Quantum Information meets Mathematical Physics

Entanglement, recoupling and entropy

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SRS
SwissMAP Research
Station in Les Diablerets

Quantum information

25 February 2024

Date/Time
25 February 2024 - 1 March 2024

Organized by
Matthias Christandl (University of Copenhagen), Renato Renner (ETH Zurich)

Event page & registration

Description
The concept of information permeates both our everyday life as well as our understanding of the universe. With quantum theory a
information needs a fundamental revision. This workshop will take a fresh look at this maturing subject bringing together thinkers
information, not quantum computation – although it will of course touch it as well. Focus will also be on theory, although the expe
theory.

Location
SwissMAP Research Station, Les Diablerets, Switzerland


Verifiable Quantum Advantage
Assumption-based
Oracle-based

Quantum@Math.Copenhagen

Centre for the Mathematics of Quantum Theory

UNIVERSITY OF COPENHAGEN

- 7 faculty
- 30 students and postdocs

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Innovation Fund Denmark

erc

- Embedded in a strong collaborative environment
  - Quantum for Life Center (2021-)

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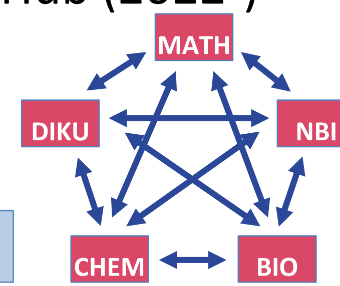
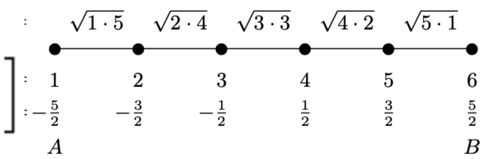
  - UCPH Quantum Hub (2022-)

HUM JUR SAMF SUND TEOL



Construct a quantum wire

$$H_G = \sum_{(n,n+1)\in E(G)} \frac{J_n}{2}\left[\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y\right]$$



https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.92.187902

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PHYSICAL REVIEW LETTERS

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Perfect State Transfer in Quantum Spin Networks

Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl
Phys. Rev. Lett. **92**, 187902 – Published 4 May 2004

Article References Citing A

ABSTRACT

We propose a class c
fixed period of time. I
networks do not requ
networks of identical
distance for hypercul
qubits, then perfect s

PHYSICAL REVIEW LETTERS

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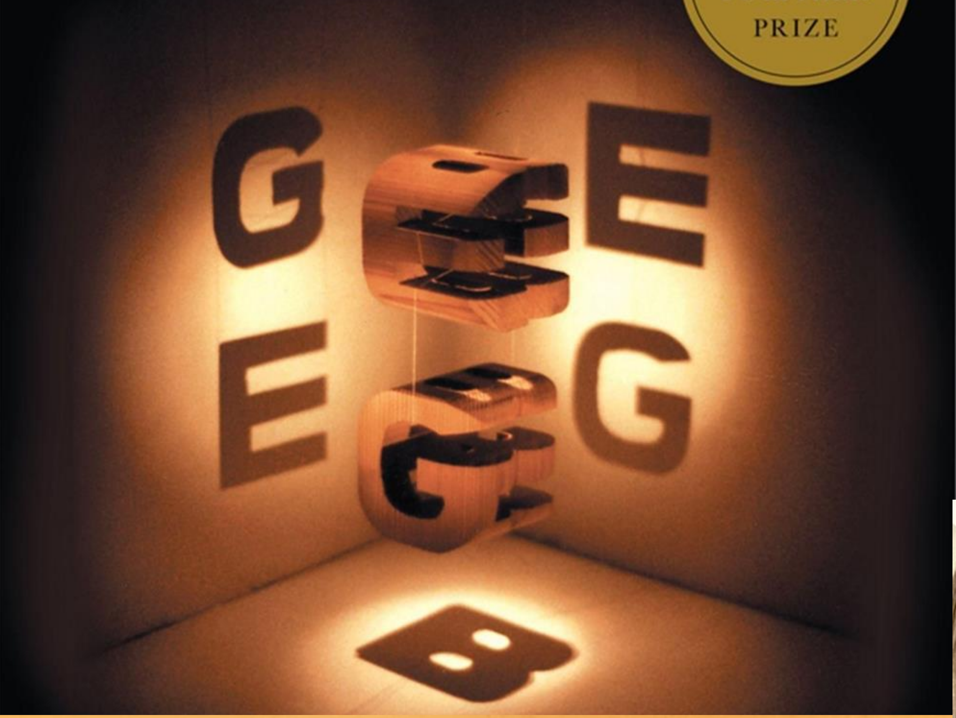
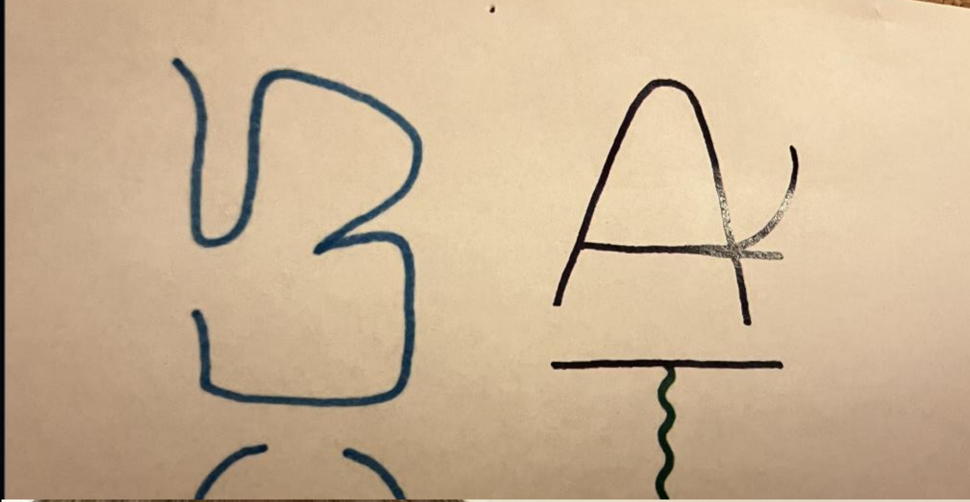
Mirror Inversion of Quantum States in Linear Registers

