Factorization of density matrices for the critical RSOS models

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HF & D Westerfeld: SciPost Phys. **11** *(2021) 057 [arXiv:2104.06269]*

D Westerfeld, M Großpietsch, H Kakuschke & HF (2023), in preparation

Physical properties of many-body systems

To characterize the properties of a quantum mechanical system we need to solve the spectral problem for the commuting integrals of motion (Hamiltonian, momentum, ...): *all* (or at least low energy) eigenvalues and eigenstates

$\rightsquigarrow$ for integrable models: spectrum of the transfer matrix

For a many-body system this leaves us with the formidable task to obtain information on measurable observables – in many cases few-point correlation functions, one- or two-particle reduced density matrices (RDM),...

Alternatively, a quantum system can be characterized in terms of *all* correlation functions of local operators or generating functionals thereof

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle, \qquad n = 1..\text{system size}\,.$$

Given that we are often interested in such correlation functions for small n this set appears to contain much more information than what is needed.

For **many-body systems with two-particle interaction**: is it possible to use (all) **few-particle** RDMs as a basis for the calculation of physical quantities? Under which conditions?
(**Coulson's challenge** to density functional methods)

Correlation functions in integrable models

Results for correlation functions in certain **integrable systems** (in particular the six-vertex models and relatives) have been obtained based on

- representation theory of quantum algebras [Jimbo,Miki,Miwa,Nakashiki (1992)]
- functional equations of q-Knizhnik-Zamolodchikov (qKZ) type [Jimbo,Miwa (1996)]
- algebraic Bethe ansatz, multiple integrals [Kitanine,Maillet,Terras (2000);...;Göhmann,Klümper,Seel (2004)]

The numerical evaluation of these expressions continued to be a challenge which was met by their **factorization**

- of multiple integral representations of density matrices on short segments into single ones [Boos,Göhmann,Klümper,Suzuki (2006)],
- of N-point correlation functions into terms containing nearest neighbour two-point ones ("physical part") and structure functions ("algebraic part", independent of the model parameters) [Boos,Miwa,Jimbo,Smirnov,Takeyama (2006)]

These properties have been proven for the XXZ model using the fermionic basis approach. (Discrete) functional equations of reduced qKZ type can be derived from local properties of the Boltzmann weights and are solved by the factorized ansatz. [Boos,Miwa,Jimbo,Smirnov,Takeyama; Wuppertal group ...]

Correlation functions in integrable models

Reduced density matrices of the Heisenberg spin chain for N adjacent sites can be written in terms of a nearest neighbour two-point function ω ("phyiscal part") and a set of recursively defined 'structure functions' $f_{N;I,J}$ of the spectral parameters λ_k ("algebraic part") [Boos *et al.* (2006)]:

$$D_N(\lambda_1,\ldots,\lambda_N) = \sum_{m=0}^{[N/2]} \sum_{I,J} \left(\prod_{p=1}^{m} \omega(\lambda_{i_p},\lambda_{j_p}) \right) f_{N;I,J}(\lambda_1,\ldots,\lambda_N)$$

where $I=(i_1,\ldots,i_m)$ and $J=(j_1,\ldots,j_m)$ such that $I\cap J=\varnothing$, $1\le i_p<j_p\le N$ and $i_1<\cdots<i_m$.

e.g.

$$\begin{aligned} D_2(\lambda_1,\lambda_2) &= f_{2,\varnothing,\varnothing}(\lambda_1,\lambda_2) + \omega(\lambda_1,\lambda_2)\, f_{2,\{1\},\{2\}}(\lambda_1,\lambda_2) \\ D_3(\lambda_1,\lambda_2,\lambda_3) &= f_{3,\varnothing,\varnothing}(\lambda_1,\lambda_2,\lambda_3) + \omega(\lambda_1,\lambda_2)\, f_{3,\{1\},\{2\}}(\lambda_1,\lambda_2,\lambda_3) \\ &\quad + \omega(\lambda_2,\lambda_3)\, f_{3,\{2\},\{3\}}(\lambda_1,\lambda_2,\lambda_3) + \omega(\lambda_1,\lambda_3)\, f_{3,\{1\},\{3\}}(\lambda_1,\lambda_2,\lambda_3) \\ &\quad \ldots \end{aligned}$$

Q: Is this "factorization" a generic property of (Yang-Baxter) integrable models?

Correlation functions in integrable models

Here: "interaction around the **face**" **models**

Partial results:

- Local height probabilities (LHPs) have been computed using Baxter's corner transfer matrix [Andrews,Baxter,Forrester (1984)]
- Correlation functions of vertex operators related to quantum group symmetries satisfy qKZ equations [Foda,Jimbo,Miwa,Miki,Nakayashiki (1994)]
- Multiple integral representations for multi-point LHPs by bosonization of the vertex operator algebra (massive case) [Lukyanov,Pugai (1996)]
- models with 'dynamical' R-matrix allow transfer of some concepts from vertex models (algebraic BA, SoV) [Felder,Varchenko (1996); Levy-Bencheton&Terras (2013/14); Levy-Bencheton,Terras,Niccoli (2016)]

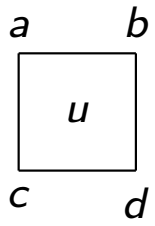
Some obstacles:

- proper description of local operators ("inverse problem")?
- Hilbert space is not a tenser product of local spaces, in general no algebraic Bethe Ansatz, no Slavnov determinant for scalar products, ...

Face models

Local Boltzmann weights depend on the spins a, b, c, d on the vertices surrounding an elementary face and a spectral parameter u:

$$W\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle| u\right) =$$



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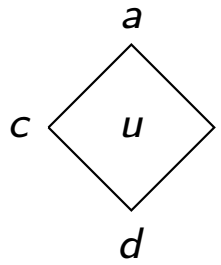

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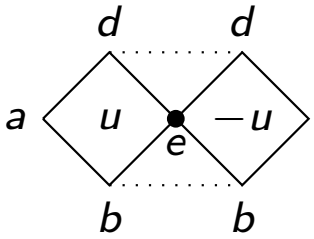
The spins take values from some discrete set $\mathfrak{S}$ subject to an adjacency condition on the allowed values on neighbouring sites. We assume:

unitarity

$$\sum_{e\in\mathfrak{S}}$$



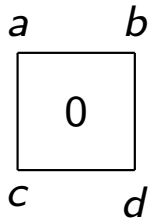
$= \rho(u)\rho(-u)\delta_{ac},$

crossing

$$W\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle| u\right) = W\left(\begin{matrix} b & d \\ a & c \end{matrix}\middle| \lambda - u\right),$$

