On the influence of boundary conditions on the critical behaviour of the staggered six-vertex model

Sascha Gehrmann

based the joint works with H. Frahm: arXiv:2111.00850, arXiv:2209.06182

Les Diablerets, 03-12 February 2023

Leibniz Universität Hannover

Motivation

- Study of 2D exactly solvable lattice models were important for understanding critical phenomena e.g.
  1. Ising Model
  2. Six/eight-vertex model

Leibniz Universität Hannover

Motivation

- Study of 2D exactly solvable lattice models were important for understanding critical phenomena e.g.
  1. Ising Model
  2. Six/eight-vertex model

- Their scaling limits are described by compact field theories (free compact fermions/bosons and massiv Thirring model)

Motivation

- Study of 2D exactly solvable lattice models were important for understanding critical phenomena e.g.
    1. Ising Model
    2. Six/eight-vertex model

- Their scaling limits are described by compact field theories (free compact fermions/bosons and massiv Thirring model)

- Non-compact QFTs appear in modern interesting problems such as
    1. High Energy Theory (AdS/CFT, Strings [Schomerus '05])
    2. Condensed Matter Physics (IQHE [Zirnbauer '99])

Leibniz Universität Hannover

Motivation

- Study of 2D exactly solvable lattice models were important for understanding critical phenomena e.g.
  1. Ising Model
  2. Six/eight-vertex model

- Their scaling limits are described by compact field theories (free compact fermions/bosons and massiv Thirring model)

- Non-compact QFTs appear in modern interesting problems such as
  1. High Energy Theory (AdS/CFT, Strings [Schomerus '05])
  2. Condensed Matter Physics (IQHE [Zirnbauer '99])

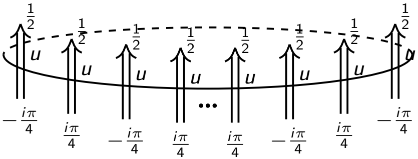
- Non-compact lattice models are significant harder to study [Faddeev,Korchemsky '95; Bytsko, Teschner '07]

Leibniz Universität Hannover

Motivation

- Staggered six-vertex model exhibits a **continuous spectrum of scaling dimensions (hallmark of non-compact theories)**
  [Jacobsen, Saleur ’05] (see also [Essler, Frahm, Saleur ’05])

$\tau^{\mathrm{pbc}}(u) =$



$$q = e^{i\gamma}, \quad \gamma \in (0, \tfrac{\pi}{2})$$

$$H \propto \sum_{k=1}^{2} \frac{\mathrm{d}}{\mathrm{d}u}\Big|_{u=0} \log\left(\tau^{\mathrm{pbc}}(u + (-1)^k \tfrac{i\pi}{4})\right)$$

Leibniz Universität Hannover

Hamitonian

$$H = \frac{4}{\sin(2\gamma)} \left[\sum_{j=1}^{2L} 2\sin^2(\gamma)\sigma_j^z\sigma_{j+1}^z - \left(\sigma_j^x\sigma_{j+2}^x + \sigma_j^y\sigma_{j+2}^y + \sigma_j^z\sigma_{j+2}^z\right) \right.$$

$$\left. -2i\sin(\gamma)\left(\sigma_j^x\sigma_{j+1}^y + \sigma_j^y\sigma_{j+1}^x\right)\left(\sigma_{j-1}^z - \sigma_{j+2}^z\right) \right]$$

Hamitonian

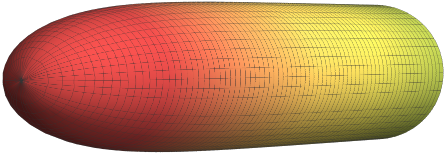
$$H = \frac{4}{\sin(2\gamma)}\left[\sum_{j=1}^{2L} 2\sin^2(\gamma)\sigma_j^z\sigma_{j+1}^z - \left(\sigma_j^x\sigma_{j+2}^x + \sigma_j^y\sigma_{j+2}^y + \sigma_j^z\sigma_{j+2}^z\right)\right.$$
$$\left. -2i\sin(\gamma)\left(\sigma_j^x\sigma_{j+1}^y + \sigma_j^y\sigma_{j+1}^x\right)\left(\sigma_{j-1}^z - \sigma_{j+2}^z\right)\right]$$

These terms break self-adjointness

Scaling limit of the staggered six-vertex model

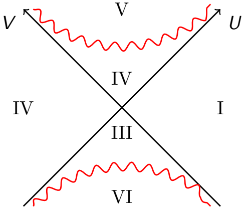
[Ikhlef, Jacobsen, Saleur '11]: Scaling limit relate to the 2D Euclidean black hole sigma model [Witten '91]

$$\mathcal{L}_{EBH} = \frac{\partial_\mu U \partial^\mu U^*}{1+UU^*}$$



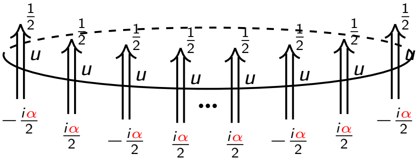
[Bazhanov, Kotousov, Koval, Lukyanov '21] revised: scaling limit described by the 2D Lorentzian black hole sigma model

$$\mathcal{L}_{LBH} = \frac{\partial_\mu U \partial^\mu V}{1-UV}$$



Motivation

- Generalisations of the underlying spin chain have been studied

$\tau^{\rm pbc}(u) =$ 

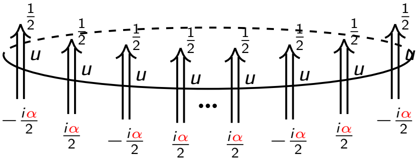
$q = e^{i\gamma}, \quad \gamma \in (0, \pi)$

$\alpha \in (0, \pi)$

Leibniz Universität Hannover

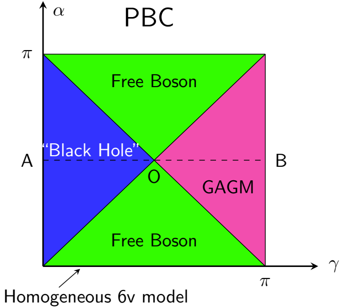
Motivation

- Generalisations of the underlying spin chain have been studied

$\tau^{\mathrm{pbc}}(u) =$ 

$q = e^{i\gamma}, \quad \gamma \in (0, \pi)$

$\alpha \in (0, \pi)$

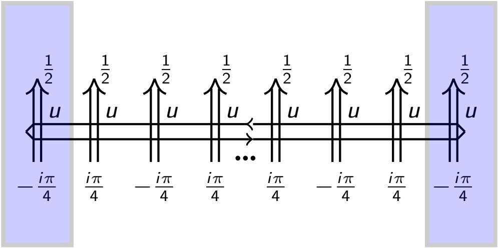


- AO Line [Jacobsen, Saleur '05; Ikhlef, Jacobsen, Saleur '06'11; Frahm, Martins '12; Candu, Ikhlef '13; Bazhanov, Kotousov, Koval, Lukyanov '19'20'21]
- Whole BH region [Frahm, Seel '14]
- OB line [Ikhlef, Jacobsen, Saleur '09]
- Whole GAGM region (1 compact boson + 2 Majorana fermions)+ conj. general inhom. 6-vertex [Kotousov, Lukyanov '21]

Leibniz Universität Hannover

Open Boundary Conditions

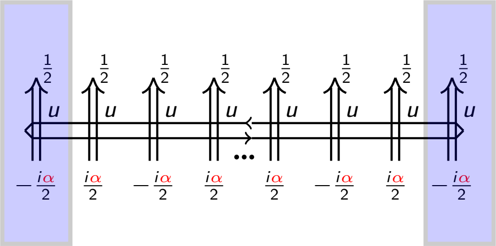
Direction of further study: Varying the boundary conditions.



Presence of continuum highly boundary condition dependent:
[Robertson, Jacobsen, Saleur '20,'21] (see also [Nepomechie, Retore '21])

Subject of study

$\mathbb{Z}_2$-staggered six-vertex model with $U_q(\mathfrak{sl}(2))$ symmetry



Hamiltonian

$$H = -\sin(\alpha)\left[\sum_{j=2}^{2L-1} \frac{e_{j,j+1}e_{j-1,j}}{\sin(\alpha+(-1)^j\gamma)} + \frac{e_{j-1,j}e_{j,j+1}}{\sin(\alpha-(-1)^j\gamma)}\right] - 2\sum_{j=1}^{2L-1} e_{j,j+1}\,,$$

$$e_{j,j+1} = \frac{1}{2}\left[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \cos(\gamma)\sigma_j^z\sigma_{j+1}^z - \cos(\gamma) + i\sin(\gamma)(\sigma_j^z - \sigma_{j+1}^z)\right]$$

Has $U_q(\mathfrak{sl}(2))$ symmetry!

Leibniz Universität Hannover

$\mathfrak{su}(2)$

- Generators of $\mathfrak{su}(2)$ obey

$$[S^z, S^\pm] = \pm S^\pm \qquad [S^+, S^-] = 2S^z$$

e.g.

$$S^z = \sum_{i=1}^{2L} \sigma_i^z, \qquad S^\pm = \sum_{i=1}^{2L} \sigma_i^\pm$$

$U_q(\mathfrak{sl}(2))$

- Generators of $U_q(\mathfrak{sl}(2))$ obey

$$\left[S^z_q, S^\pm_q\right] = \pm S^\pm_q, \quad \left[S^+_q, S^-_q\right] = [2S^z]_q, \quad [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad q = e^{i\gamma}$$

e.g.

$$S^z_q = \sum_{i=1}^{2L} \sigma^z_i \quad S^\pm_q = \sum_{n=1}^{2L} e^{\pm(-1)^{n+1}\frac{i\alpha}{2}} e^{i\gamma(\frac{1}{2}\sigma^z_1 + \ldots + \frac{1}{2}\sigma^z_{n-1})} \sigma^\pm_n e^{-i\gamma(\frac{1}{2}\sigma^z_{n+1} + \ldots + \frac{1}{2}\sigma^z_{2L})}$$

and

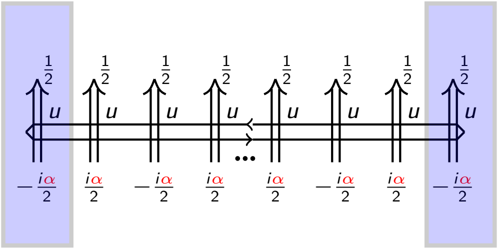
$$\left[H, S^z_q\right] = \left[H, S^\pm_q\right] = 0$$

- Simplifies analysis: Can only focus on $S^z_q = 0$ sector for the whole spectrum!