Claudio Albanese, Matthias Christandl, Nilanjana Datta, and Artur Ekert
Phys. Rev. Lett. **93**, 230502 – Published 30 November 2004

connects to
- integrability &special functions (Vinet)
- graph theory (Godsil)
- quantum random walk (Childs)

Outline

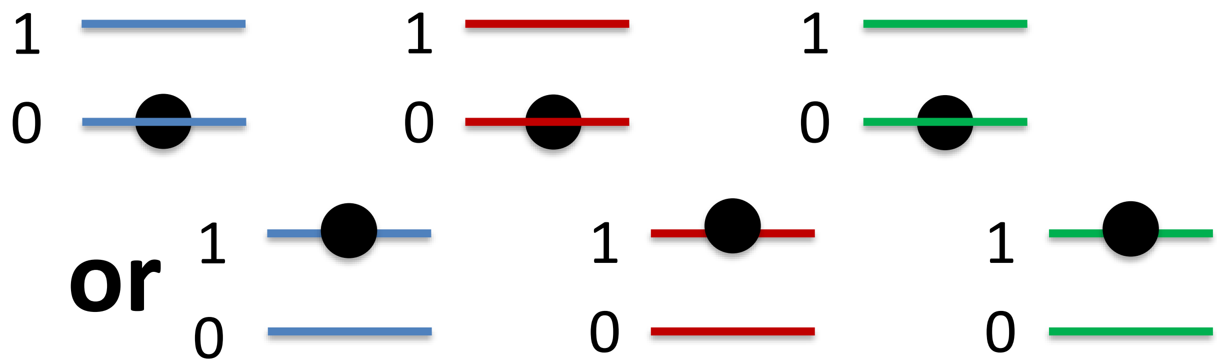
- Entanglement
  - Ch., Vrana and Zuiddam, JAMS 2023
  - https://arxiv.org/abs/1709.07851
- Recoupling and Quantum Entropy
  - Ch., Sahinoglu and Walter, AHP 2018
  - https://arxiv.org/abs/1210.0463


PRIZE


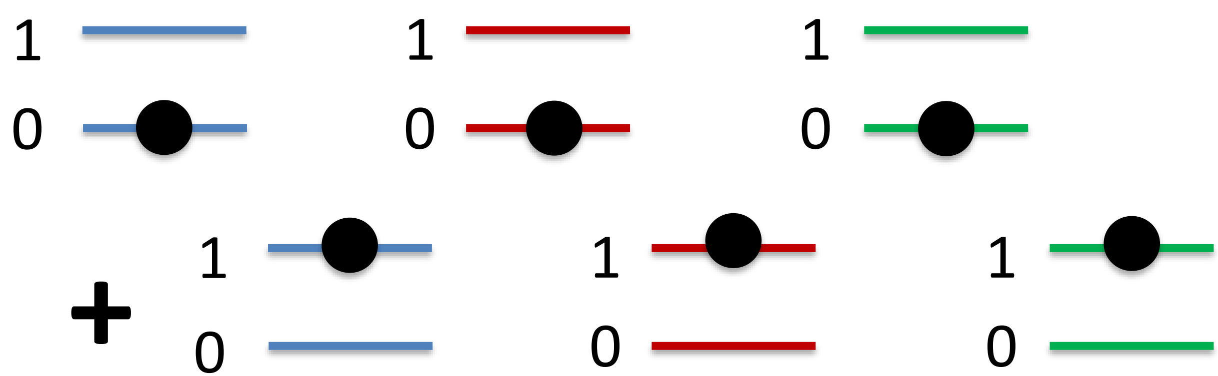
Entanglement

Quantum states

State of a classical system (3 bits)



State of a quantum system (3 qubits)



$$e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

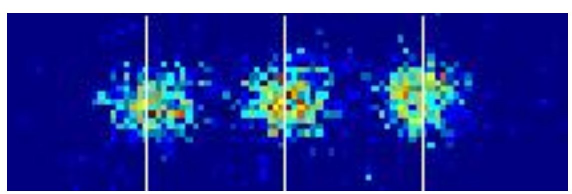
$$e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

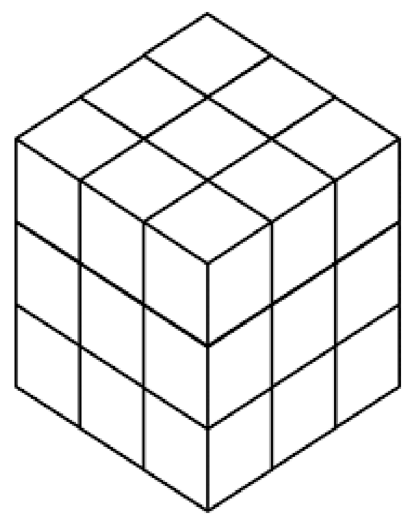
$$t = e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1$$

Quantum state=tensor

$$t \in \mathbf{C}^d \otimes \mathbf{C}^d \otimes \mathbf{C}^d$$

$$t = \sum_{i,j,k=1}^{d} t_{ijk} e_i \otimes e_j \otimes e_k$$

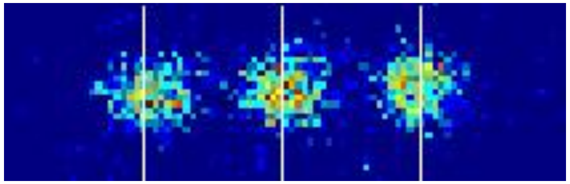


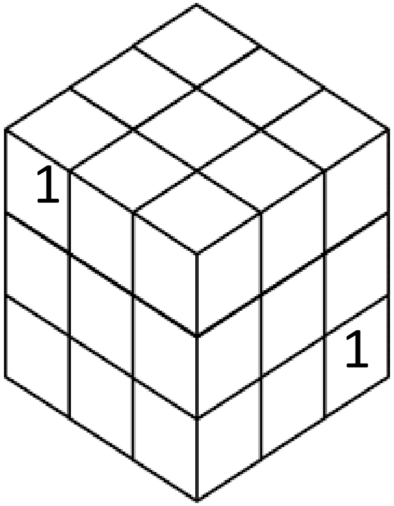


GHZ state = unit tensor

Greenberger-Horne-Zeilinger

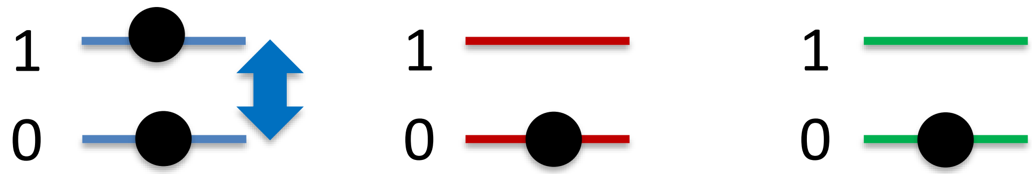
$$\langle r \rangle = \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i$$