initial condition

$$W\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle| 0\right) =$$

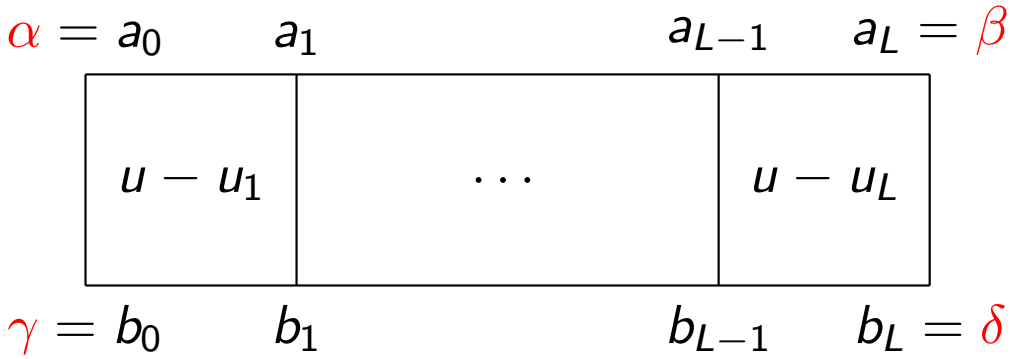


$= \delta_{b,c}\,.$

Face models

With these local weights we define single row operators on $\mathcal{H}^L = \mathrm{span}\{|a_0..a_L\rangle\}$

$$\langle \boldsymbol{a}| T^{\alpha\beta}_{\gamma\delta}(u)|\boldsymbol{b}\rangle =$$



Inhomogeneities $\{u_i \in \mathbb{C}\}_{i=1}^{L}$ parameterize local variations in the interactions.

Periodic boundary conditions in the horizontal direction can be imposed by requiring $\alpha = \beta$, $\gamma = \delta$.

The trace $t(u) = \sum_{\alpha\beta} T^{\alpha\alpha}_{\beta\beta}(u)$ is the **transfer matrix** of an inhomogeneous face model with periodic b.c. in the horizontal direction.

Face models – local operators

Elementary operators $(E^{\alpha}_{\beta})_n$ and $E^{\alpha_{n_1}\dots\alpha_{n_2}}_{\beta_{n_1}\dots\beta_{n_2}}$ on $\mathcal{H}^L = \mathrm{span}\{|a_0..a_L\rangle\}$

$$\langle \boldsymbol{a}| \left(E^{\alpha}_{\beta}\right)_n |\boldsymbol{b}\rangle = \delta_{a_n,\alpha}\ \delta_{b_n,\beta} \prod_{j\neq n} \delta_{a_j b_j}$$

$$= \quad \alpha = a_n \quad a_{n-1} \quad a_{n+1} \quad \beta = b_n$$

$$\langle \boldsymbol{a}| E^{\alpha_{n_1}\dots\alpha_{n_2}}_{\beta_{n_1}\dots\beta_{n_2}} |\boldsymbol{b}\rangle = \prod_{k=n_1}^{n_2} \delta_{a_k,\alpha_k}\ \delta_{b_k,\beta_k} \prod_{j\notin\{n_1\dots n_2\}} \delta_{a_j b_j}$$

$$= \quad \alpha_{n_1} = a_{n_1} \quad \dots \quad \alpha_{n_2} = a_{n_2} \quad a_{n_1-1} \quad a_{n_2+1} \quad \beta_{n_1} = b_{n_1} \quad \dots \quad \beta_{n_2} = b_{n_2}$$

act locally on a single site n or on sequences of $n_2 - n_1 + 1$ of neighbouring sites.

The "inverse problem"

Relation of local spin operators to elements of the Yang-Baxter algebra for vertex models [Kitanine,Maillet,Terras (1999), Göhmann,Korepin (2000)] and face models allowing for a formulation as dynamical vertex models [Levy-Bencheton,Terras (2013/14), Levy-Bencheton,Niccoli,Terras (2016)].

For a general face model (not using the existence of a dynamical R-matrix) the elementary operators can be expressed in terms of the single-row operators $T^{\alpha\beta}_{\gamma\delta}$ introduced above [HF,Westerfeld (2021)]:

The inverse problem

The local operator $\left(E^{\alpha}_{\beta}\right)_n$, $1 \leq n < L$, can be expressed as

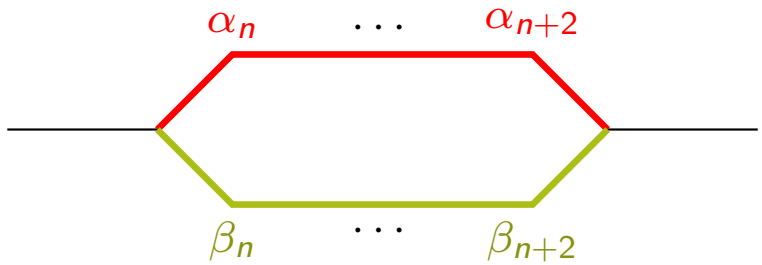
$$\left(E^{\alpha}_{\beta}\right)_n = \prod_{k,\ell=1}^{L} \frac{1}{\rho(u_k - u_\ell)} \left(\prod_{k=1}^{n-1} t(u_k)\right) T_{\alpha\beta}(u_n) T^{\alpha\beta}(u_{n+1}) \left(\prod_{k=n+2}^{L} t(u_k)\right).$$

Here $T_{\alpha\beta} \equiv \sum_{\gamma\delta} T^{\gamma\delta}_{\alpha\beta}$ and $T^{\alpha\beta} \equiv \sum_{\gamma\delta} T^{\alpha\beta}_{\gamma\delta}$.

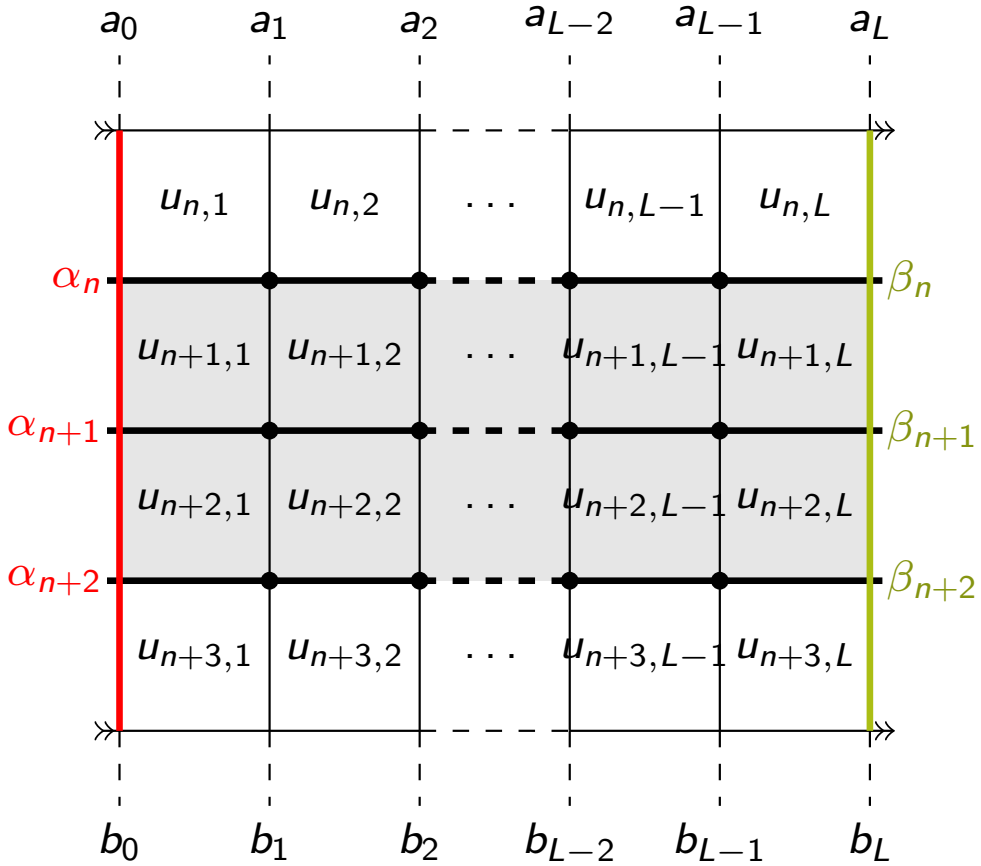
Similarly, $E^{\alpha_{n_1}\dots\alpha_{n_2}}_{\beta_{n_1}\dots\beta_{n_2}} \propto \dots \left(T_{\alpha_{n_1}\beta_{n_1}}(u_{n_1}) T^{\alpha_{n_1}\beta_{n_1}}_{\alpha_{n_1+1}\beta_{n_1+1}}(u_{n_1+1}) \dots T^{\alpha_{n_2}\beta_{n_2}}(u_{n_2+1})\right) \dots$