Leibniz Universität Hannover

Main results

$\mathbb{Z}_2$-staggered six-vertex model with $U_q(\mathfrak{sl}(2))$ symmetry



- Demonstrate that $\gamma < \alpha < \pi - \gamma$ has no effect on the presence of a continuum for the open case (see [Frahm, Seel '14] for the periodic case).
- Explain the drastic influence of boundary conditions on the spectrum

General Strategy

- For a critical open spin chain one expects [Cardy '86]

$$E \asymp Le_\infty + f_\infty + \frac{\pi v_F}{L}\left(-\frac{c}{24} + h_n + d\right)$$

- Extract scaling dimensions by large L asymptotic of energies
- Integrability enables study of larger L in comparison to non-integrable systems

Leibniz Universität Hannover

Methodology

1. Obtain by direct diagonalization of H the first few low energy eigenstates for small lattice sites.
2. Solve the Bethe-Ansatz equation and identify the corresponding Bethe-Root configuration
3. Use the Bethe Ansatz to construct RG-trajectory to $2L \gg 1$ by fixing the root pattern (without explicit construction/diagonalization of H).

Bethe-Ansatz

- Model can be solved by Bethe-Ansatz [Kulish, Sklyanin '91]
- Bethe-State is height-weight states in $U_q(\mathfrak{sl}(2))$ representation
- Highest weight $S = L - M$

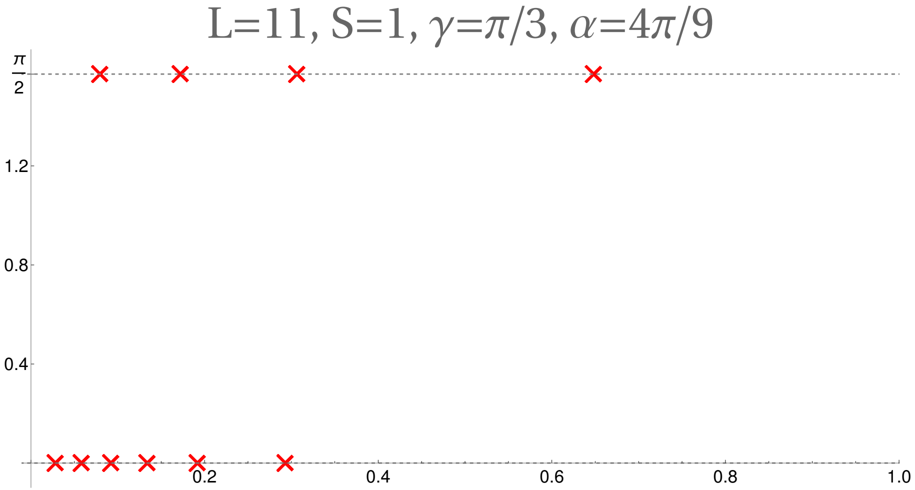
$$E = \sum_{j=1}^{M} \epsilon_0(v_j) = \sum_{j=1}^{M} \frac{4\sin(\gamma)(\cos(\alpha)\cosh(2v_j) - \cos(\gamma))}{(\cosh(2v_j) - \cos(\alpha - \gamma))(\cosh(2v_j) - \cos(\alpha + \gamma))}$$

where

$$\left(\frac{\cosh(2v_m + i\gamma) - \cos(\alpha)}{\cosh(2v_m - i\gamma) - \cos(\alpha)}\right)^{2L} = \prod_{k=1,\neq m}^{M} \frac{\sinh(v_m - v_k + i\gamma)}{\sinh(v_m - v_k - i\gamma)} \frac{\sinh(v_m + v_k + i\gamma)}{\sinh(v_m + v_k - i\gamma)}.$$

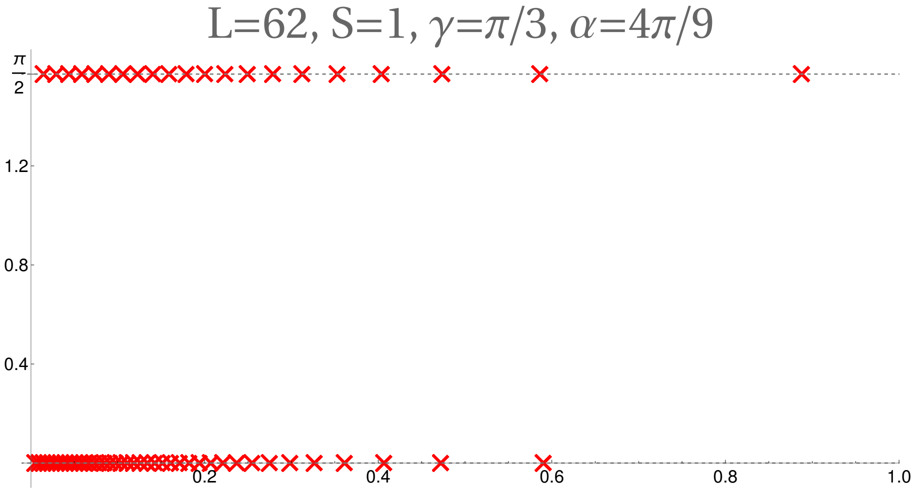
Leibniz Universität Hannover

Groundstate



Leibniz Universität Hannover

Groundstate



Root Density approach for the ground state

- Roots $x_j, y_j + \frac{i\pi}{2}$ become dense
- Describe roots by two densities $\rho^{x,y}(x) = \sigma^{x,y}(x) + \frac{1}{L}\tau^{x,y}(x)$

$$\sigma^x(x) = \frac{2\sin(\frac{\pi(\alpha-\gamma)}{\pi-2\gamma})}{\pi - 2\gamma} \frac{1}{\cosh(\frac{2\pi y}{\pi-2\gamma}) - \cos(\frac{\pi(\alpha-\gamma)}{\pi-2\gamma})}$$

$$\sigma^y(x) = \frac{2\sin(\frac{\pi(\alpha-\gamma)}{\pi-2\gamma})}{\pi - 2\gamma} \frac{1}{\cosh(\frac{2\pi x}{\pi-2\gamma}) + \cos(\frac{\pi(\alpha-\gamma)}{\pi-2\gamma})}$$

$$\tau^{x,y}(x) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega e^{i\omega x} \frac{\sinh(\frac{3\gamma-\pi}{4}\omega)}{\sinh(\frac{\gamma\omega}{4})\cosh(\frac{2\gamma-\pi}{4}\omega)}$$

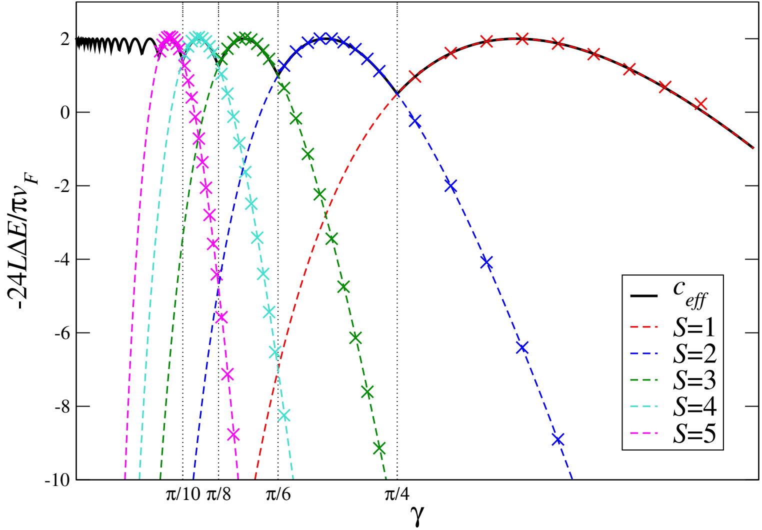
Number of Bethe Roots:

$$\frac{2M^{0}_{GS}+1}{L} = 2\cdot\frac{\pi-\alpha-\gamma}{\pi-2\gamma} + \frac{1}{L}\left(\frac{3}{2}-\frac{\pi}{2\gamma}\right) + \mathcal{O}\Big(\frac{1}{L^2}\Big),$$
$$\frac{2M^{\frac{\pi}{2}}_{GS}+1}{L} = 2\cdot\frac{\alpha-\gamma}{\pi-2\gamma} + \frac{1}{L}\left(\frac{3}{2}-\frac{\pi}{2\gamma}\right) + \mathcal{O}\Big(\frac{1}{L^2}\Big).$$

$$S = L - M \Longrightarrow S^{GS} = \left[-\frac{1}{2}+\frac{\pi}{2\gamma}\right],$$

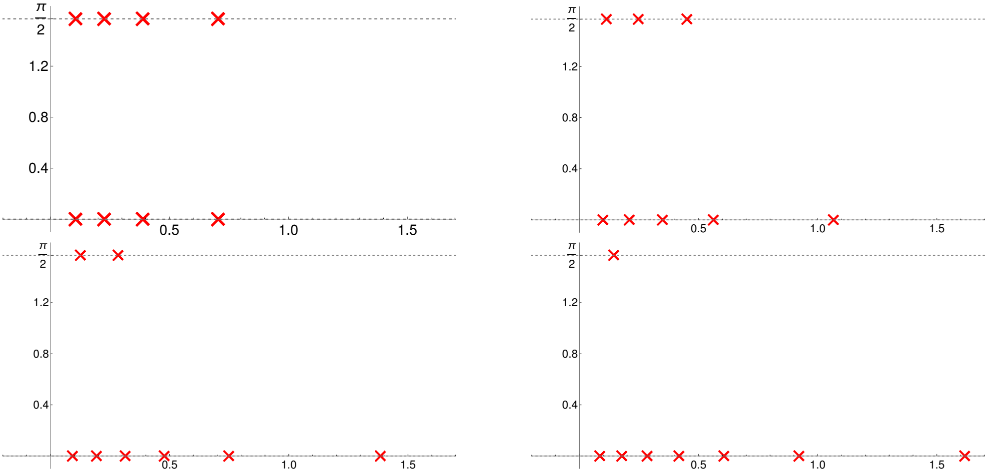
where the brackets indicate the rounding. Inverting this relation we obtain a range of anisotropies γ for which the ground state is realized in the sector with spin S^{GS}:

$$\frac{\pi}{2S^{GS}+2} < \gamma < \frac{\pi}{2S^{GS}}.$$



$$\frac{\pi}{2S^{GS}+2} < \gamma < \frac{\pi}{2S^{GS}}$$

Excitations: one class of states



$$L = 10,\ S = 2,\ \gamma = \frac{\pi}{4.9} \text{ and } \alpha = \pi/2$$

Low energy spectrum

- Expanding energy around the ground state energy in the limit $L \to \infty$ gives

$$h_{eff} = \left(-\frac{1}{12} + \frac{\gamma}{4\pi} \left(2S + 1 - \frac{\pi}{\gamma} \right)^2 + \frac{1}{4} \frac{(dN - dN_{GS})^2}{\tilde{Z}_D^2} \right),$$