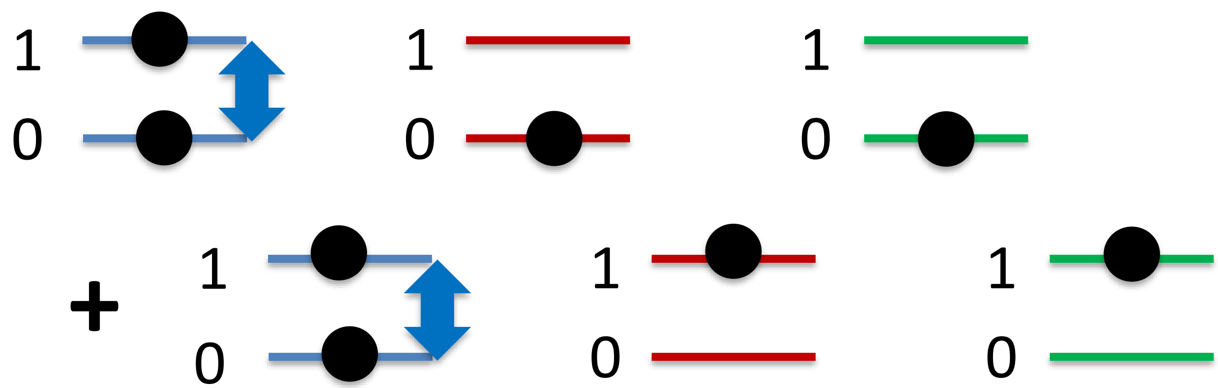


Local operations

Local trans- formation: Flip first bit



Local trans- formation: Flip first qubit



$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

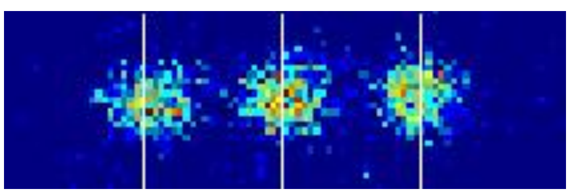
$$t = e_0 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_1$$

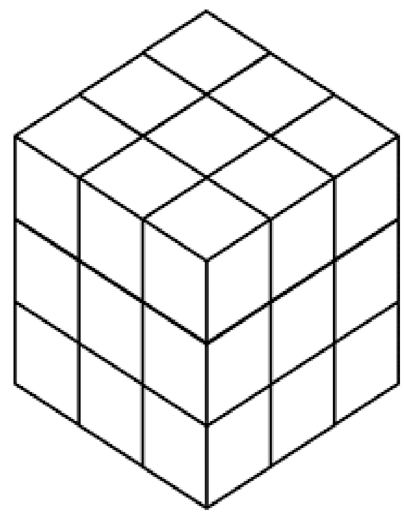
stochastic

Local operations=restrictions

$$t \geq t' \text{ if } (a \otimes b \otimes c)\; t = t'$$

$$\text{for some matrices } a, b, c$$





Linear combination of slices

3 qubits

Greenberger-Horne-Zeilinger
GHZ-state

$$e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1$$

W-state

$$\approx e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_0$$

Einstein-Podolsky-Rosen
(EPR)-state

$$e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_0$$

$$e_0 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_1$$

$$e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_0 \otimes e_1$$

$$e_0 \otimes e_0 \otimes e_0$$

unentangled state

free operations

Resource theory of tensors

valuable resource

- Restriction $\quad t \geq t' \text{ if } (a \otimes b \otimes c)\ t = t'$ for some matrices a, b, c
- Unit $\quad \langle r \rangle = \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i$
- Rank $\quad R(t) = \min\{r : \langle r \rangle \geq t\}$
- Subrank $\quad Q(t) = \max\{r : t \geq \langle r \rangle\}$

Restriction

$$t \geq t' \text{ if } (a \otimes b \otimes c)\ t = t'$$

for some matrices a, b, c

$$t \cong t' \text{ if } t \geq t' \text{ and } t' \geq t$$

$$\text{iff } (a \otimes b \otimes c)\ t = t'$$

for invertible a, b, c

$$\text{iff } G.t = G.t'$$

if concise, i.e. cannot
be embedded
in smaller dimensions

$G = GL(d) \times GL(d) \times GL(d)$

Deciding restriction $\Longleftrightarrow$ Classifying orbits and their relations

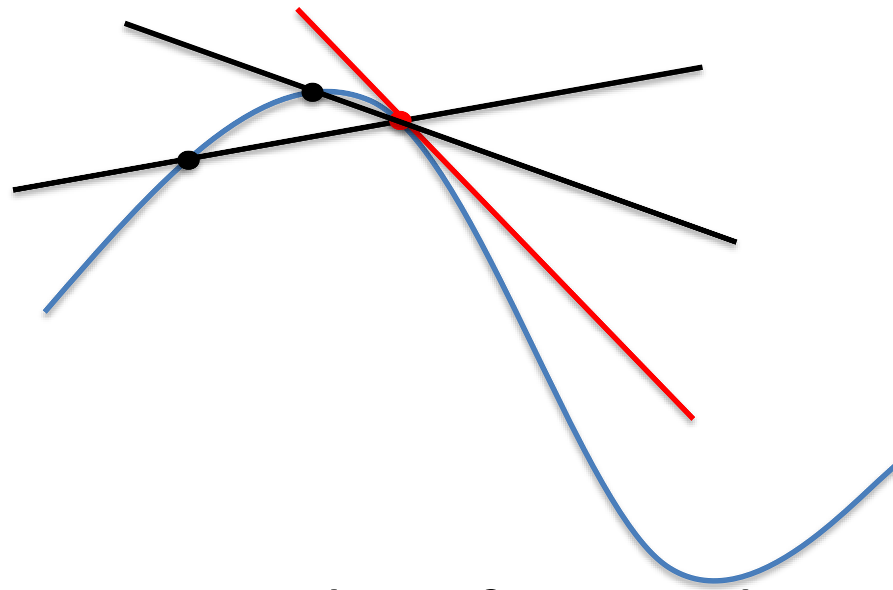
GHZ state

Degeneration

W state

$$(e_0 + \epsilon e_1)^{\otimes 3} - e_0^{\otimes 3}$$
$$= \epsilon(e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_0) + O(\epsilon^2)$$

$$t \trianglerighteq t' \text{ if } t_\epsilon \underset{\epsilon \mapsto 0}{\longrightarrow} t', t \geq t_\epsilon$$