The "inverse problem"

$$E^{\alpha_n \dots \alpha_{n+2}}_{\beta_n \dots \beta_{n+2}} =$$



$\propto$



$$u_{k,\ell} = u_k - u_\ell$$

The N-site reduced density matrix

Generic correlation functions can be expressed through **reduced density matrices**

$$\frac{1}{\langle\Phi|\Phi\rangle}\,\langle\Phi|\,E^{\alpha_{n_1}\ldots\alpha_{n_2}}_{\beta_{n_1}\ldots\beta_{n_2}}\,|\Phi\rangle\,.$$

Here $|\Phi\rangle \in \mathcal{H}^L$ is any right (left) eigenvector $|\Phi\rangle$ ($\langle\Phi|$) corresponding to a particular eigenvalue $\Lambda(u)$ of the transfer matrix.

Let D_N, ($\underline{\boldsymbol{\alpha}} = (\alpha_{n_1}..\alpha_{n_2})$ with $n_2 = n_1 + N - 2$)),

$$D_N(\lambda_{n_1},\ldots,\lambda_{n_2+1})^{\{\underline{\boldsymbol{\alpha}}\}\{\underline{\boldsymbol{\beta}}\}} \equiv \frac{\langle\Phi|\prod_{k=n_1}^{n_2+1} T^{\alpha_{k-1}\beta_{k-1}}_{\alpha_k\beta_k}(\lambda_k)|\Phi\rangle}{\langle\Phi|\Phi\rangle\prod_{k=n_1}^{n_2+1}\Lambda(\lambda_k)}\,,$$

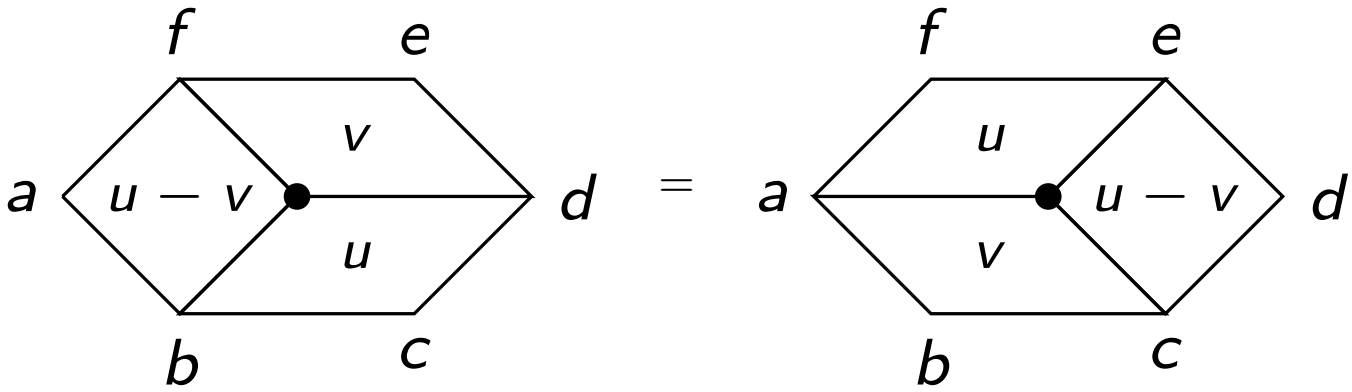
D_N is the density matrix of a generalized problem depending on independent spectral parameters λ_k. Adjusting them one obtains the N-site RDM above

$$D_N(\lambda_{n_1},\ldots,\lambda_{n_2+1})^{\{\underline{\boldsymbol{\alpha}}\}\{\underline{\boldsymbol{\beta}}\}}\Big|_{\lambda_k=u_k,\,k=n_1,\ldots,n_2+1} = \frac{1}{\langle\Phi|\Phi\rangle}\,\langle\Phi|E^{\alpha_{n_1}\ldots\alpha_{n_2}}_{\beta_{n_1}\ldots\beta_{n_2}}|\Phi\rangle\,,$$

NB: Periodic b.c. imply $\alpha_{n_{1,2}} = \beta_{n_{1,2}}$: blocks $[\alpha_{n_1}\alpha_{n_2}]$ of non-zero entries in D_N.

Integrable face models

Integrability, i.e. commuting transfer matrices, is garanteed by (the face version of) the **Yang-Baxter equation** (YBE)



With the **Yang-Baxter operator**

$$\langle \underline{\boldsymbol{\alpha}} | W_i(u) | \underline{\boldsymbol{\beta}} \rangle \equiv W \left(\begin{matrix} \alpha_{i-1} & \beta_i \\ \alpha_i & \alpha_{i+1} \end{matrix} \middle| u \right) \prod_{j \neq i} \delta_{\alpha_j \beta_j} \,,$$

the YBE can be written as

$$W_i(u) W_{i+1}(u+v) W_i(v) = W_{i+1}(v) W_i(u+v) W_{i+1}(u) \,.$$

Integrable models: functional equations for RDMs

A. Consequences of the Yang-Baxter equation

The YBE allows to reorder the arguments of $D_N(\lambda_1, \dots \lambda_N)$:

Intertwining

$$W_i(\lambda_{i+1} - \lambda_i)\, D_N(\lambda_1, .., \lambda_i, \lambda_{i+1}, .., \lambda_N) = D_N(\lambda_1, .., \lambda_{i+i}, \lambda_i, .., \lambda_N)\, W_i(\lambda_{i+1} - \lambda_i)\,.$$

Using crossing and unitary one can show (see also [Morin Duchesne,Hagendorf,Cantini (2020)])