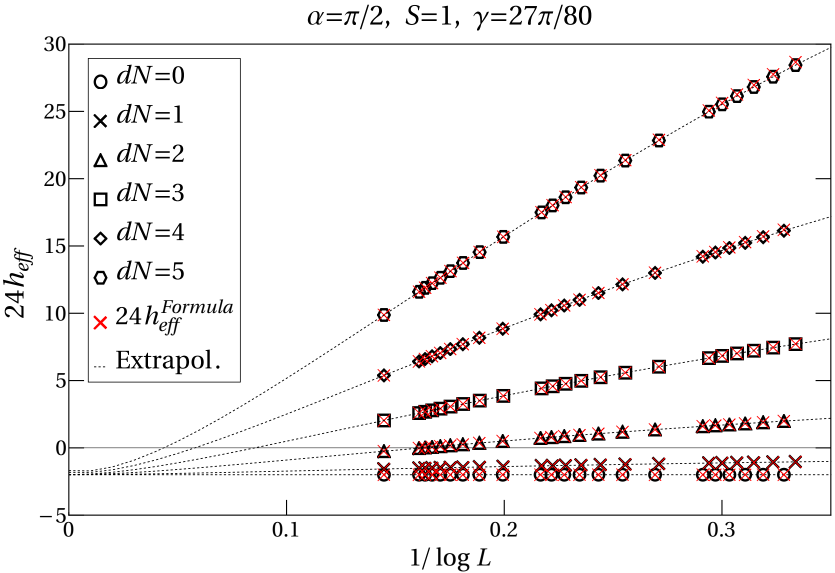
where

$$dN = M^0 - M^{\frac{\pi}{2}}, \quad \tilde{Z}_D = \lim_{\omega \to 0} \left(1 - \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{i\omega x} \left(K_0(x) - K_1(x) \right) \right)^{-1}.$$

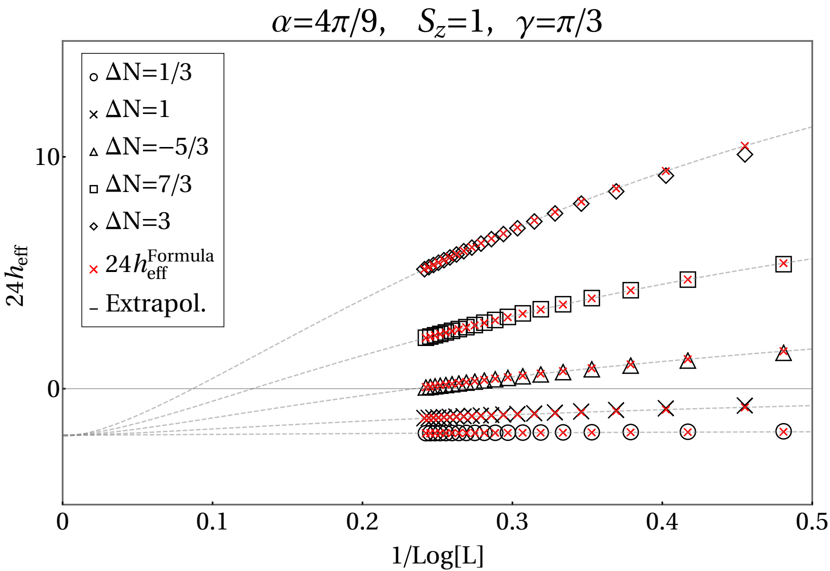
- Penultimate term vanishes formally since $\tilde{Z}_D = \infty$ (singular!)
$\Longrightarrow$ [see talk Klümper Saturday]

Leibniz Universität Hannover

Numeric results



Numeric Results



Scaling limit

Logarithmic corrections for fixed $dN = 0, 1, 2, 3.....$

$$s \asymp \frac{\pi}{2} \frac{dN - dN_{GS}}{\log(L)} \xrightarrow{L \to \infty} 0$$

Scaling limit

Logarithmic corrections for fixed $dN = 0, 1, 2, 3.....$

$$s \asymp \frac{\pi}{2} \frac{dN - dN_{GS}}{\log(L)} \xrightarrow{L\to\infty} 0$$

Define scaling limit by introducing L-dependent disbalance $(dN - dN_{GS})(L)$:

$$s_* = \underset{L\to\infty}{\text{slim}}\ s \qquad s_* \in \mathbb{R}$$

$$\underset{L\to\infty}{\text{slim}} \frac{L}{v_F \pi} \left(E - Le_\infty - f_\infty\right) = \left(-\frac{1}{12} + \frac{\gamma}{\pi} \cdot p^2 + \frac{\gamma}{\pi - 2\gamma} \cdot s_*^2\right)$$

where

$$p = \frac{1}{2}\left(2S + 1 - \frac{\pi}{\gamma}\right)$$

Leibniz Universität Hannover

Further results

- Definition of an operator on the lattice measuring s (Quasi momentum operator) [Frahm, Seel '14, Ikhlef Jacobsen, Saleur'12] :

$$s = \frac{\pi - 2\gamma}{4\pi\gamma}\left(\mathcal{K} - \mathcal{K}_{Thermo}\right), \quad \mathcal{K} = \sum_{i=1}^{M} 2\log\left[\frac{\cosh(2v_i) - \cos(\alpha+\gamma)}{\cosh(2v_i) - \cos(\alpha-\gamma)}\right]$$

- Comparison with primaries of the Black Hole CFT [Maldecena, Ooguri '00; Hanany, Prezas, Troost '02; Ribault, Schomerus '04]

- Identification of discrete states

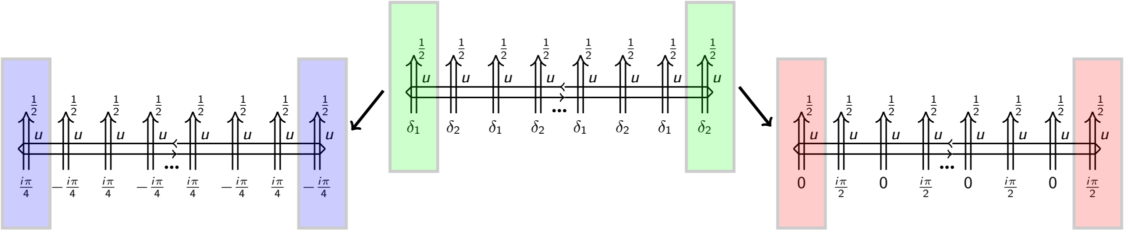
Leibniz Universität Hannover

Influence of different boundary conditions

How can such rich structure disappear by choosing different boundary conditions?
[Robertson, Jacobsen, Saleur '21 vs '20]

$$H = -\frac{1}{\cos(\gamma)}\left[\sum_{j=2}^{2L-1} e_{j,j+1}e_{j-1,j} + e_{j-1,j}e_{j,j+1}\right] - 2\sum_{j=1}^{2L-1} e_{j,j+1} - \frac{1}{\cos(\gamma)}\left(e_{1,2} + e_{2L-1,2L}\right),$$

Change of boundary condition $\Longleftrightarrow$ change of staggering in the six-vertex model



- Study flow with $\delta_{1,2} = \frac{i\vartheta}{2} \pm \frac{i\pi}{4}$, **Phase transitions at** $\vartheta = -\gamma, \frac{\pi}{2} - \gamma$

Summary of main points and directions of further study

- $U_q(sl(2))$-Symmetry is spontaneously broken!
- A continuum is present for the alternating case (generalisation to $\gamma < \alpha < \pi - \gamma$)
- Disappearance of continuum can be interpreted as bulk phenomena

---

- Systematic study via advanced techniques like ODE/IQFT correspondence as in [Bazhanov, Kotousov, Koval, Lukyanov '21]
- Considering higher rank models

Thank you!

Leibniz Universität Hannover

Summary of main points and directions of further study

- $U_q(sl(2))$-Symmetry is spontaneously broken!
- A continuum is present for the alternating case (generalisation to $\gamma < \alpha < \pi - \gamma$)
- Disappearance of continuum can be interpreted as bulk phenomena

---

- Systematic study via advanced techniques like ODE/IQFT correspondence as in [Bazhanov, Kotousov, Koval, Lukyanov '21]
- Considering higher rank models

Leibniz Universität Hannover

Back-Up Slides

The Ingredients

Standard *XXZ*-*R*-matrix:

$$R(u)=\begin{pmatrix}\sinh(u+i\gamma) & 0 & 0 & 0\\ 0 & \sinh(u) & \sinh(i\gamma) & 0\\ 0 & \sinh(i\gamma) & \sinh(u) & 0\\ 0 & 0 & 0 & \sinh(u+i\gamma)\end{pmatrix}$$

And the following boundary matrices :

$$K^-(u)=\begin{pmatrix}e^u & 0\\ 0 & e^{-u}\end{pmatrix}, \qquad K^+(u)=\begin{pmatrix}e^{-u-i\gamma} & 0\\ 0 & e^{u+i\gamma}\end{pmatrix}$$

$$R_{23}(v)R_{13}(u)R_{12}(u-v)=R_{12}(u-v)R_{13}(u)R_{23}(v)$$
$$R_{12}(u-v)K_1^-(u)R_{12}(u+v)K_2^-(v)=K_2^-(v)R_{12}(u+v)K_1^-(u)R_{12}(u-v)$$

Leibniz Universität Hannover

Open boundary conditions for symmetric R-matrices

YBE: $R_{23}(v)R_{13}(u)R_{12}(u-v) = R_{12}(u-v)R_{13}(u)R_{23}(v)$

PBC: $\tau^{\mathrm{PBC}}(u) = \mathrm{tr}_0\left(R_{0L}(u)\cdots R_{01}(u)\right) \overset{YBE}{\Longrightarrow} \left[\tau^{\mathrm{PBC}}(u), \tau^{\mathrm{PBC}}(v)\right] = 0$

OBC: $\tau^{OBC}(u) = \mathrm{tr}_0\left(\overset{0}{K}_+(u)R_{0L}(u)\cdots R_{01}(u)\overset{0}{K}_-(u)R_{01}(u)\cdots R_{0L}(u)\right)$

Reflectionalgebras or BYBE:

- $R_{12}(u-v)\overset{1}{K}_-(u)R_{12}(u+v)\overset{2}{K}_-(v) = \overset{2}{K}_-(v)R_{12}(u+v)\overset{1}{K}_-(u)R_{12}(u-v)$
- $R_{12}(-u+v)\overset{1}{K}{}^t_+(u)R_{12}(-u-v-2i\gamma)\overset{2}{K}{}^t_+(v) = \overset{2}{K}{}^t_+(v)R_{12}(-u-v-2i\gamma)\overset{1}{K}{}^t_+(u)R_{12}(-u+v)$

$\overset{BYBE+YBE}{\Longrightarrow} \left[\tau^{\mathrm{OBC}}(u), \tau^{\mathrm{OBC}}(v)\right] = 0$

Hamiltonian limit is given by $H = A\,\frac{\mathrm{d}}{\mathrm{d}u}\tau^{\mathrm{OBC}}(u)\Big|_{u=0} + \mathrm{B}$

The Alternating Staggered Model

Possibility to include arbitrary inhomogeneities:

- $\tau^{OBC}(u) = \mathrm{tr}_0 \left(\overset{0}{K}_+(u) R_{0L}(u+\delta_L) \cdots R_{01}(u+\delta_1) \overset{0}{K}_-(u) R_{01}(u-\delta_1) \cdots R_{0L}(u-\delta_L) \right)$

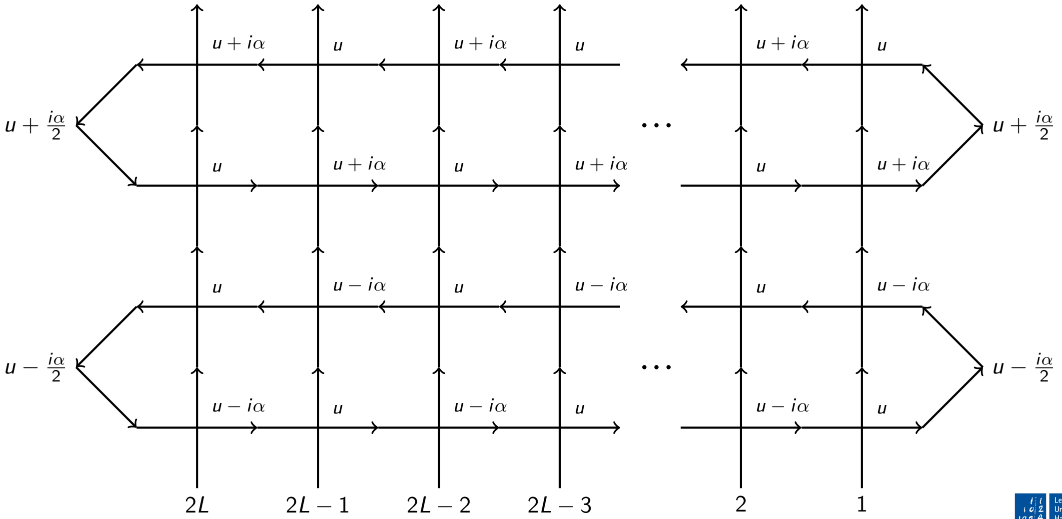
We will focus on the staggering $\pm\frac{i\alpha}{2}$ in horizontal

- $\tau^{OBC}(u) = \mathrm{tr}_0 \left(\overset{0}{K}_+(u) R_{02L}(u+\frac{i\alpha}{2}) \cdots R_{01}(u-\frac{i\alpha}{2}) \overset{0}{K}_-(u) R_{01}(u+\frac{i\alpha}{2}) \cdots R_{02L}(u-\frac{i\alpha}{2}) \right)$

as well as in the vertical direction via

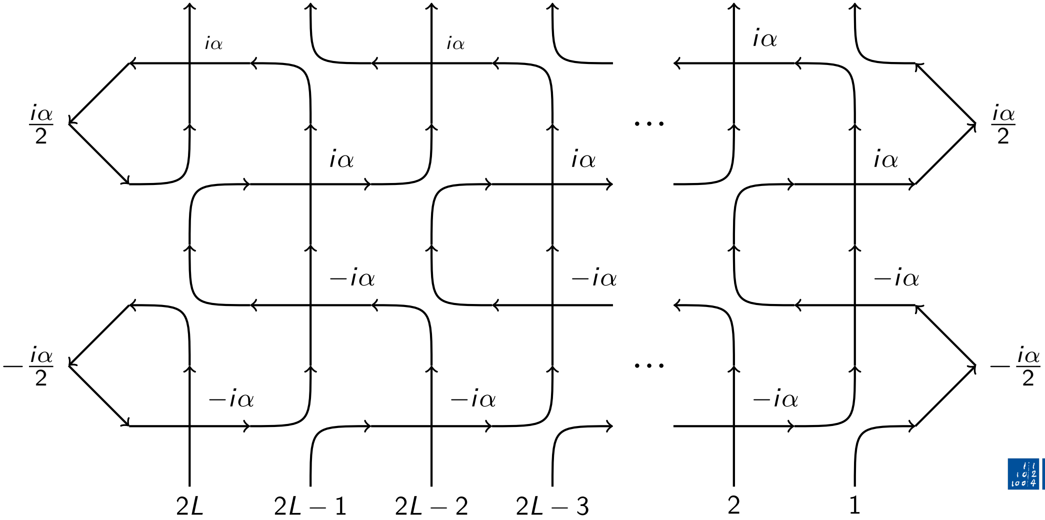
- $\mathcal{T}(u) = \tau^{OBC}(u+\frac{i\alpha}{2}) \tau^{OBC}(u-\frac{i\alpha}{2})$

The Staggered Model



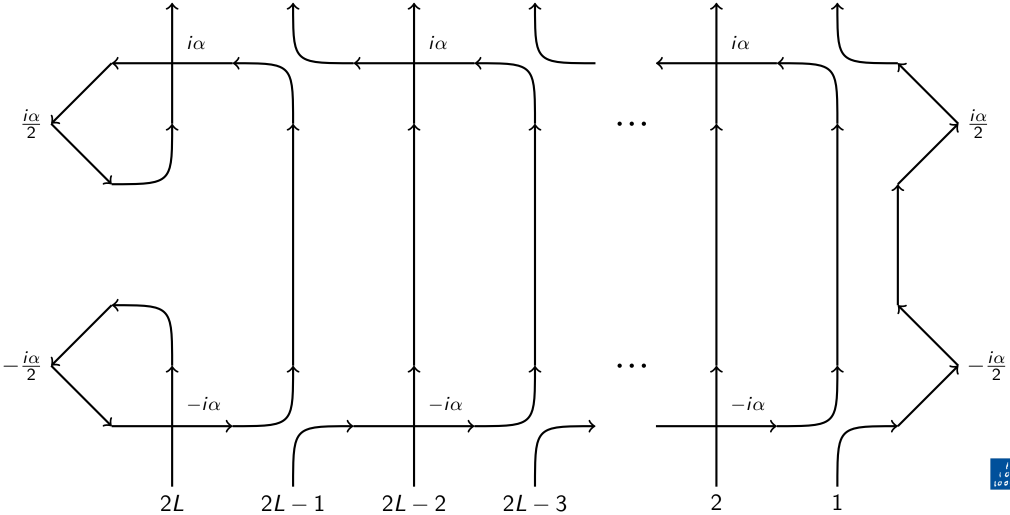
The Staggered Model

Evaluated at zero this gives:



The Staggered Model

Evaluated at zero this gives:



Hamiltonian $H = A\frac{\mathrm{d}}{\mathrm{d}u}\tau(u+\frac{i\alpha}{2})\tau(u-\frac{i\alpha}{2})|_{u=0} + B$

$$
\begin{aligned}
H \propto & 2\sin^2(\gamma)\sum_{j=1}^{2L-1}\cos(\gamma)\sigma_j^z\sigma_{j+1}^z + 2\cos(\alpha)(\sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+) \\
& + \cos(\gamma)\sin^2(\alpha)\sum_{j=1}^{2L-2}\sigma_j^z\sigma_{j+2}^z + 2(\sigma_j^+\sigma_{j+2}^- + \sigma_j^-\sigma_{j+2}^+) \\
& - \sin(\alpha)\sin(2\gamma)\sum_{j=1}^{2L-2}(-1)^{j+1}\sigma_j^z\sigma_{j+1}^+\sigma_{j+2}^- + (-1)^j\sigma_j^z\sigma_{j+1}^-\sigma_{j+2}^+ + (-1)^{j+1}\sigma_j^+\sigma_{j+1}^-\sigma_{j+2}^z + (-1)^j\sigma_j^-\sigma_{j+1}^+\sigma_{j+2}^z \\
& - \sin(\gamma)\sin(2\alpha)\sum_{j=1}^{2L-2}(-1)^{j+1}\sigma_j^-\sigma_{j+1}^z\sigma_{j+2}^+ + (-1)^j\sigma_j^+\sigma_{j+1}^z\sigma_{j+2}^- \\
& - \cos(\gamma)\sin^2(\alpha)(\sigma_1^z\sigma_2^z + \sigma_{2L-1}^z\sigma_{2L}^z) \\
& - (i\sin(\alpha)\cos(2\gamma) - i\sin(\alpha)e^{2i\alpha})(\sigma_1^+\sigma_2^- + \sigma_{2L-1}^+\sigma_{2L}^-) \\
& - (i\sin(\alpha)\cos(2\gamma) - i\sin(\alpha)e^{-2i\alpha})(\sigma_1^-\sigma_2^+ + \sigma_{2L-1}^-\sigma_{2L}^+) \\
& - 2\sin(\alpha-\gamma)\sin(\alpha+\gamma)\sinh(i\gamma)(\sigma_1^z - \sigma_{2L}^z)
\end{aligned}
$$