Deciding degeneration $\Leftrightarrow$ Classifying orbit closures and their relations

Deciding degeneration

- Orbit closures are G-invariant algebraic varieties

$$t \not\trianglerighteq t' \text{ iff there exists}$$

$$G-\text{covariant polynomial } f : f(t) \neq f(t')$$

$$f(t) = 0, \text{ but } f(t') \neq 0$$

- Example: $e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1$

$$\downarrow \quad \cancel{\uparrow}$$

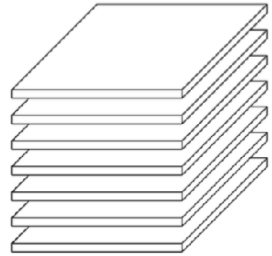
f=Cayley hyperdeterminant

$$\approx \; e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_0$$

Local spectra (moment map)

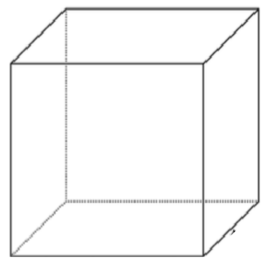
$$t'_A \in \mathbf{C}^d \otimes (\mathbf{C}^d \otimes \mathbf{C}^d)$$

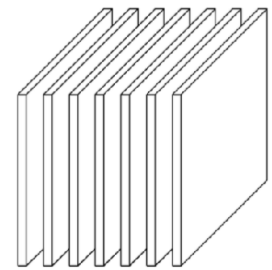
$$\lambda_A = \text{ singular values } (t'_A)^2$$



normalised

$$t' \in \mathbf{C}^d \otimes \mathbf{C}^d \otimes \mathbf{C}^d$$





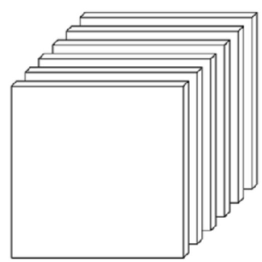
$$t'_B \in \ldots$$

$$\lambda_B = \text{ singular values } (t'_B)^2$$

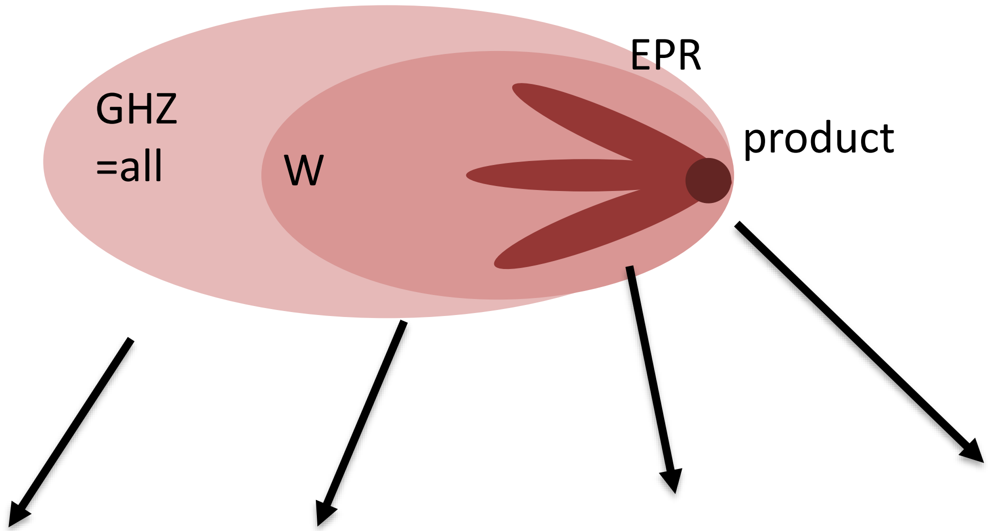
ordered probability distribution
=spectrum of reduced density operator

$$t'_C \in (\mathbf{C}^d \otimes \mathbf{C}^d) \otimes \mathbf{C}^d$$

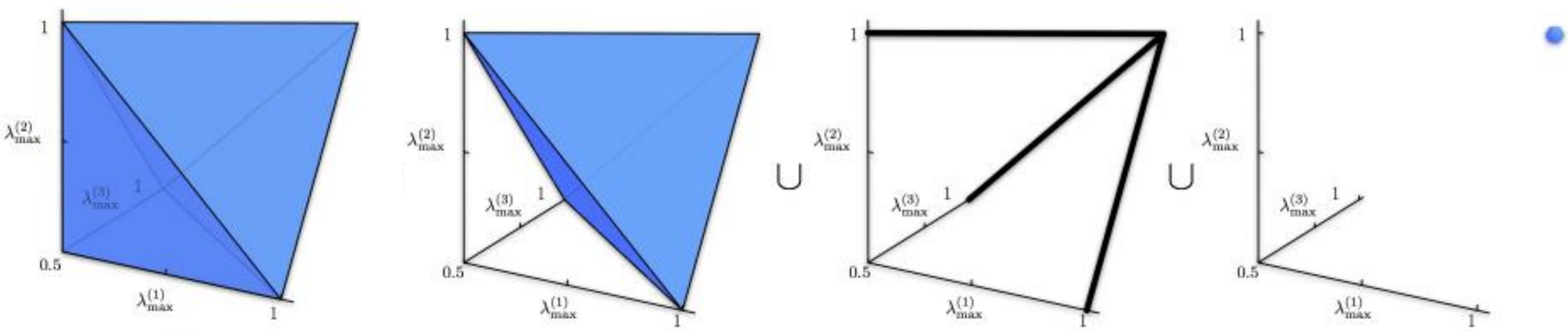
$$\lambda_C = \text{ singular values } (t'_C)^2$$



