sum_j \left[ \sigma_j^- \sigma_{j+2}^+ + \sigma_j^+ \sigma_{j+2}^- \right] \frac{1 - U \sigma_j^z}{2}, \]
Wave function is given by

$$|\psi\rangle = \sum_{x_1 \leq x_2 \leq \ldots \leq x'_N} \sum_{\mathcal{P} \in S_{N'}} e^{i \sum_{j=1}^{N'} q_{\mathcal{P}_j} x_j} \prod_{j \leq k} S_{a_j, a_k}(q_j, q_k) \prod_{j=1}^{N'} a^p_{x_j} |\Omega\rangle$$

$$A^a_j = \begin{cases} \sigma^{-} & \text{if } a = 1 \\ \sigma^{-} \sigma^{-} & \text{if } a = 2. \end{cases}$$

Here scattering matrices are

$$S_{1,1}(q_1, q_2) = -e^{-i(q_1 - q_2)},$$
$$S_{2,2}(q_1, q_2) = e^{-i(q_1 - q_2)} S_{XXZ}(q_1, q_2),$$
$$S_{1,2}(q_1, q_2) = e^{-i(q_1 - 2q_2)},$$

where $S_{XXZ}$ is an ordinary scattering matrix of magnons in XXZ chains.
Yang-Baxter integrability:

\[ R_{12}(\lambda_1, \lambda_2)R_{23}(\lambda_2, \lambda_3)R_{13}(\lambda_1, \lambda_3) = R_{13}(\lambda_1, \lambda_3)R_{23}(\lambda_2, \lambda_3)R_{12}(\lambda_1, \lambda_2) \]

\[ R_{B,A}(\nu, \mu) \mathcal{L}_{B,j}(\nu) \mathcal{L}_{A,j}(\mu) = \mathcal{L}_{A,j}(\mu) \mathcal{L}_{B,j}(\nu) R_{B,A}(\nu, \mu) \]

- **Lax operator** could be derived (rather guessed) from the set of condition (self-commutativity of the transfer matrix \([t(u), t(v)] = 0\) could be checked on a final chain)

\[ \mathcal{L}_{a,b,j}(u) = \mathcal{L}^{(XXZ)}_{a,j}(u) P^\bullet + P^\circ \]

3rd index denotes quantum space, the 1st and the 2nd – auxiliary spaces (i.e. \(A, B = a \otimes b\))

- **R-matrix** could be fixed then from the \(RLL\)-relation
- **Yang-Baxter equation** provides the final check (!) for the \(R\)-matrix
$R(\lambda, \mu) = \begin{pmatrix} E_{11} + E_{44}\rho_1 & E_{21} + E_{43}\rho_2 & E_{31} & E_{41}\rho_5 \\ E_{12} & E_{22} + E_{44}\rho_6 & E_{32} & E_{42}\rho_5 \\ E_{13} + E_{24}\rho_2 & E_{23}\rho_3 & E_{33} + E_{44}\rho_6 & E_{21}\rho_4 + E_{43}\rho_5 \\ E_{14}\rho_5 & E_{13}\rho_4 + E_{24}\rho_5 & E_{34}\rho_5 & E_{11}\rho_7 + (E_{22} + E_{33})\rho_6 + E_{44} \end{pmatrix}$

- Matrix has a non-difference form
- Diagonal elements (Cartans) are “degenerated” (Gauss decomposition issues?)
- Rational limit ($\Delta = 1$)

$\tilde{R}_{12,34}(u, \nu) = 1 + \frac{u - \nu}{u - \nu + 1} \left( h_{234} + h_{123} + \frac{u}{u + 1} h_{234} h_{123} + \frac{\nu}{\nu - 1} h_{123} h_{234} \right)$

$\tilde{\mathcal{L}}_{a,b,j}(u) = \tilde{\mathcal{L}}^{(XXX)}_{a,j}(u) P_b^\bullet + P_b^\circ$, \quad $\tilde{R}_{ab,cd}(u) = \mathcal{P}_{a,c} \mathcal{P}_{b,d} R_{ab,cd}(u)$
Conclusions

- Folded-XXZ is only one model in a big family of hard-rod models
- Moreover, each given $R$-matrix assume existence of (in general infinite) family of Lax operators (so, infinite family of models)
- Folded-XXZ Hamiltonian arises as a crystal limit of usual XXZ (algebra symmetry $\mathfrak{gl}(2)$). Immediate generalization for higher rank algebra related cases turns out to be possible (crystal limit in Perk-Schultz models)
- All of such models posses quite degenerate spectrum
- RLL expansion of these models in terms of Drinfeld currents is quite challenging task because of the “degeneracy of Cartans” but desirable task (algebraic Bethe ansatz)
\[ \rho_1 = \frac{\sinh(\lambda - \mu) \sinh(\mu)}{\sinh(\lambda - \mu + \eta) \sinh(\mu - \eta)}, \quad \rho_2 = -\frac{\sinh(\lambda - \mu) \sinh(\eta)}{\sinh(\lambda - \mu + \eta) \sinh(\mu - \eta)}, \]
\[ \rho_3 = \frac{1}{\sinh(\lambda - \mu + \eta)} \left( \frac{\sinh(\eta) \sinh(\eta + \mu)}{\sinh(\eta + \lambda)} + \frac{\sinh(\lambda - \mu) \sinh(\mu)}{\sinh(\mu - \eta)} \right), \]
\[ \rho_4 = \frac{\sinh(\lambda - \mu) \sinh(\eta)}{\sinh(\lambda - \mu + \eta) \sinh(\lambda + \eta)}, \quad \rho_5 = \frac{\sinh(\eta)}{\sinh(\lambda - \mu + \eta)}, \]
\[ \rho_6 = \frac{\sinh(\lambda - \mu)}{\sinh(\lambda - \mu + \eta)}, \quad \rho_7 = \frac{\sinh(\lambda - \mu) \sinh(\lambda)}{\sinh(\lambda - \mu + \eta) \sinh(\lambda + \eta)}. \]

Rational limit \( \lambda \to \eta \lambda, \mu \to \eta \mu, \eta \to 0 \)
The energy is carried only by the particles (!), and the effect of the domain walls is only a change in the available volume.