Leibniz Universität Hannover

Hamiltonian $H = A\frac{\mathrm{d}}{\mathrm{d}u}\tau(u+\frac{i\alpha}{2})\tau(u-\frac{i\alpha}{2})|_{u=0} + B$

$$
\begin{aligned}
H =& 2\sin^2(\gamma)\sum_{j=1}^{2L-1}\cos(\gamma)\sigma_j^z\sigma_{j+1}^z + 2\cos(\alpha)(\sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+)\\
&-\cos(\gamma)\sin^2(\alpha)\sum_{j=1}^{2L-2}\sigma_j^z\sigma_{j+2}^z + 2(\sigma_j^+\sigma_{j+2}^- + \sigma_j^-\sigma_{j+2}^+)\\
&-\sin(\alpha)\sin(2\gamma)\sum_{j=1}^{2L-2}(-1)^{j+1}\sigma_j^z\sigma_{j+1}^+\sigma_{j+2}^- + (-1)^j\sigma_j^z\sigma_{j+1}^-\sigma_{j+2}^+ + (-1)^{j+1}\sigma_j^+\sigma_{j+1}^-\sigma_{j+2}^z + (-1)^j\sigma_j^-\sigma_{j+1}^+\sigma_{j+2}^z\\
&-\sin(\gamma)\sin(2\alpha)\sum_{j=1}^{2L-2}(-1)^{j+1}\sigma_j^-\sigma_{j+1}^z\sigma_{j+2}^+ + (-1)^j\sigma_j^+\sigma_{j+1}^z\sigma_{j+2}^-\\
&-\cos(\gamma)\sin^2(\alpha)(\sigma_1^z\sigma_2^z + \sigma_{2L-1}^z\sigma_{2L}^z)\\
&-(i\sin(\alpha)\cos(2\gamma) - i\sin(\alpha)e^{2i\alpha})(\sigma_1^+\sigma_2^- + \sigma_{2L-1}^+\sigma_{2L}^-)\\
&+(i\sin(\alpha)\cos(2\gamma) - i\sin(\alpha)e^{-2i\alpha})(\sigma_1^-\sigma_2^+ + \sigma_{2L-1}^-\sigma_{2L}^+)\\
&-2\sin(\alpha-\gamma)\sin(\alpha+\gamma)\sinh(i\gamma)(\sigma_1^z - \sigma_{2L}^z) \qquad \xrightarrow{\alpha\to 0}\sim H_{XXZ}^{\mathrm{OBC}}\ \text{homogeneous}
\end{aligned}
$$

Hamiltonian $H = A\frac{\mathrm{d}}{\mathrm{d}u}\tau(u+\frac{i\alpha}{2})\tau(u-\frac{i\alpha}{2})|_{u=0} + B$

$$
\begin{aligned}
H =& 2\sin^2(\gamma)\sum_{j=1}^{2L-1}\cos(\gamma)\sigma_j^z\sigma_{j+1}^z + 2\cos(\alpha)(\sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+)\\
&-\cos(\gamma)\sin^2(\alpha)\sum_{j=1}^{2L-2}\sigma_j^z\sigma_{j+2}^z + 2(\sigma_j^+\sigma_{j+2}^- + \sigma_j^-\sigma_{j+2}^+)\\
&-\sin(\alpha)\sin(2\gamma)\sum_{j=1}^{2L-2}(-1)^{j+1}\sigma_j^z\sigma_{j+1}^+\sigma_{j+2}^- + (-1)^j\sigma_j^z\sigma_{j+1}^-\sigma_{j+2}^+ + (-1)^{j+1}\sigma_j^+\sigma_{j+1}^-\sigma_{j+2}^z + (-1)^j\sigma_j^-\sigma_{j+1}^+\sigma_{j+2}^z\\
&-\sin(\gamma)\sin(2\alpha)\sum_{j=1}^{2L-2}(-1)^{j+1}\sigma_j^-\sigma_{j+1}^z\sigma_{j+2}^+ + (-1)^j\sigma_j^+\sigma_{j+1}^z\sigma_{j+2}^-\\
&-\cos(\gamma)\sin^2(\alpha)(\sigma_1^z\sigma_2^z + \sigma_{2L-1}^z\sigma_{2L}^z)\\
&-(i\sin(\alpha)\cos(2\gamma) - i\sin(\alpha)e^{2i\alpha})(\sigma_1^+\sigma_2^- + \sigma_{2L-1}^+\sigma_{2L}^-)\\
&+(i\sin(\alpha)\cos(2\gamma) - i\sin(\alpha)e^{-2i\alpha})(\sigma_1^-\sigma_2^+ + \sigma_{2L-1}^-\sigma_{2L}^+)\\
&-2\sin(\alpha-\gamma)\sin(\alpha+\gamma)\sinh(i\gamma)(\sigma_1^z - \sigma_{2L}^z) \quad \xrightarrow{\gamma\to 0}\sim H_{XXX}^{\mathrm{PBC}}
\end{aligned}
$$

ferromagnetic!

Leibniz Universität Hannover

BAE and Energies

- Solve the system via algebraic Bethe-Ansatz [Kulish, Sklyanin '91] with BAE

$$\left(\frac{\sinh\left(v_m-\frac{i\alpha}{2}+\frac{i\gamma}{2}\right)\sinh\left(v_m+\frac{i\alpha}{2}+\frac{i\gamma}{2}\right)}{\sinh\left(v_m+\frac{i\alpha}{2}-\frac{i\gamma}{2}\right)\sinh\left(v_m-\frac{i\alpha}{2}-\frac{i\gamma}{2}\right)}\right)^{2L}=\prod_{k=1,\neq m}^{M}\frac{\sinh(v_m-v_k+i\gamma)}{\sinh(v_m-v_k-i\gamma)}\frac{\sinh(v_m+v_k+i\gamma)}{\sinh(v_m+v_k-i\gamma)},$$

and single transfermatrix eigenvalues:

$$\begin{aligned}\Lambda(u)\propto&\frac{\sinh(2u+2i\gamma)}{\sinh(2u+i\gamma)}\left(\sinh(u+\frac{i\alpha}{2}+i\gamma)\sinh(u-\frac{i\alpha}{2}+i\gamma)\right)^{2L}\prod_{m=1}^{M}\frac{\sinh(u-v_m-\frac{i\gamma}{2})\sinh(u+v_m-\frac{i\gamma}{2})}{\sinh(u-v_m+\frac{i\gamma}{2})\sinh(u+v_m+\frac{i\gamma}{2})}\\&+\frac{\sinh(2u)}{\sinh(2u+i\gamma)}\left(\sinh(u+\frac{i\alpha}{2})\sinh(u-\frac{i\alpha}{2})\right)^{2L}\prod_{m=1}^{M}\frac{\sinh(u-v_m+\frac{3i\gamma}{2})\sinh(u+v_m+\frac{3i\gamma}{2})}{\sinh(u-v_m+\frac{i\gamma}{2})\sinh(u+v_m+\frac{i\gamma}{2})}\end{aligned}\tag{1}$$

and energies

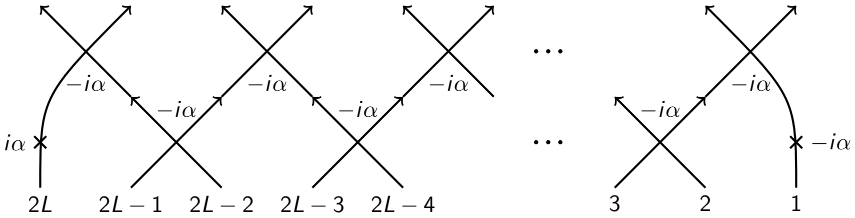
$$E=\sum_{j=1}^{M}\epsilon(v_j)=\sum_{j=1}^{M}\frac{4\sin(\gamma)(\cos(\alpha)\cosh(2v_j)-\cos(\gamma))}{(\cosh(2v_j)-\cos(\alpha-\gamma))(\cosh(2v_j)-\cos(\alpha+\gamma))}$$

- Note BAE invariant under $v_m\to -v_m$ and energies under $\alpha\to\pi-\alpha$

Leibniz Universität Hannover

Quasi Momentum

$$K = \log\left(\frac{\tau\left(u - \frac{i\alpha}{2}\right)}{\tau\left(u + \frac{i\alpha}{2}\right)}\right)\Bigg|_{u=0} \qquad \mathcal{K} = \sum_{i=1}^{M} \underbrace{2\log\left[\frac{\cosh(2v_i) - \cos(\alpha+\gamma)}{\cosh(2v_i) - \cos(\alpha-\gamma)}\right]}_{k_0(v_i)} + C,$$



Densities and Integration Boundaries

- One class of Bethe-Roots describing the low energy physics in the parameter range $\gamma < \alpha < \pi - \gamma$ are:

$$v_j^x = x_j \qquad v_j^y = y_j + \frac{i\pi}{2} \qquad x_j, y_j \in \mathbb{R}$$

Leibniz Universität Hannover

Densities and Integration Boundaries

- One class of Bethe-Roots describing the low energy physics in the parameter range $\gamma < \alpha < \pi - \gamma$ are:

$$v_j^x = x_j \qquad v_j^y = y_j + \frac{i\pi}{2} \qquad x_j, y_j \in \mathbb{R}$$

- Logarithmic Bethe Equations and Euler-Maclaurin yields the integral equations:

$$\rho^x(x) = \sigma_0^x(x) + \frac{1}{L}\tau_0^x(x) + \frac{1}{24L^2}\eta_0^x + \int_{-\infty}^{\infty} \mathrm{d}x' K_0\left(x - x'\right)\rho^x(x') + \int_{-\infty}^{\infty} \mathrm{d}x' K_1\left(x - x'\right)\rho^y(x')$$

$$\rho^y(y) = \sigma_0^y(y) + \frac{1}{L}\tau_0^y(y) + \frac{1}{24L^2}\eta_0^y + \int_{-\infty}^{\infty} \mathrm{d}x' K_1\left(y - x'\right)\rho^x(x') + \int_{-\infty}^{\infty} \mathrm{d}x' K_0\left(y - x'\right)\rho^y(x')$$

where

$$K_0(x) = \frac{1}{2\pi}\phi'(x,\gamma), \quad K_1(x) = -\frac{1}{2\pi}\psi'(x,\gamma)$$

$$\phi(x,y) = 2\arctan\left(\tanh(x)\cot(y)\right), \quad \psi(x,y) = 2\arctan\left(\tanh(x)\tan(y)\right)$$

Leibniz Universität Hannover

Results for the densities

- For the bulk contribution we recover the results for the PBC:

$$\sigma^x(x) = \frac{2\sin(\frac{\pi(\alpha-\gamma)}{\pi-2\gamma})}{\pi - 2\gamma} \frac{1}{\cosh(\frac{2\pi y}{\pi-2\gamma}) - \cos(\frac{\pi(\alpha-\gamma)}{\pi-2\gamma})}$$

$$\sigma^y(y) = \frac{2\sin(\frac{\pi(\alpha-\gamma)}{\pi-2\gamma})}{\pi - 2\gamma} \frac{1}{\cosh(\frac{2\pi y}{\pi-2\gamma}) + \cos(\frac{\pi(\alpha-\gamma)}{\pi-2\gamma})},$$

- For the self dual case $\alpha = \frac{\pi}{2}$ both densities are equal
- Densities are vanishing for $\alpha = \gamma$ and $\alpha = \pi - \gamma$.
- The surface contribution is the same for both roots:

$$\tau^i(x) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega x} \frac{\sinh(\frac{3\gamma-\pi}{4}\omega)}{\sinh(\frac{\gamma\omega}{4}) \cosh(\frac{2\gamma-\pi}{4}\omega)}$$