YB-reduction

$$\langle \underline{\alpha}|W_N(\lambda)\, D_N(\lambda_1, .., u, u+\lambda)|\underline{\beta}\rangle$$
$$= \delta_{\alpha_N \alpha_{N-2}} \delta_{\beta_N \beta_{N-2}} \langle \alpha_0 .. \alpha_{N-2}|D_{N-2}(\lambda_1, .., \lambda_{N-2})|\beta_0 .. \beta_{N-2}\rangle \, \frac{\prod_{i=1}^{L} \rho(u-u_i)\rho(u_i-u)}{\Lambda(u)\Lambda(u+\lambda)}$$

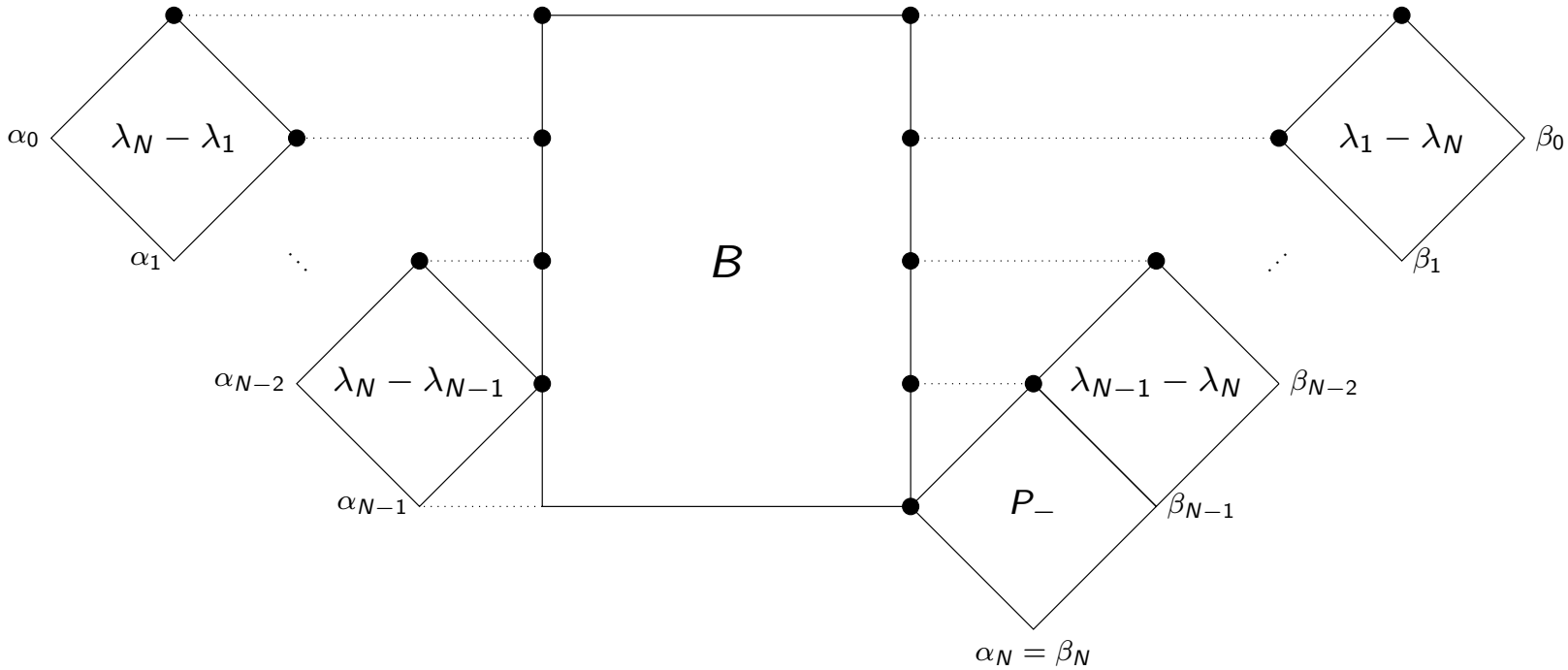
for arbitrary u.
NB: the extra factor on the RHS becomes 1 for $u \in \{u_j\}$ due to the inversion relation for the transfer matrix eigenvalues.

Integrable models: functional equations for RDMs

B. Discrete difference equation

For another functional equation we introduce the linear operator $A_N(\lambda_1, \ldots, \lambda_N)$ on $\mathcal{V}^N = \mathrm{span}\{|\alpha_0..\alpha_N\rangle\}$: the action of A_N on an operator $B \in \mathrm{End}\mathcal{V}^N$ is

$$\left(A_N(\lambda_1, \ldots, \lambda_N)[B]\right)^{\{\underline{\boldsymbol{\alpha}}\}\{\underline{\boldsymbol{\beta}}\}} = \frac{\delta_{\alpha_0\beta_0}\delta_{\alpha_N\beta_N}}{\prod_{i=1}^{N}\rho(\lambda_i-\lambda_N)\rho(\lambda_N-\lambda_i)} \times$$



(The operator P_- is related to the Boltzmann weight at the crossing parameter.)

Integrable models: functional equations for RDMs

Acting with A_N on the N-site RDM we obtain a discrete difference equation

reduced qKZ-type difference equation

The density operator $D_N(\lambda_1, \ldots, \lambda_N)$ is a solution of the functional equation

$$A_N(\lambda_1, \ldots, \lambda_N)[D_N(\lambda_1, \ldots, \lambda_{N-1}, \lambda_N)] = D_N(\lambda_1, \ldots, \lambda_{N-1}, \lambda_N + \lambda)$$

if λ_N is equal to one of the inhomogeneities, i.e. $\lambda_N \in \{u_i\}_{i=1}^{L}$.

For the proof one considers the action of A_N on $D_{N+1}(\lambda_1, \ldots, \lambda_N, \lambda_N + \lambda)$. Performing partial traces over $\alpha_N = \beta_N$ and $\alpha_{N+1} = \beta_{N+1}$, respectively, and using the YBE, unitarity and initial condition for the Boltzmann weights the theorem is obtained (cf. [Aufgebauer,Klümper (2012)] for the six-vertex model) .

NB: For the face models considered here it is straightforward to show that the restriction on λ_N can be dropped for matrix elements of the RDM where α_{N-1} is a leaf node on the adjacency graph.

Integrable models: functional equations for RDMs

C. Recurrence relations

Partial traces

Let $\underline{\boldsymbol{\alpha}} = (\alpha_1 \dots \alpha_N)$, $\underline{\boldsymbol{\beta}} = (\beta_1 \dots \beta_N)$ with $\beta_N = \alpha_N$. Performing the partial trace over α_N one finds

$$\sum_{\alpha_N} \langle \underline{\boldsymbol{\alpha}} | D_N(\lambda_1, .., \lambda_{N-1}, \lambda_N) | \underline{\boldsymbol{\beta}} \rangle = \langle \underline{\boldsymbol{\alpha}'} | D_{N-1}(\lambda_1, .., \lambda_{N-1}) | \underline{\boldsymbol{\beta}'} \rangle$$

where $\underline{\boldsymbol{\alpha}'} = (\alpha_1 \dots \alpha_{N-1})$, $\underline{\boldsymbol{\beta}'} = (\beta_1 \dots \beta_{N-1})$.