Entanglement polytopes



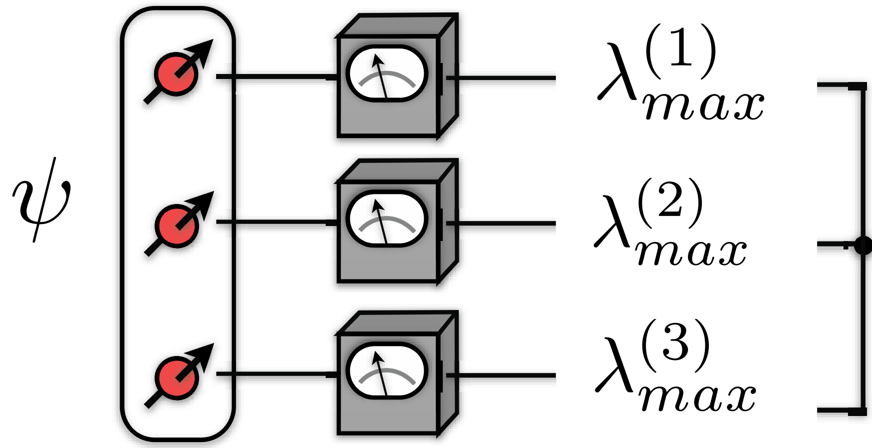
marginal
polytope

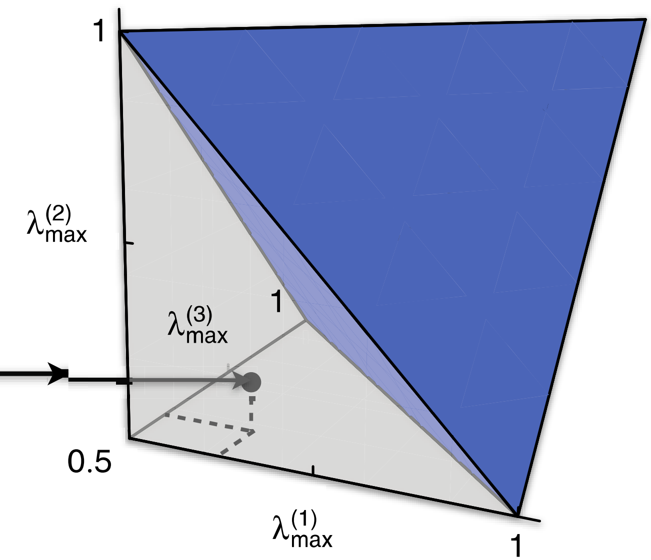


Ch-Mitchison, Klyachko,
Daftuar-Hayden (2004)
based in part on Kirwan

Walter-Doran-Gross-Ch,
Sawicki-Oszmaniec-Kus (2010) based on Brion

Experimental Detection





- if measured value
  - not in W-polytope
  - Then must be in GHZ-class!
- easy test for entanglement!

A little more partial information?

- Orbit closures are G-invariant algebraic varieties

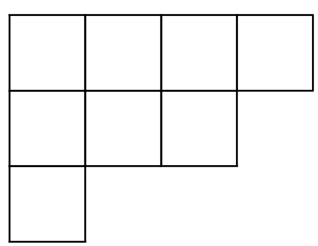
$$t \ntrianglerighteq t' \text{ iff there exists}$$

$$G-\text{covariant polynomial } f : f(t) \neq f(t')$$

$$f(t) = 0, \text{ but } f(t') \neq 0$$

- f's come in types indexed by 3 Young diagrams

$$\lambda_A =$$



boxes=degree

Weyl's construction

S_n acts $GL(d)$ acts

- Schur-Weyl duality

$$(\mathbf{C}^d)^{\otimes n} \cong \bigoplus_{\lambda} [\lambda] \otimes V_\lambda$$

- P_{λ_A} orthogonal projector onto λ_A component

$$\underbrace{(P_{\lambda_A} \otimes P_{\lambda_B} \otimes P_{\lambda_C})}_{=:P_\lambda} t^{\otimes n}$$

$$= \left(\sum_i v_i v_i^* \right) t^{\otimes n} = \sum_i v_i^* f_i(t)$$

Relaxation

- Orbit closures are G-invariant algebraic varieties

$$t \ntrianglerighteq t' \text{ iff there exists}$$

$$G - \text{covariant polynomial } f :$$

$$f(t) = 0, \text{ but } f(t') \neq 0$$

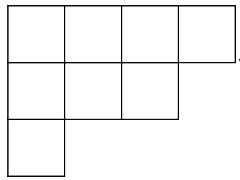
$$\text{if there is } \lambda \text{ s.th.}$$

$$P_\lambda t^{\otimes n} = 0 \text{ but } P_\lambda t'^{\otimes n} \neq 0$$

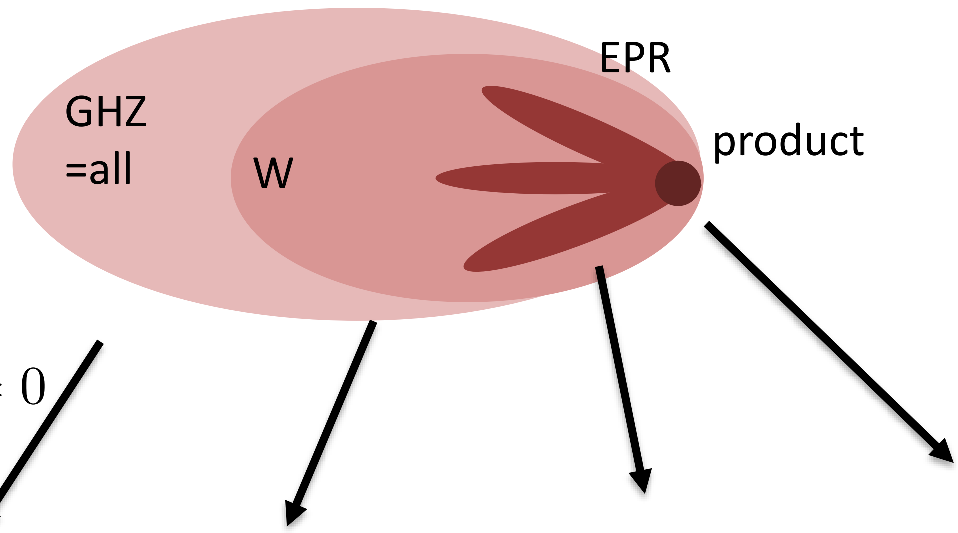
occurrence obstructions (Geometric Complexity Theory)
Mulmuley-Sohoni, Strassen, Bürgisser-Ikenmeyer, …