Number of Bethe Roots:

$$\frac{2M_{GS}^{0}+1}{L} = 2\cdot\frac{\pi-\alpha-\gamma}{\pi-2\gamma} + \frac{1}{L}\left(\frac{3}{2}-\frac{\pi}{2\gamma}\right) + \mathcal{O}\Big(\frac{1}{L^2}\Big)\,,$$

$$\frac{2M_{GS}^{\frac{\pi}{2}}+1}{L} = 2\cdot\frac{\alpha-\gamma}{\pi-2\gamma} + \frac{1}{L}\left(\frac{3}{2}-\frac{\pi}{2\gamma}\right) + \mathcal{O}\Big(\frac{1}{L^2}\Big)\,.$$

$$\Longrightarrow S^{GS} = \left[-\frac{1}{2}+\frac{\pi}{2\gamma}\right],$$

where the brackets indicate the rounding. Inverting this relation we obtain a range of anisotropies γ for which the ground state is realized in the sector with spin S^{GS}:

$$\frac{\pi}{2S^{GS}+2} < \gamma < \frac{\pi}{2S^{GS}}\,.$$

Thermodynamics Quantities and CFT

Using the densities we obtain:

$$e_\infty = -2\int_{-\infty}^{\infty} d\omega \frac{\sinh(\frac{\gamma\omega}{2})\left(\sinh\left(\frac{\pi\omega}{2}-\frac{\omega\gamma}{2}\right)\cosh(\frac{\omega\pi}{2}-\alpha\omega)-\sinh(\frac{\gamma\omega}{2})\right)}{\sinh(\frac{\omega\pi}{2})\sinh((\frac{\pi-2\gamma}{2})\omega)},$$

$$v_F = \frac{2\pi}{\pi-2\gamma} \qquad f_\infty = \cdots \qquad k_\infty = \cdots \qquad k_s = \cdots \qquad \mathcal{K}_{thermo} = Lk_\infty + k_s + \mathcal{O}(\frac{1}{L})$$

Relation to CFT:

$$\frac{L}{\pi v_F}(E(L) - Le_\infty - f_\infty) = \underbrace{-\frac{c}{24} + h_n + d}_{h_{eff}} \tag{2}$$

Leibniz Universität Hannover

Taking Small Excitations into Account

- Expand energy around the ground state energy in the limit $L \to \infty$ gives

$$h_{eff} = \left(-\frac{1}{12} + \frac{\gamma}{4\pi}\left(2S^Z + 1 - \frac{\pi}{\gamma}\right)^2 + \frac{1}{4}\frac{(dN - dN_{GS})^2}{\tilde{Z}_D^2} + n_{ph} \right),$$

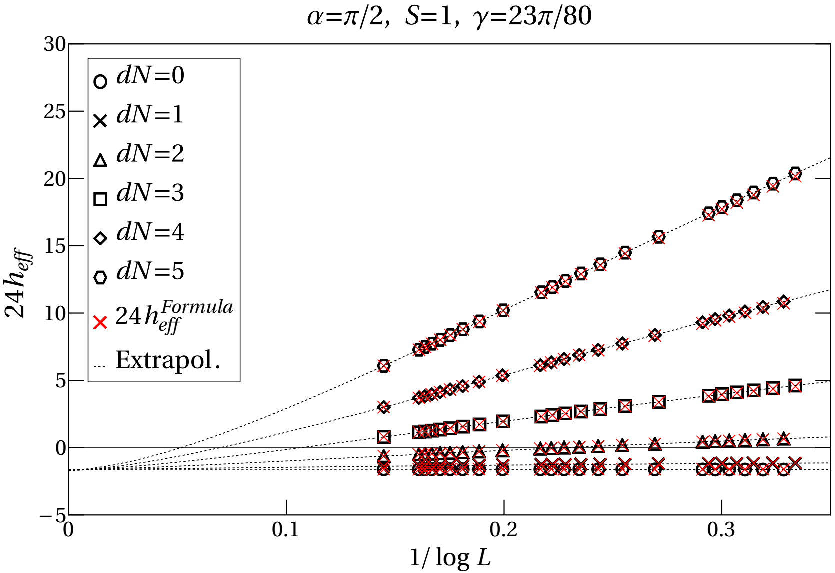
where

$$dN = M^0 - M^{\frac{\pi}{2}}, \quad \tilde{Z}_D = \lim_{\omega \to 0}\left(1 - \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{i\omega x}\left(K_0(x) - K_1(x)\right)\right)^{-1}.$$

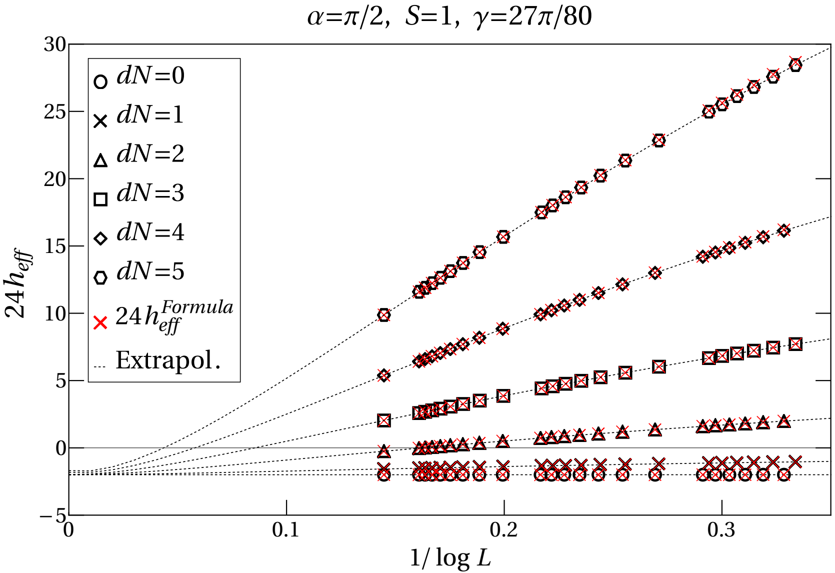
- Penultimate term vanishes formally since $\tilde{Z}_D = \infty$. Numerics shows that the decrease is actually $\propto \frac{1}{\log(L)}$

Leibniz Universität Hannover

Numeric Results



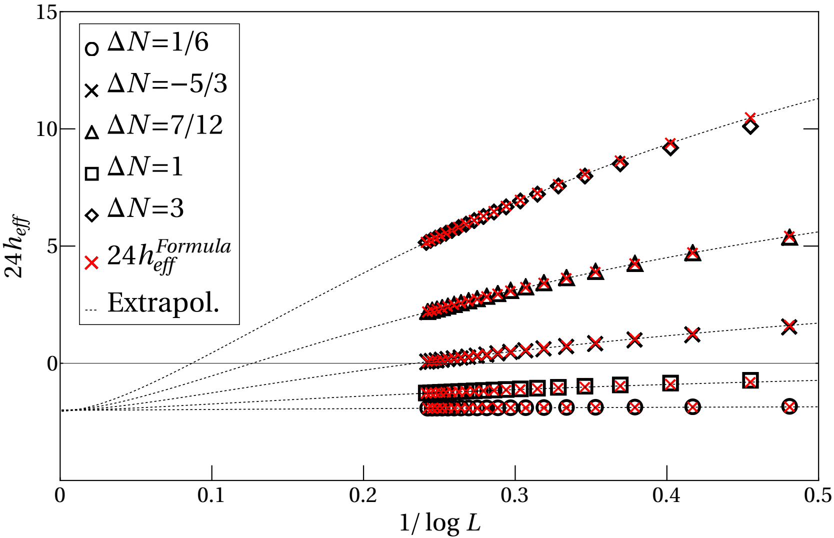
Numeric Results



Leibniz Universität Hannover

Numeric Results

$\alpha=4\pi/9,\ S=1,\ \gamma=\pi/3$



Logarithmic Bethe Equations

In logarithmic form we obtain for the BAE:

$$2\pi I_m^x = -2L\phi\left(x_m, \frac{\gamma-\alpha}{2}\right) - 2L\phi\left(x_m, \frac{\alpha+\gamma}{2}\right) - \phi(2x_m,\gamma) - \phi(x_m,\gamma) + \psi(x_m,\gamma)$$
$$+ \sum_{k=-M_0}^{M^0} \phi\left(x_m - x_k, \gamma\right) - \sum_{k=-M^{\frac{\pi}{2}}}^{M^{\frac{\pi}{2}}} \psi\left(x_m - y_k, \gamma\right)$$
$$2\pi I_m^y = 2L\psi\left(y_m, \frac{\gamma-\alpha}{2}\right) + 2L\psi\left(y_m, \frac{\alpha+\gamma}{2}\right) - \phi(2y_m) - \phi(y_m,\gamma) + \psi(y_m,\gamma)$$
$$- \sum_{k=-M^0}^{M^0} \psi\left(y_m - x_k, \gamma\right) + \sum_{k=-M^{\frac{\pi}{2}}}^{M^{\frac{\pi}{2}}} \phi\left(y_m - y_k, \gamma\right),$$

where

$$\phi(x,y) = 2\arctan\left(\tanh(x)\cot(y)\right)$$
$$\psi(x,y) = 2\arctan\left(\tanh(x)\tan(y)\right)$$

and the redefined Bethe-integers are integers.

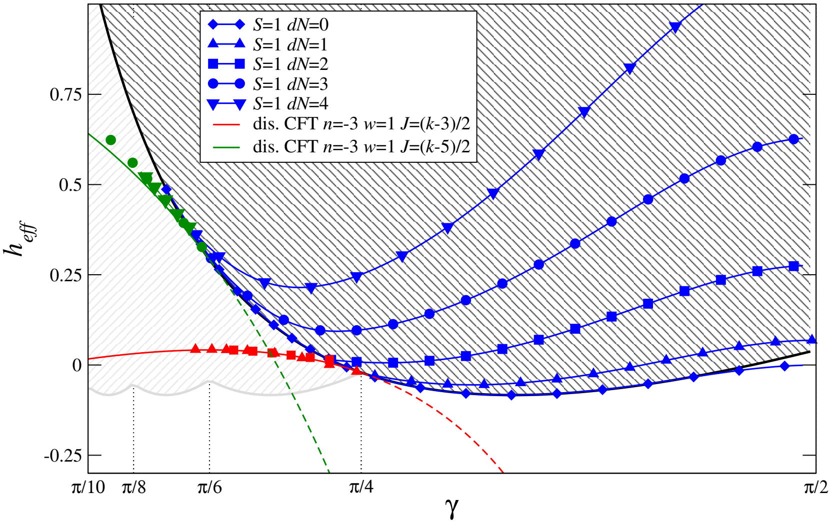
Identification with the Black Hole CFT Discrete Part

The BH CFT has also a discrete part given by

$$\frac{1}{2} < J < \frac{(k-1)}{2}\,, \quad J = \frac{|kw| - |n|}{2} - \ell, \qquad \ell = 0, 1, 2 \ldots . \tag{3}$$

These states can also be found in the spin chain where the quasi momentum becomes purely imaginary.