Asymptotics

In the limit $\lambda_N \to i\infty$ the N-site RDM is related to the $N-1$-site one by

$$\lim_{\lambda_N \to i\infty} [D_N(\lambda_1, \dots, \lambda_N)]^{\alpha_0 \dots \alpha_N, \beta_0 \dots \beta_N}$$

$$= [D_{N-1}(\lambda_1, \dots, \lambda_{N-1})]^{\alpha_0 \dots \alpha_{N-1}, \beta_0 \dots \beta_{N-1}} \frac{[D_1]^{\alpha_{N-1}\alpha_N, \beta_{N-1}\beta_N}}{\sum_\alpha [D_1]^{\alpha_{N-1}\alpha, \beta_{N-1}\alpha}} .$$

Restricted solid-on-solid (RSOS) models

For the RSOS models the spins take values from $\mathfrak{S} = \{1, ..., r-1\}$ subject to the condition that spins on neighbouring vertices differ by 1.

The weights of the critical model

$$W\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle| u\right) = \delta_{ad}\sqrt{\frac{g_b g_c}{g_a g_d}}\,\rho(u+\lambda) - \delta_{bc}\,\rho(u)$$

satisfy the YBE where

$$\rho(u) = \frac{\sin(u-\lambda)}{\sin(\lambda)}\,, \quad g_x = \frac{\sin(\lambda\, x)}{\sin(\lambda)}$$

with the crossing parameter $\lambda = \pi/r$.

The crossing symmetry acquires additional (gauge) factors

$$W\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle| u\right) = \sqrt{\frac{g_b g_c}{g_a g_d}}\, W\left(\begin{matrix} b & d \\ a & c \end{matrix}\middle| \lambda - u\right)$$

(which also enter in the operator P_- in the definition of A_N.)

Restricted solid-on-solid (RSOS) models

The spectrum of the RSOS models can be decomposed into **topological sectors** labelled by $j \in \{0, 1/2, \ldots, (r-2)/2\}$ with quantum dimension

$$d_q(j) = \frac{\sin((2j+1)\lambda)}{\sin\lambda} \,.$$

determined by the asymptotics of the transfer matrix eigenvalues [Klümper,Pearce (1992)].

Hamiltonian limit: expanding the homogeneous transfer matrix ($u_k \equiv 0$) around the shift point $u = 0$ to first order one obtains the Temperley-Lieb Hamiltonian of the one-dimensional quantum RSOS model [Bazhanov,Reshetikhin (1989)]

$$\mathcal{H} = \frac{\lambda}{4\pi \sin\lambda} \sum_{j=1}^{L} e_j \,, \qquad \langle \mathbf{a}|e_j|\mathbf{b}\rangle = W\left(\begin{matrix} a_{j-1} & a_j \\ b_j & a_{j+1} \end{matrix} \middle| \lambda\right) \prod_{k\neq j} \delta_{a_k b_k} \,.$$

The ground states of $\pm\mathcal{H}$ are in the sectors with quantum dimension $d_q(j) = 1$, i.e. $j = 0$ and, for r even, $j = (r-2)/2$.

NB: face models have attracted considerable interest recently as they can be used to describe the collective behaviour of non-Abelian anyons in topological quantum liquids (e.g. quantum Hall edge states). [Feiguin *et al.* (2007); ...]

The single site density matrix

Heisenberg model:

$$D_1(\lambda_1) = f_{1,\varnothing,\varnothing}(\lambda_1) \equiv \frac{1}{2}\,\mathbf{1}\,.$$

RSOS models:
Only the diagonal matrix elements of $D_1(\lambda_1)$ are non-zero (the D_1-blocks $[a, a+1]$ allowed by the adjacency rules are one-dimensional). They can be shown to be independent of the spectral parameter λ_1.

Moreover, for states in the topological sectors with $d_q(j) = 1$ they are found to be independent of the inhomogeneities u_k and the state chosen.

Local height probabilities of the critical model [Andrews,Baxter,Forrester (1984)]:

$$P(a) = (2/r)\sin^2(a\lambda)$$

$\succ$ the elements of D_1 are given as $(1 \le a \le r-2)$

$$\langle a, a+1|D_1(\lambda_1)|a, a+1\rangle = \frac{\sin(a\lambda)\sin((a+1)\lambda)}{r\,\cos\lambda}\,.$$

The two-site density matrix

Heisenberg model:

$$D_2(\lambda_1,\lambda_2) = f_{2,\varnothing,\varnothing}(\lambda_1,\lambda_2) + \omega(\lambda_1,\lambda_2)\, f_{2,\{1\},\{2\}}(\lambda_1,\lambda_2)$$

RSOS models in the topological sectors with $d_q(j) = 1$:

$$D_2(\lambda_1,\lambda_2) = A_2 + B_2\, f(\lambda_1,\lambda_2)\,.$$

- The non-zero elements of the diagonal matrix A_2 are obtained from the asymptotic behaviour of D_2 for $\lambda_2 \to i\infty$:

$$\langle abc|A_2|ab'c\rangle = \delta_{bb'}\,\frac{\sin a\lambda\,\sin c\lambda}{2r\cos^2\lambda}\,, \quad |a-b| = |b-c| = 1\,.$$

- Intertwining relation and partial traces reduces the number of free parameters in B_2. Fixing the normalization of $f(\lambda_1,\lambda_2)$ they follow from the rqKZ equation, e.g. in the block $[aa] = \mathrm{span}\{|a,a\mp 1,a\rangle\}$ $(1 < a < r-1)$:

$$B_2^{[aa]} = \frac{g_a}{g_1 g_3}\begin{pmatrix} g_{a-2} & 2\cos\lambda\sqrt{g_{a-1}\,g_{a+1}} \\ 2\cos\lambda\sqrt{g_{a-1}\,g_{a+1}} & g_{a+2}\end{pmatrix}\,, \quad g_a = \frac{\sin a\lambda}{\sin\lambda}\,.$$

The three-site reduced density matrices

The $N = 3$ density matrices of the RSOS models for states from the topological sectors $d_q(j) = 1$ can be factorized similar as those of the Heisenberg chain [Boos *et al.* (2006)]:

Three sites:

$$D_3(\lambda_1, \lambda_2, \lambda_3) = A_3 + B_{3|12}(\{\lambda_j\})\, f(\lambda_1, \lambda_2) + B_{3|13}(\{\lambda_j\})\, f(\lambda_1, \lambda_3) + B_{3|23}(\{\lambda_j\})\, f(\lambda_2, \lambda_3)\,,$$

The "algebraic part" is of the form ($\lambda_{ij} = \lambda_i - \lambda_j$)

$$A_3 \text{ constant, diagonal:} \qquad \langle abcd|A_3|abcd\rangle = \frac{\sin a\lambda \, \sin d\lambda}{4r \cos^3 \lambda}\,,$$

$$B_{3|12} = f^1_{12} + (\cot\lambda_{13} - \cot\lambda_{23})\, f^2 + (1 + \cot\lambda_{13}\, \cot\lambda_{23})\, f^4\,,$$

$$B_{3|23} = f^1_{23} + (\cot\lambda_{12} - \cot\lambda_{13})\, f^2 + (1 + \cot\lambda_{12}\, \cot\lambda_{13})\, f^4\,,$$

$$B_{3|13} = f^1_{13} + (\cot\lambda_{23} - \cot\lambda_{12})\, f^2 + (1 - \cot\lambda_{23}\, \cot\lambda_{12})\, f^4\,,$$

where $f^1_{ij} = \left(f^1_{ij}\right)^\top$, $f^2 = -\left(f^2\right)^\top$ and $f^4 = \left(f^4\right)^\top$ are real matrices.