Entanglement polytopes

another relaxation



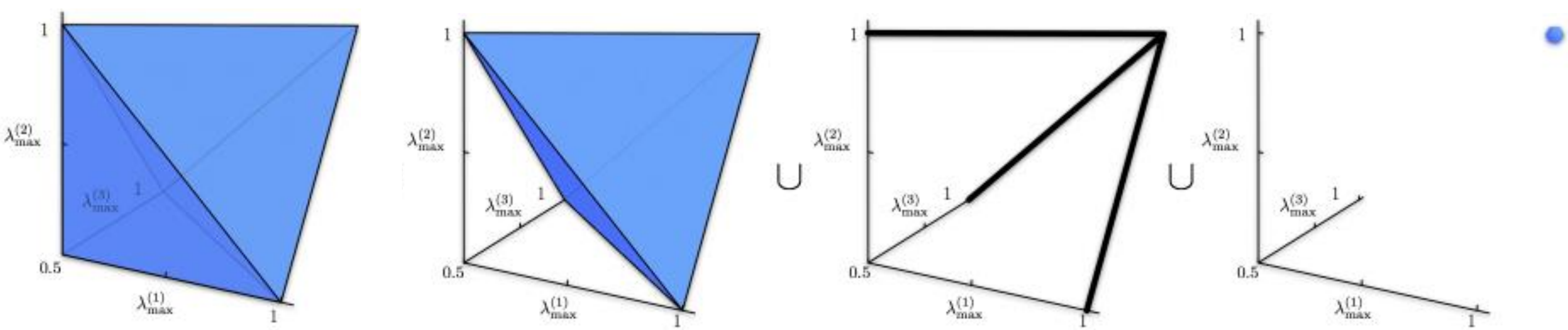
$$\longrightarrow \left(\frac{4}{8},\frac{3}{8},\frac{1}{8}\right)$$



$g_\lambda \neq 0$

Kronecker
= marginal
polytope

$P_\lambda t^{\otimes n} \neq 0$



A small observation

$$d = 2^n$$

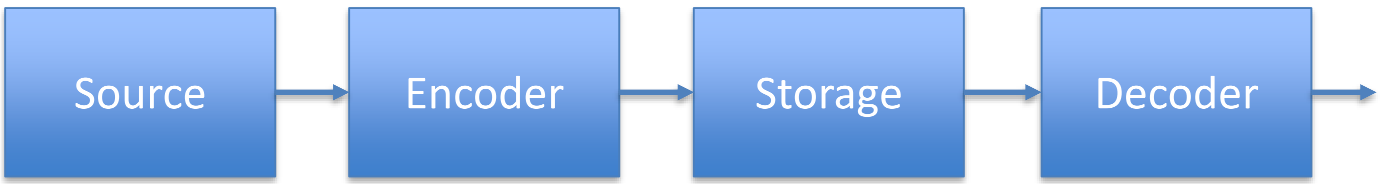
$$e_i = e_{i_1 i_2 \cdots i_n} = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}$$

$$\sum_{i=1}^{d} e_i \otimes e_i = \left(\sum_{i_1=1}^{2} e_{i_1} \otimes e_{i_1}\right) \otimes \left(\sum_{i_2=1}^{2} e_{i_2} \otimes e_{i_2}\right) \otimes \cdots \otimes \left(\sum_{i_n=1}^{2} e_{i_n} \otimes e_{i_n}\right)$$

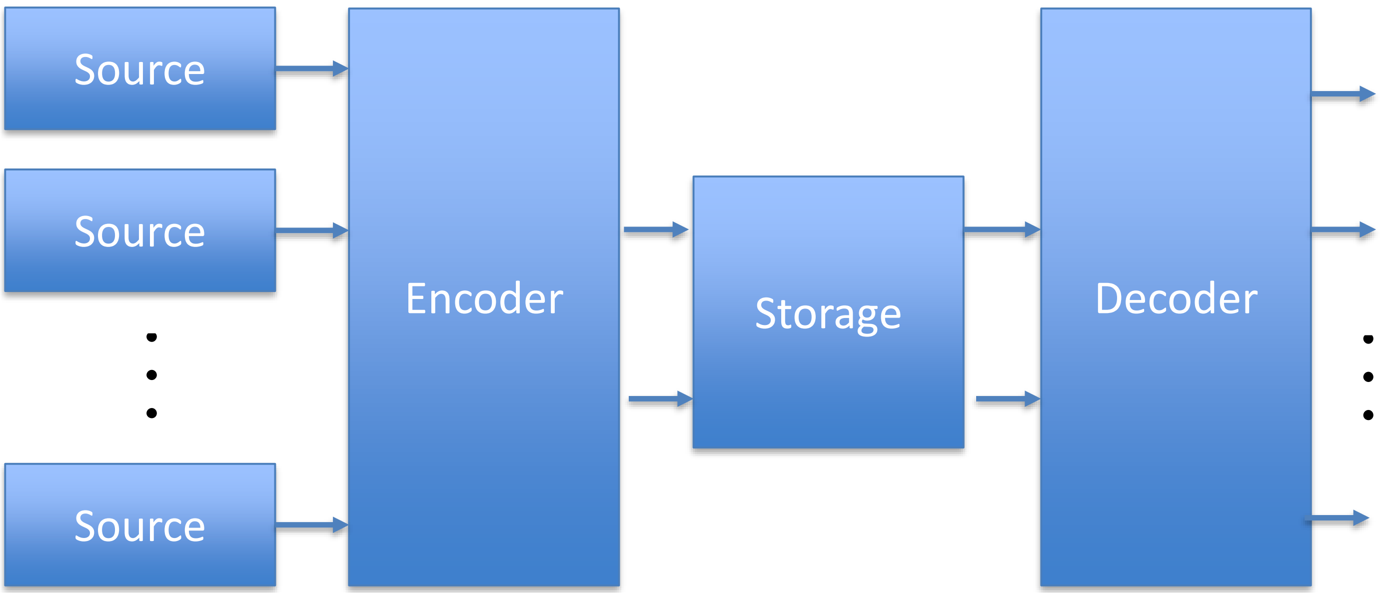
$$= (e_0 \otimes e_0 + e_1 \otimes e_1)^{\otimes n}$$

$$\langle d \rangle = \sum_{i=1}^{d} e_i \otimes e_i \otimes e_i = (e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1)^{\otimes n} = \langle 2 \rangle^{\otimes n}$$