Transmutation of Continuous State to Discrete States



Defining a Quantum number

- Idea: Parameterise logarithmic correction by the quasi momentum operator as done in the (quasi)-periodic case [Frahm, Seel '14, Ikhlef Jacobsen, Saleur'12] :

$$K = \left(\prod_{i=1}^{L} c_{2i-1,2i}(-\alpha) \right) e^{-i\alpha/2\sigma_1^z} \left(\prod_{i=1}^{L-1} c_{2i,2i+1}(-\alpha) \right) e^{i\alpha/2\sigma_{2L}^z},$$

$$c_{i,j}(\alpha) = \frac{i}{2}(\sin(\alpha-\gamma)+\sin(\gamma)) + \frac{i}{2}(\sin(\alpha-\gamma)-\sin(\gamma))\sigma_i^z\sigma_j^z - \frac{i}{2}\sin(\alpha)(\sigma_i^+\sigma_j^- + \sigma_i^-\sigma_j^+)$$

- Using the variable we obtain

$$h_{\text{eff}} = -\frac{1}{12} + \frac{\gamma}{4\pi}\left(2S+1-\frac{\pi}{\gamma}\right)^2 + \frac{\gamma s^2}{\pi-2\gamma} + n_{ph}$$

Comparision with the Black Hole CFT Continuous Part

The BH CFT has a continuous part of spectrum which has following central charge and scaling dimensions [Ribault, Schomerus '04]:

$$c_{BH} = 2 + \frac{6}{(k-2)}\,, \quad h_{BH} = \frac{(n+wk)^2}{4k} - \frac{J(J-1)}{k-2} \quad \text{with} \quad J = \frac{1}{2} + i\tilde{s}, \quad \tilde{s} \in \mathbb{R}_0^+\,, \tag{4}$$

$$h_{\text{eff}}^{BH} = -\frac{1}{12} + \frac{(n+wk)^2}{4k} - \frac{1}{k-2}\left(J - \frac{1}{2}\right)^2 + d\,, \tag{5}$$

Compared these with the one of the spin chain:

$$h_{\text{eff}} = -\frac{1}{12} + \frac{\gamma}{4\pi}\left(2S + 1 - \frac{\pi}{\gamma}\right)^2 + \frac{\gamma s^2}{\pi - 2\gamma}$$

$$k = \frac{\pi}{\gamma}, \qquad n = -2S - 1, \qquad w = 1, \qquad \left(J - \frac{1}{2}\right)^2 = (is)^2,$$

Leibniz Universität Hannover

Background The Black Hole CFT:

- The Lagrangian is given by

$$L(g) = \frac{k}{8\pi} \int_{\Sigma} \sqrt{h} h^{ij} \mathrm{Tr}\left(g^{-1}\partial_i g g^{-1}\partial_j g\right) + \underbrace{ik\Gamma}_{\text{Wess-Zumino-Term}}$$

Leibniz Universität Hannover

Background The Black Hole CFT:

- The Lagrangian is given by

$$L(g) = \frac{k}{8\pi}\int_{\Sigma}\sqrt{h}h^{ij}\,\mathrm{Tr}\left(g^{-1}\partial_i g g^{-1}\partial_j g\right) + \underbrace{ik\Gamma}_{\text{Wess-Zumino-Term}}$$

- Invariant under $SL(2)\times SL(2)$ via $g \to agb^{-1}$ with $a, b \in SL(2)$

Background The Black Hole CFT:

- The Lagrangian is given by

$$L(g) = \frac{k}{8\pi}\int_\Sigma \sqrt{h}h^{ij}\mathrm{Tr}\left(g^{-1}\partial_i g g^{-1}\partial_j g\right) + \underbrace{ik\Gamma}_{\text{Wess-Zumino-Term}}$$

- Invariant under $SL(2)\times SL(2)$ via $g \to agb^{-1}$ with $a, b \in SL(2)$
- Gauging $U(1)$-subgroup generated by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

leads to after going to complex coordinates and fixing the gauge by setting

$$g = \cosh(r) + \sinh(r)\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$$

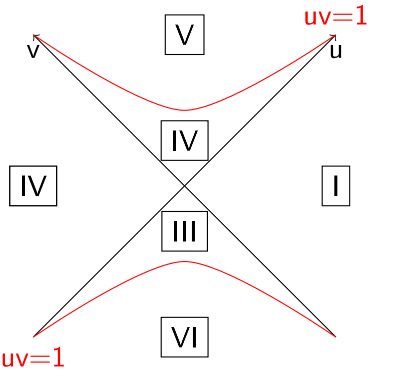
to

$$I(r,\theta) = \frac{k}{2\pi}\int \partial_z r \partial_{\bar{z}} + \tanh^2(r)\partial_z\theta\partial_{\bar{z}}\theta$$

Leibniz Universität Hannover

Metric

- $$ds^2 = dr^2 + \tanh^2(r)d\theta^2$$
$$= -\frac{dudv}{1-uv} \implies R = \frac{4}{1-uv}$$



-

Scaling Dimensions:

The scaling Dimensions for the PBC and the Black-Hole CFT are given by

$$h = \frac{(m - k\omega)^2}{4k} + \frac{s^2 + \frac{1}{4}}{k - 2}$$

$$h = \frac{(m + k\omega)^2}{4k} + \frac{s^2 + \frac{1}{4}}{k - 2}$$

$$k = \frac{\pi}{\gamma}$$

Densities

$$e_\infty = -2\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\sinh(\frac{\gamma\omega}{2})\left(\sinh\left(\frac{\pi\omega}{2}-\frac{\omega\gamma}{2}\right)\cosh(\frac{\omega\pi}{2}-\alpha\omega)-\sinh(\frac{\gamma\omega}{2})\right)}{\sinh(\frac{\omega\pi}{2})\sinh((\frac{\pi-2\gamma}{2})\omega)},$$

$$f_\infty = -\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\cosh\left(\frac{1}{4}(\pi-2\alpha)\omega\right)\sinh\left(\frac{1}{4}(3\gamma-\pi)\omega\right)\cosh\left(\frac{\gamma\omega}{4}\right)}{\cosh\left(\frac{1}{4}(\pi-2\gamma)\omega\right)\sinh\left(\frac{\pi\omega}{4}\right)} - \frac{4\sin(2\gamma)}{\cos(2\alpha)-\cos(2\gamma)}$$

$$k_\infty = 4\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\sinh(\frac{\omega\gamma}{2})\sinh(\frac{\pi\omega}{2}-\alpha\omega)\sinh(\frac{\pi-\gamma}{2}\omega)}{\omega\sinh(\frac{\omega\pi}{2})\sinh(\frac{\pi-2\gamma}{2}\omega)},$$

$$k_s = 2\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\sinh(\frac{3\gamma-\pi}{4}\omega)\cosh(\frac{\gamma\omega}{4})\sinh(\frac{\pi-2\alpha}{4}\omega)}{\omega\sinh(\frac{\omega\pi}{4})\cosh(\frac{2\gamma-\pi}{4}\omega)}$$
$$-\log\left(\frac{\cos(\alpha+\gamma)-1}{\cos(\alpha-\gamma)-1}\right)-\log\left(\frac{\cos(\alpha+\gamma)+1}{\cos(\alpha-\gamma)+1}\right).$$

Leibniz Universität Hannover

R-matrix

$$R_{i,j}(u)R_{j,i}(-u) = \xi(u)\mathbf{1}\,, \tag{6}$$
$$R_{i,j}^{t_i t_j}(u) = R_{j,i}(u)\,, \tag{7}$$
$$R_{i,j}(u) = V_i R_{i,j}^{t_j}(-u-\eta)V_i^{-1} \tag{8}$$
$$R_{i,j}^{t_i}(u)M_i R_{i,j}^{t_j}(-u-2\eta)M_i^{-1} = \xi(u+\eta)\,\mathbf{1}\,, \tag{9}$$
$$M_i^{-1}R_{i,j}(u)M_i = M_j R_{i,j}(u)M_j^{-1}\,. \tag{10}$$
$$R_{i,j}(u+p) = f_p G_i R_{i,j}(u)G_i^{-1}, \tag{11}$$

Leibniz Universität Hannover

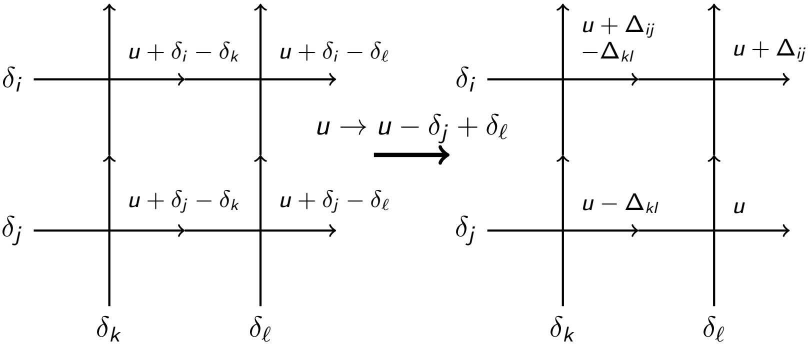
Composite R matrix

$$\mathbb{R}_{i,j|k,\ell}(u, \Delta_{ij}, \Delta_{k\ell}) = R_{i,\ell}(u + \Delta_{ij}) R_{i,k}(u + \Delta_{ij} - \Delta_{k\ell}) R_{j,\ell}(u) R_{j,k}(u - \Delta_{k\ell}) \,, \tag{12}$$

$$\mathbb{R}_{i,j|k,\ell}(u - v, \Delta_{ij}, \Delta_{k\ell}) \mathbb{R}_{i,j|m,n}(u, \Delta_{ij}, \Delta_{mn}) \mathbb{R}_{k,\ell|m,n}(v, \Delta_{k\ell}, \Delta_{mn}) = \tag{13}$$

$$\mathbb{R}_{k,\ell|m,n}(v, \Delta_{k\ell}, \Delta_{mn}) \mathbb{R}_{i,j|m,n}(u, \Delta_{ij}, \Delta_{mn}) \mathbb{R}_{i,j|k,\ell}(u - v, \Delta_{ij}, \Delta_{k\ell}) \,, \tag{14}$$