The three-site reduced density matrices

The asymptotic relations between $D_3(\lambda_1, \lambda_2, \lambda_3)$ and D_2 for $\lambda_j \to i\infty$ allow to determine the structure constants f_{ij}^1 and f^2, e.g.

$$\left[f_{12}^1\right]^{\alpha_0\ldots\alpha_2\alpha_3,\alpha_0\ldots\beta_2\alpha_3} = \delta_{\alpha_2\beta_2}\,[B_2]^{\alpha_0\ldots\alpha_2,\alpha_0\ldots\alpha_2}\,\frac{D_1^{[\alpha_2\alpha_3]}}{\sum_\alpha D_1^{[\alpha_2\alpha]}}$$

$$= \delta_{\alpha_2\beta_2}\,[B_2]^{\alpha_0\ldots\alpha_2,\alpha_0\ldots\alpha_2}\,\frac{\sin\alpha_3\lambda}{2\sin\alpha_2\lambda\,\cos\lambda}\,.$$

Relations between the elements of f^4 follow from the intertwining and partial traces. The remaining ones can be determined using rqKZ equation, e.g.

$$r = 4: \quad f^4 \equiv 0\,,$$

$$r = 5: \quad \langle 1234|f^4|1234\rangle = -\frac{1}{4}(3\sqrt{5}-5)$$

$$r = 6: \quad \langle 1234|f^4|1234\rangle = -\frac{1}{4}$$

$$r = 7: \quad \langle 1234|f^4|1234\rangle = -\frac{2-2\cos(2\pi/7)}{2+4\cos(2\pi/7)}$$

The $N > 3$-site reduced density matrices

Four sites: For $N > 3$ products of the nearest neighbour function f appear in the matrix elements of the generalized density matrices, e.g.

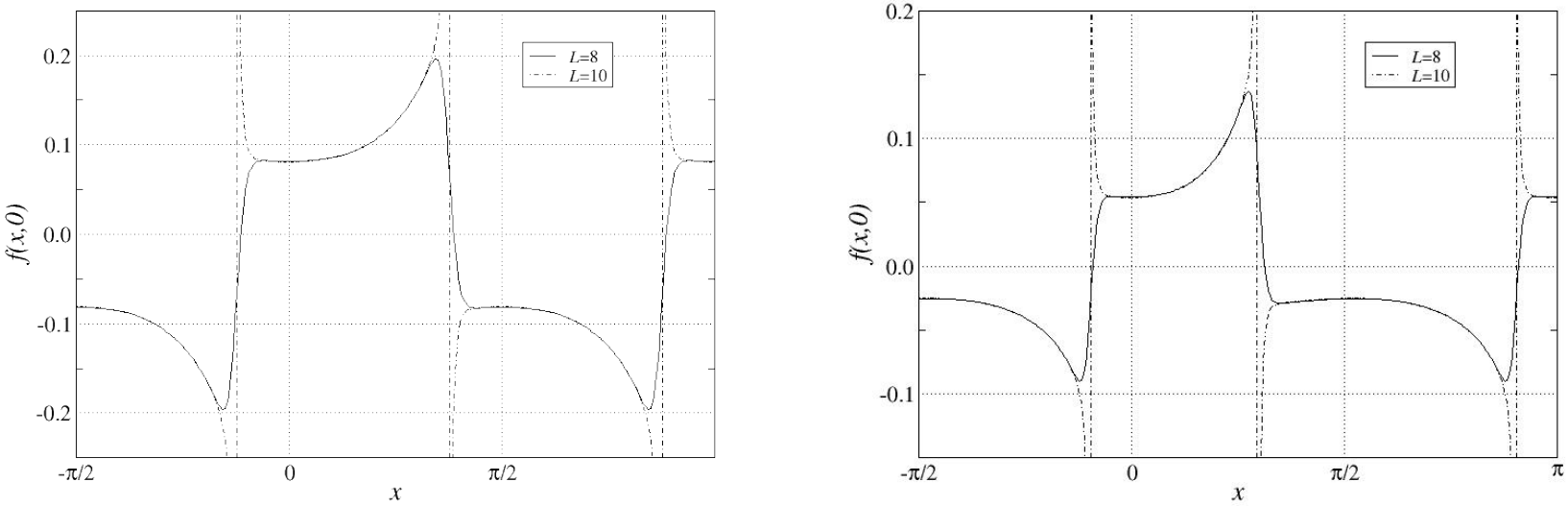
$$\begin{aligned} D_4^{[ab]}(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = & f_0(\{\lambda_j\}) \\ & + f_{12}(\{\lambda_j\})\, f(\lambda_1,\lambda_2) + f_{13}(\{\lambda_j\})\, f(\lambda_1,\lambda_3) \\ & + f_{14}(\{\lambda_j\})\, f(\lambda_1,\lambda_4) + f_{23}(\{\lambda_j\})\, f(\lambda_2,\lambda_3) \\ & + f_{24}(\{\lambda_j\})\, f(\lambda_2,\lambda_4) + f_{34}(\{\lambda_j\})\, f(\lambda_3,\lambda_4) \\ & + f_{12;34}(\{\lambda_j\})\, f(\lambda_1,\lambda_2)\, f(\lambda_3,\lambda_4) \\ & + f_{13;24}(\{\lambda_j\})\, f(\lambda_1,\lambda_3)\, f(\lambda_2,\lambda_4) \\ & + f_{14;23}(\{\lambda_j\})\, f(\lambda_1,\lambda_4)\, f(\lambda_2,\lambda_3)\,. \end{aligned}$$

The nearest neighbour function f (which is already determined in D_2) contains all the information on the specific model (in particular system size and inhomogeneities $\{u_k\}$ and the state of the system. Hence one can build a system of *linear* equations fixing the structure functions $f_{i_1,j_1;i_2,j_2;\ldots}(\{\lambda_j\})$ for a given set of spectral parameters $\{\lambda_j\}$.