(Quantum) information theory



Shannon: storage cost= all bits



Shannon: storage cost= H(X) bits/symbol

Asymptotic resource theory

- Asymp. restriction $t \gtrsim t'$ if $t^{\otimes n+o(n)} \geq t'^{\otimes n}$
- Unit $$\langle r \rangle = \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i$$
- Asymp. rank $$\tilde{R}(t) := \lim_{n \to \infty} R(t^{\otimes n})^{\frac{1}{n}}$$
- Asymp. subrank $$\tilde{Q}(t) := \lim_{n \to \infty} Q(t^{\otimes n})^{\frac{1}{n}}$$

Strassen’s spectral theorem

$$t \gtrsim t' \text{ iff } F(t) \geq F(t') \text{ for all } F:$$

F monotone — under restriction $F(s) \geq F(s')$ for all $s \geq s'$

F normalised — $F(\langle r \rangle) = r$

F multiplicative — $F(s \otimes s') = F(s) \cdot F(s')$

F additive — $F(s \oplus s') = F(s) + F(s')$

$$\tilde{R}(t) = \max_F F(t)$$

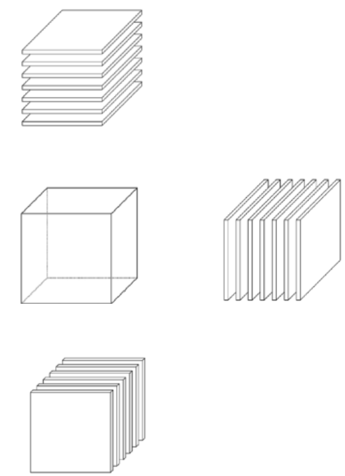
$$\tilde{Q}(t) = \min_F F(t)$$

$\Rightarrow$ easy

$\Leftarrow$ difficult

every *F* is an obstruction

What are the F’s?



- Existence non-constructive
  - Compact space worth of them
  - Gauge points: ranks of slicings
  - What are the others?

- Theorem also true for subclasses of tensors
  - Oblique tensor
  - Strassen’s support functionals
  - Conjecture (Strassen): they are all

Quantum functionals

$$\theta = (\theta_A, \theta_B, \theta_C) \text{ probability distribution e.g. } \theta_A = \theta_B = \theta_C = \frac{1}{3}$$

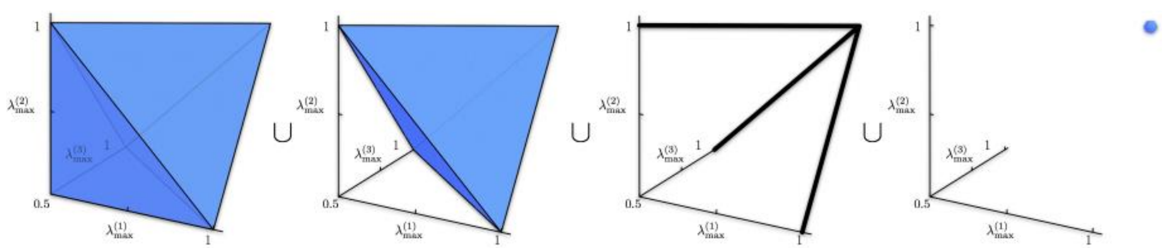
$$E_\theta(t) := \max_{\lambda \in \Delta(t)} \{\theta_A H(\lambda_A) + \theta_B H(\lambda_B) + \theta_C H(\lambda_C)\}$$

entanglement polytope

$$F_\theta(t) := 2^{E_\theta(t)}$$

quantum functionals

Measures distance to origin (relative entropy distance)



$$E_{(\frac{1}{3},\frac{1}{3},\frac{1}{3})} \qquad 1 \qquad h\left(\frac{1}{3}\right) \approx 0.92 \qquad \frac{2}{3} \qquad 0$$

Quantum functionals

$$E_\theta(t) := \max_{\lambda \in \Delta(t)} \{\theta_A H(\lambda_A) + \theta_B H(\lambda_B) + \theta_C H(\lambda_C)\}$$

$$F_\theta(t) := 2^{E_\theta(t)}$$

easy, since polytope gets smaller under restriction
quantum functional gets smaller

F_θ monotone

easy, since polytope of unit tensor
contains uniform point $F(\langle r \rangle) = r$

F_θ normalised

F_θ multiplicative

similar to multiplicativity, see paper

F_θ additive

Multiplicativity

$$F_\theta(t \otimes t') = F_\theta(t) \cdot F_\theta(t')$$

$$\Big\Updownarrow \quad F_\theta(t) := 2^{E_\theta(t)}$$

$$E_\theta(t \otimes t') = E_\theta(t) + E_\theta(t')$$

$\geq$	$\leq$
easy	more difficult

$$E_\theta(t \otimes t') \geq E_\theta(t) + E_\theta(t')$$

$$E_\theta(t) := \max_{\lambda \in \Delta(t)} \{\theta_A H(\lambda_A) + \theta_B H(\lambda_B) + \theta_C H(\lambda_C)\}$$

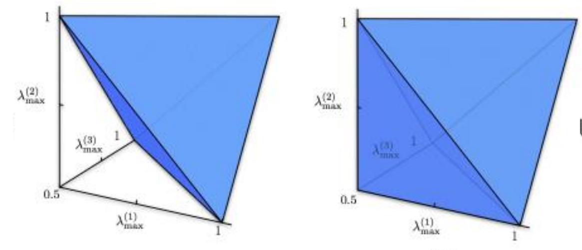
Lemma: $\Delta(t \otimes t') \supseteq \Delta(t) \otimes \Delta(t')$