Motivation of $\mathbb{R}$



Factorisation of Reflection Matrices

Alternating case:

$$\mathbb{K}^{-}_{i,j}(u, 2\delta_0) = P_{i,j} K^{-}_{j}(u+\delta_0) R_{i,j}(2u) K^{-}_{i}(u-\delta_0)\,, \tag{15}$$

$$\mathbb{K}^{+}_{i,j}(u, 2\delta_0) = \frac{1}{\xi(2u+\eta)} P_{i,j} K^{+}_{j}(u-\delta_0) M_i R_{i,j}(-2u-2\eta) M_i^{-1} K^{+}_{i}(u+\delta_0)\,. \tag{16}$$

Quasi periodic case:

$$\overline{\mathbb{K}}^{-}_{i,j}\left(u, -\frac{p}{2}\right) = G_i^{-1} K^{-}_{i}\left(u+\frac{p}{2}\right) R_{j,i}\left(2u+\frac{p}{2}\right) K^{-}_{j}(u) R_{i,j}\left(-\frac{p}{2}\right)\,, \tag{17}$$

$$\overline{\mathbb{K}}^{+}_{i,j}\left(u, -\frac{p}{2}\right) = \frac{1}{\xi\left(2u+\frac{p}{2}+\eta\right)} R_{j,i}\left(\frac{p}{2}\right) K^{+}_{j}(u) M_i R_{i,j}\left(-2u-\frac{p}{2}-2\eta\right) M_i^{-1} K^{+}_{i}\left(u+\frac{p}{2}\right) G_i\,. \tag{18}$$

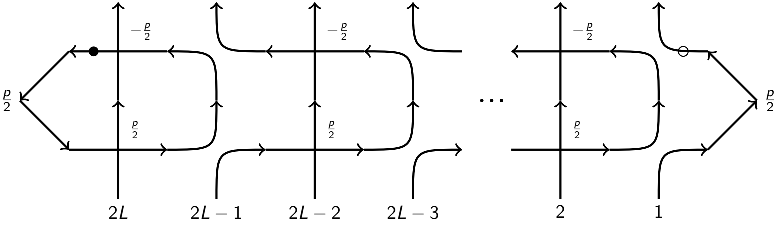
Leibniz Universität Hannover

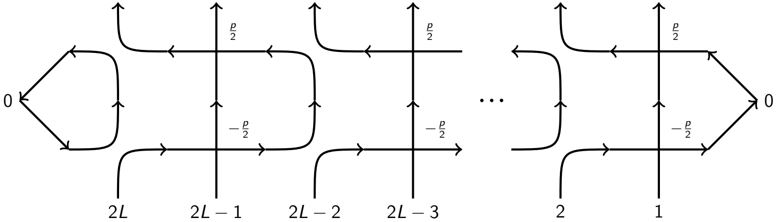
Reflection equations

$$\mathbb{R}_{i,j|k,\ell}(u-v,\theta,\theta)\mathbb{K}^{-}_{i,j}(u,\theta)\mathbb{R}_{k,\ell|i,j}(u+v,\theta,\theta)\mathbb{K}^{-}_{k,\ell}(v,\theta) \\ = \mathbb{K}^{-}_{k,\ell}(v,\theta)\mathbb{R}_{i,j|k,\ell}(u+v,\theta,\theta)\mathbb{K}^{-}_{i,j}(u,\theta)\mathbb{R}_{k,\ell|i,j}(u-v,\theta,\theta) \tag{19}$$

$$\mathbb{R}_{i,j|k,\ell}(-u+v,-\theta,-\theta)\left(\mathbb{K}^{+}_{i,j}(u,\theta)\right)^{t_i t_j}\mathbb{M}^{-1}_{i,j}\mathbb{R}_{k,\ell|i,j}(-u-v-2\eta,-\theta,-\theta)\mathbb{M}_{i,j}\left(\mathbb{K}^{+}_{k,\ell}(v,\theta)\right)^{t_k t_\ell} = \\ \left(\mathbb{K}^{+}_{k,\ell}(v,\theta)\right)^{t_k t_\ell}\mathbb{M}_{i,j}\mathbb{R}_{i,j|k,\ell}(-u-v-2\eta,-\theta,-\theta)\mathbb{M}^{-1}_{i,j}\left(\mathbb{K}^{+}_{i,j}(u,\theta)\right)^{t_i t_j}\mathbb{R}_{k,\ell|i,j}(-u+v,-\theta,-\theta)\,. \tag{20}$$

QI Trivial as

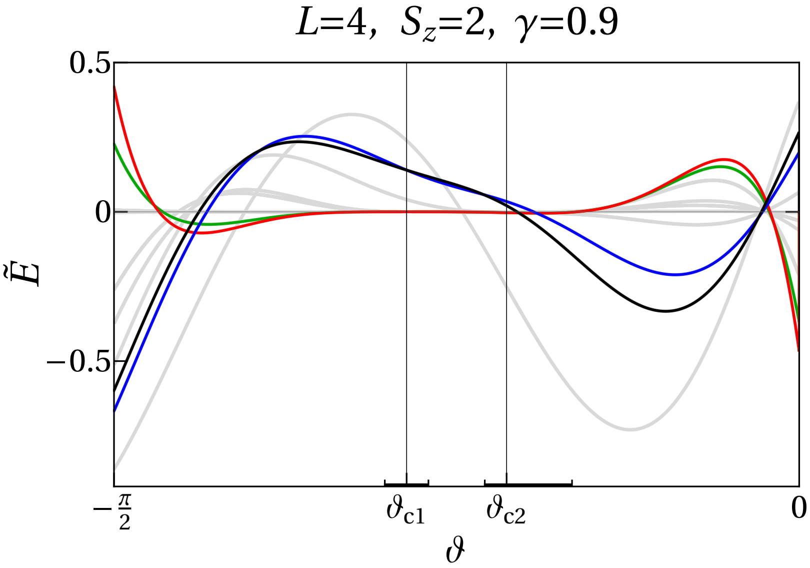




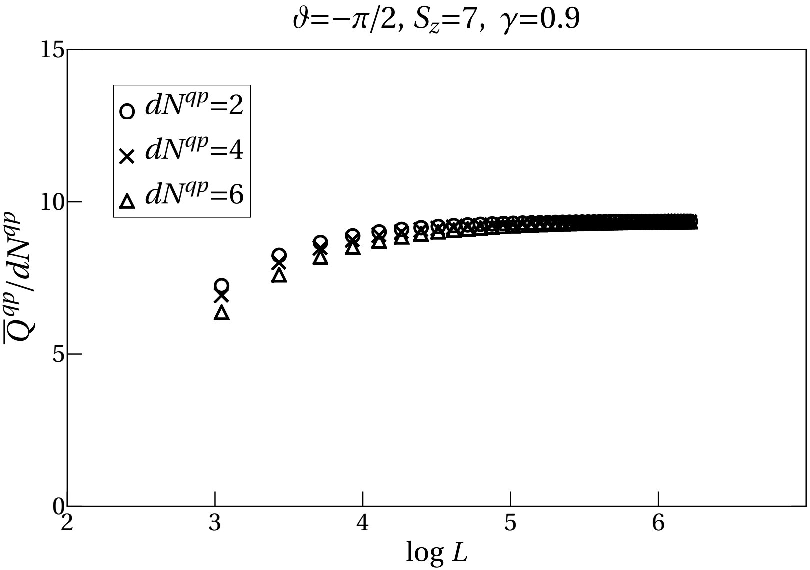
Non Local H Energies

$$E = \Bigg(-4L\cot(2\gamma) + L\frac{2\sin(2\vartheta)}{\sin(2\gamma)\sin(2(\gamma+\vartheta))} - \frac{2\sin(\vartheta)}{\cos(\gamma)\cos(\gamma+\vartheta)} + \frac{2\sin(\vartheta)}{\cos(2\gamma)\cos(2\gamma+\vartheta)}$$
$$+\frac{2\tan(\gamma)}{\cos(2\gamma)}\Bigg) \times \left(\prod_{m=1}^{M}\frac{\cos(2(\gamma-\vartheta)) + \cosh(4v_m)}{\cos(2(\gamma+\vartheta)) + \cosh(4v_m)} - 1\right)$$
$$-4\sin(2\gamma)\sum_{k=1}^{M}\left\{\frac{\cos(2\gamma)+\cos(\vartheta)\cosh(4v_k)}{(\cos(2(\gamma+\vartheta))+\cosh(4v_k))^2}\right\} \times \prod_{\substack{m=1\\ m\neq k}}^{M}\frac{\cos(2(\gamma-\vartheta)) + \cosh(4v_m)}{\cos(2(\gamma+\vartheta)) + \cosh(4v_m)}. \tag{21}$$

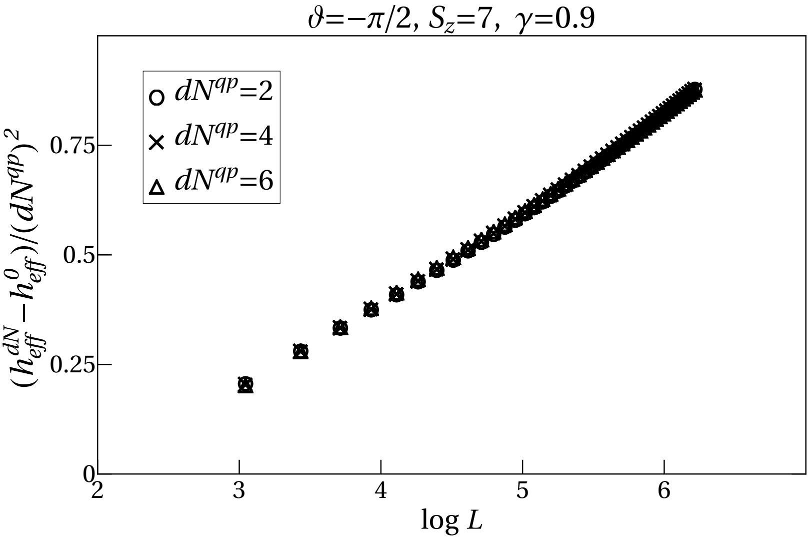
Spectral Flow



Alternative Quasi Momentum



FS.pdf


$\vartheta=-\pi/2,\ S_z=7,\ \gamma=0.9$
$dN^{qp}=2$
$dN^{qp}=4$
$dN^{qp}=6$
$(h_{eff}^{dN}-h_{eff}^{0})/(dN^{qp})^2$
0.75
0.5
0.25
0
2
3
4
5
6
$\log L$