The nearest-neighbour function $f(u, v)$

$$D_2(\lambda_1, \lambda_2) = A_2 + B_2\, f(\lambda_1, \lambda_2)\,.$$

The two-site function $f(u, v) = f(v, u)$ can be obtained from the explicit form of D_2 in terms of the operators $T^{\alpha\beta}_{\gamma\delta}$ (for small systems) or by solving the functional equations:



$f(x, 0)$ from D_2 for $r = 4$ and 5 (ground state of the homogeneous model)

The nearest-neighbour function $f(u,v)$ $(r=4)$

$r=4$ ($\sim$ Ising): the YB reduction $D_3 \to D_1(=const.)$ for the homogeneous model, $u_k \equiv 0$, can be solved using the exact expression for $\Lambda(u)\Lambda(u+\lambda)$ with the transfer matrix eigenvalues $\Lambda(u)$ of the Ising model [O'Brien,Pearce,Warnaar (1996)]

$$\sin(2(u-v))f(u,v)+\cos(2(u-v))f(u,v+\frac{\pi}{4}) = \frac{1}{2(1+y\tan^L(2u))}-\frac{1}{4}\,,$$

valid for all $u,v\in\mathbb{C}$. $y=\pm 1$ is the eigenvalue of the height reflection operator.

Using Fourier methods one obtains $f(u,v)$ for arbitrary L.

In the thermodynamic limit $L\to\infty$ one finds $f(u,v)=f(u-v)$ and obtains two explicit solutions which are regular in a finite interval around $x=0$:

$$f(x)=\frac{1}{\pi\sin 2x}\begin{cases} x & \text{for } -\frac{\pi}{8}<x<\frac{3\pi}{8} \\ -x & \text{for } -\frac{3\pi}{8}<x<\frac{\pi}{8}\end{cases}$$

These solutions correspond to the ground states of $\pm\mathcal{H}$ resp.

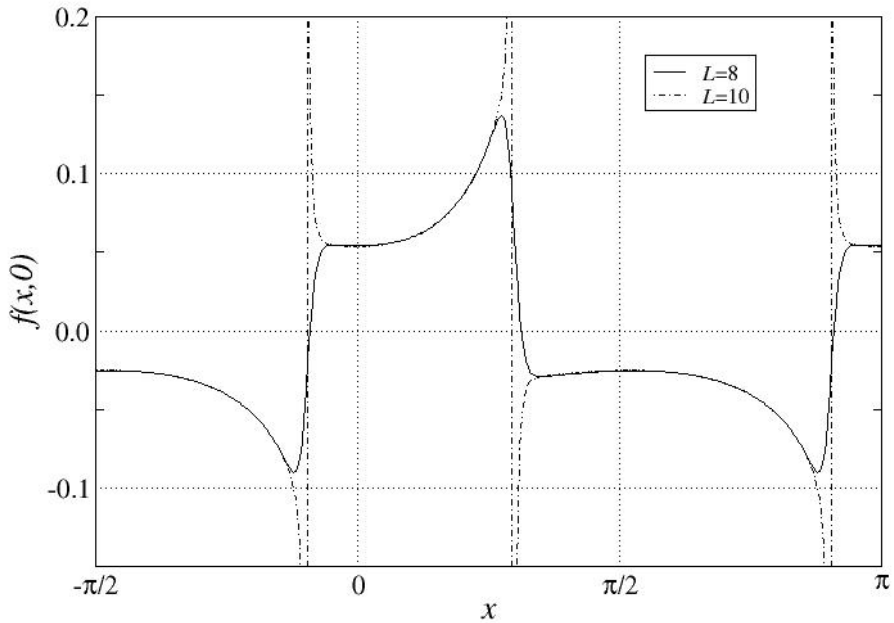
The nearest-neighbour function $f(u,v)$ $(r \geq 5)$

For $r \geq 5$ the function f can be determined either (a) from the from the explicit form of D_2 in terms of the operators $T^{\alpha\beta}_{\gamma\delta}$ (for small systems) or (b) by solving the functional equation

$$f(u, v+\lambda) = \frac{\sin^2\lambda \, \tan^2\lambda \, (1+2\cos(2\lambda))}{r\,\phi_r(u-v)} - \frac{\phi_r(u-v-\lambda)}{\phi_r(u-v)} f(u,v)\,,$$

(valid at $v \in \{u_k\}$) where $\phi_r(x) = \cos 2x - \cos 2\lambda$.

finite size data for $f(x,0)$
(in the ground state of the
$r = 5$ homogeneous model)



The nearest-neighbour function $f(u,v)$

In the ground state $f(u,v)$ is an analytical function of v in the strips $\mathcal{S}_1 = \{v \in \mathbb{C} : -\lambda/2 \lesssim \mathrm{Re}(v) \lesssim 3\lambda/2\}$ (the physical strip in regime III/IV) and $\mathcal{S}_2 = \{v \in \mathbb{C} : -\pi + 3\lambda/2 \lesssim \mathrm{Re}(v) \lesssim -\lambda/2\}$
and vanishes for $\lambda \to i\infty$. Assuming that the functional eq. holds throughout these strips for $L \to \infty$ it can be solved using Fourier methods. Based on our results for $r \leq 9$ we find that $f(u,v)$ depends on $x = u - v$ only and we conjecture for $x \in \mathcal{S}_1$

$$f_r(x) = \frac{\phi_r(x)}{\sin rx} \begin{cases} \left(\frac{\tan\lambda}{\pi}\, x + \sum_{j=1}^{(r-4)/2} a_j \sin 2jx \right) & \text{for } r \geq 4 \text{ even} \\ \left(\sum_{j=0}^{(r-5)/2} a_j \sin(2j+1)x \right) & \text{for } r \geq 5 \text{ odd} \end{cases} .$$

(regular in $-2\lambda < x < 2\lambda$!)
The a_j can be determined from the functional eq. For $r = 5$ we obtain

$$f_{r=5}(u,v) = (3-\sqrt{5}) \frac{\sin(u-v-\lambda)\sin(u-v)\sin(u-v+\lambda)}{\sin 5(u-v)}$$

The nearest-neighbour function $f(u, v)$

Similarly, we can solve the functional equation for $x \in \mathcal{S}_2$ and $x - \pi \in \mathcal{S}_2$.
For $r = 4, 5$ we have obtained explicit expressions for these other branches in the thermodynamic limit:

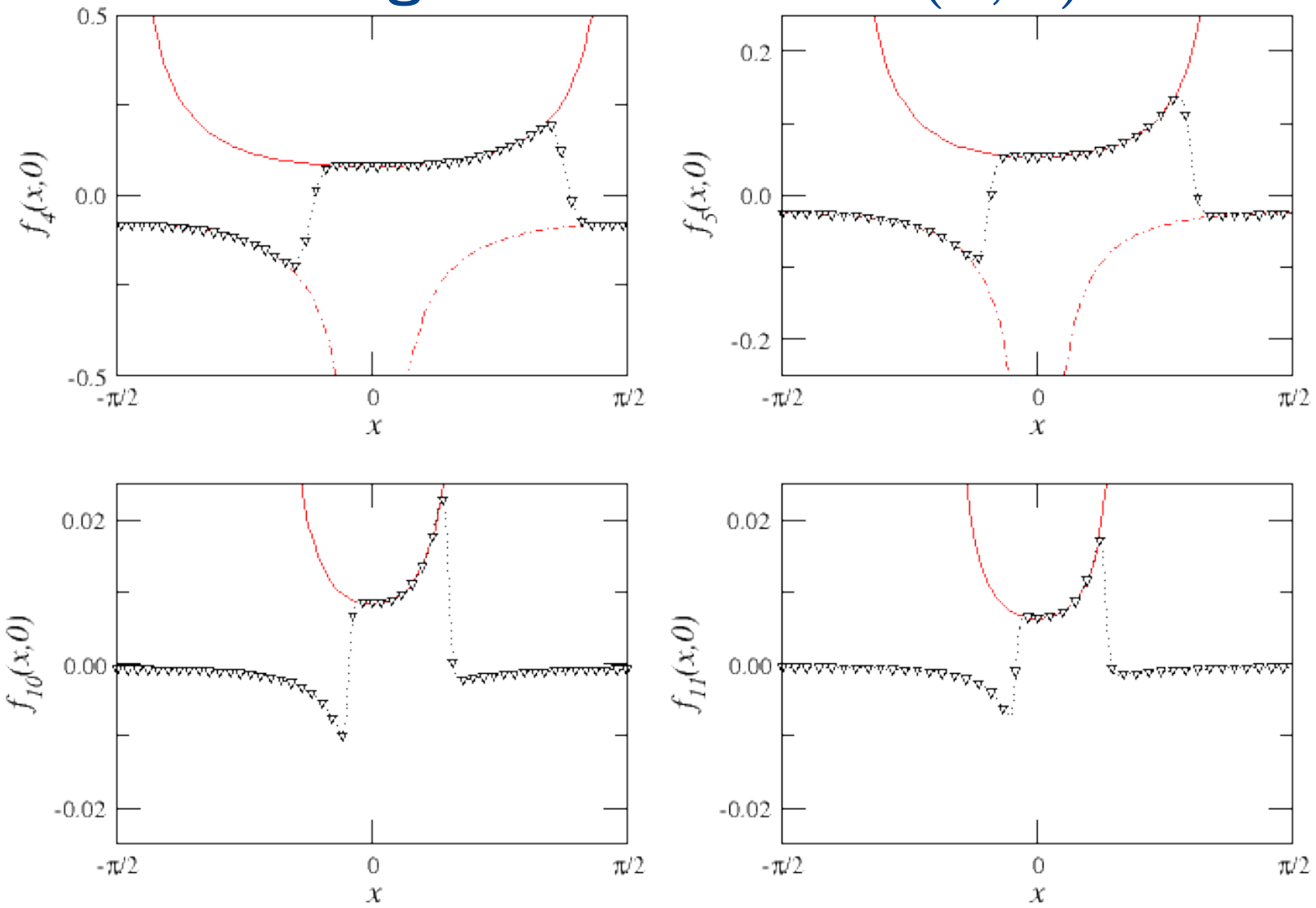
$$\widetilde{f}_{4,\pm}(x) = \left(x \pm \frac{\pi}{2}\right) \frac{1}{2\pi \sin 2x} ,$$
$$\widetilde{f}_{5,\pm}(x) = \frac{\phi_5(x)\ (\sin x \pm \sin 2\lambda)}{4 \cos^2 \lambda\ \sin 5x} .$$

Note that the solutions $\widetilde{f}_{r,\pm}(x)$ have period 2π and are regular in the intervals $-\pi < x < 0$ and $0 < x < \pi$, respectively.

In the thermodynamic limit we find

$$f(x) = \begin{cases} \widetilde{f}_{r,+}(x) & \text{for } -\pi/2 < x < -\lambda/2 \\ f_r(x) & \text{for } -\lambda/2 < x < 3\lambda/2 \\ \widetilde{f}_{r,-}(x) & \text{for } \ 3\lambda/2 < x < \pi/2 \end{cases}$$

The nearest-neighbour function $f(u, v)$



nearest neighbour function $f_r(x)$ for the ground state of the homogeneous model for small systems (from explicit form of the D_2 in terms of Boltzmann weights) and the different branches in the thermodynamic limit (red).

Multi-point local height probabilities

Given these "factorized" expressions the physical correlation functions of the (homogeneous) RSOS models are obtained in the limit $\lambda_j \to 0$, $j = 1..N$. In that limit the elements of the density matrices can be expressed in terms of the nearest neighbour correlation function $f(u, v)$ and its derivatives at $(u, v) = (0, 0)$.

In the simplest case, $N = 2$

$$\langle abc|D_2(0,0)|def\rangle = \langle abc|A_2|def\rangle + \langle abc|B_2|def\rangle\, f(0,0)$$

In this formulation the energy the quantum RSOS model in a given state is

$$\frac{E}{L} = \frac{\lambda}{4\pi \sin\lambda} \mathrm{tr}\left(D_2(0,0)\, e_j\right) = \frac{\lambda}{2\pi}\left(\frac{1}{\sin 2\lambda} + \frac{r\cos\lambda}{\sin^3\lambda}\, f(0,0)\right) .$$

For the $L \to \infty$ ground state this agrees with the energy density [Bazhanov,Reshetikhin 1989]

$$\frac{E}{L} \to \frac{\lambda}{2\pi}\cot\lambda - \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x\, \frac{\sinh x\, \sinh(r-3)x}{\sinh 2x\, \sinh rx} .$$

Multi-point local height probabilities

four neighbouring points: probability for the sequence (1234) of heights

$$\langle 1234|D_3(0,0,0)|1234\rangle = \frac{\sin\lambda \sin 4\lambda}{4r\cos^3\lambda} + \left(\frac{3\sin\lambda}{\sin 3\lambda} - 3 + 2f^4\right) f_r(0,0)$$
$$+ f^4\ \partial_u\partial_v f_r(u,v)|_{u=v=0} - \frac{1}{2} f^4\, \partial_u^2\ f_r(u,v)|_{u=v=0}$$

In the thermodynamic limit:

$$r=5: \quad P_5(1234) = \frac{1}{10}\,\frac{25-11\sqrt{5}}{11+5\sqrt{5}} = 0.001818062...$$
$$r=6: \quad P_6(1234) = \frac{1}{6^3}\left(2-\frac{3\sqrt{3}}{\pi}\right) = 0.001601913...$$
$$r=7: \quad P_7(1234) = \quad ... \quad = 0.001212539...$$

Summary & Outlook

- local operators in generic face models have been expressed in terms of generalized RDMs (inverse problem)
- discrete difference equations of reduced quantum Knizhnik-Zamolodchikov type for RDMs for Yang-Baxter integrable face models and, moreover certain reduction relations relating RDMs of different order.
- RDMs of RSOS models have factorization properties similar to those of the Heisenberg spin chain. In certain topological sectors the RDMs are given in terms of a single function $f(u, v)$ containing all information about the model (L, inhomogeneities) and state

Future directions:

- recursive definition of algebraic part of D_N for larger N
- factorization properties of correlation functions in topological sectors with $d_q \neq 1$ (RSOS), for the elliptical case, and in face models based on higher rank symmetries