Proof:
$$\begin{aligned}
\Delta(t \otimes t') &= \{\lambda(\tau) : t \otimes t' \trianglerighteq \tau\} \\
&\supseteq \{\lambda(s \otimes s') : t \otimes t' \trianglerighteq s \otimes s'\} \\
&= \{\lambda(s) \otimes \lambda(s') : t \trianglerighteq s, t' \trianglerighteq s'\} \\
&= \Delta(t) \otimes \Delta(t')
\end{aligned}$$

product distribution

qed

$$E_\theta(t \otimes t') \le E_\theta(t) + E_\theta(t')$$



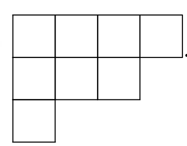
Lemma:

$$\Delta(t \otimes t') \subseteq \Delta(t) \otimes_{\mathrm{Kron}} \Delta(t')$$

$$:= \{(\alpha, \beta, \gamma) : (a, b, c) \in \Delta(t), (a', b', c') \in \Delta(t'),$$
$$(a, a', \alpha) \& (b, b', \beta) \& (c, c', \gamma) \in \mathrm{Kron}\}$$

Proof: $0 \neq (P_{n\alpha} \otimes P_{n\beta} \otimes P_{n\gamma})\, t^{\otimes n} \otimes t'^{\otimes n}$

$$= [P_{n\alpha} \otimes P_{n\beta} \otimes P_{n\gamma}] \Big[\underbrace{(\sum P_{na})}_{=id} \otimes (\sum P_{nb}) \otimes (\sum P_{nc}) \otimes (\sum P_{na'}) \otimes (\sum P_{nb'}) \otimes (\sum P_{nc'}) \Big] [t^{\otimes n} \otimes t'^{\otimes n}]$$



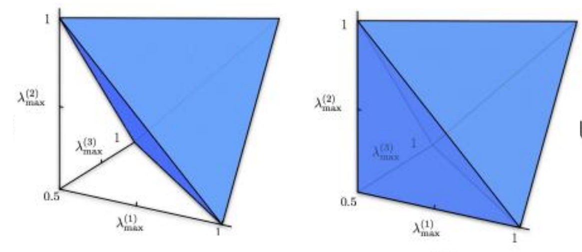
$$P_{n\alpha}(P_{na} \otimes P_{na'}) \neq 0$$
$$P_{n\beta}(P_{nb} \otimes P_{nb'}) \neq 0$$
$$P_{n\gamma}(P_{nc} \otimes P_{nc'}) \neq 0$$

$$(P_{na} \otimes P_{nb} \otimes P_{nc}) t^{\otimes n} \neq 0$$
$$(P_{na'} \otimes P_{nb'} \otimes P_{nc'}) t'^{\otimes n} \neq 0$$

qed

$$E_\theta(t \otimes t') \leq E_\theta(t) + E_\theta(t')$$

$$E_\theta(t) := \max_{\lambda \in \Delta(t)} \{\theta_A H(\lambda_A) + \theta_B H(\lambda_B) + \theta_C H(\lambda_C)\}$$



Lemma:

$$\Delta(t \otimes t') \subseteq \Delta(t) \otimes_{\mathrm{Kron}} \Delta(t')$$

$$:= \{(\alpha, \beta, \gamma) : (a, b, c) \in \Delta(t), (a', b', c') \in \Delta(t'), \\ (a, a', \alpha) \& (b, b', \beta) \& (c, c', \gamma) \in \mathrm{Kron}\}$$

Subadditivity
v. Neumann entropy

Lemma: If $(a, a', \alpha) \in \mathrm{Kron}$, then $H(\alpha) \leq H(a) + H(a')$

Proof: $$\theta_A H(\alpha) + \theta_B H(\beta) + \theta_C H(\gamma) \leq \theta_A \left(H(a) + H(a')\right) \\ + \theta_B \left(H(b) + H(b')\right) \\ + \theta_C \left(H(c) + H(c')\right)$$

Subadditivity of E

optimal

qed

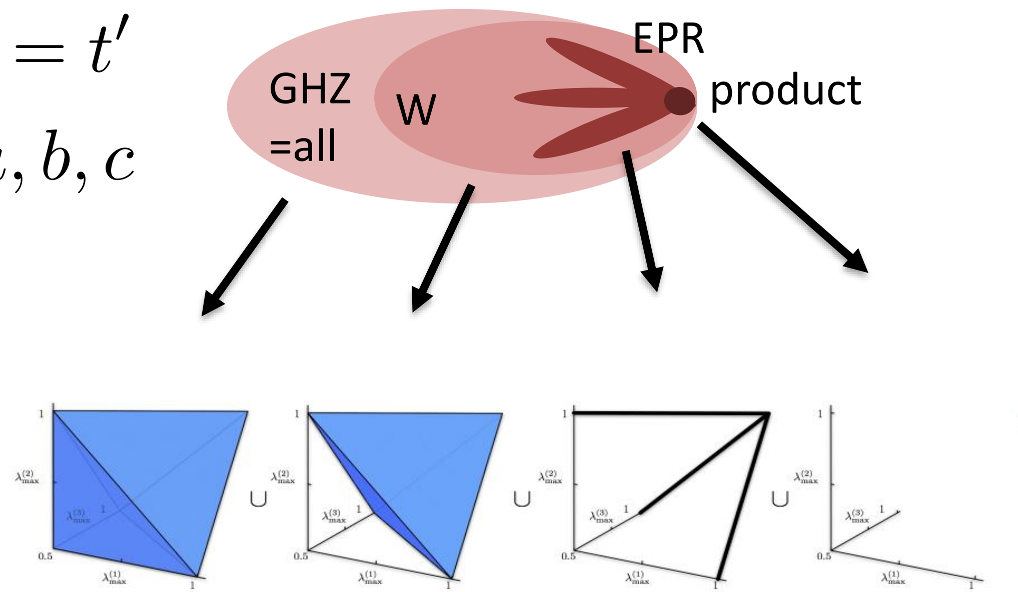
Quantum functionals

- First family of universal spectral points
- Extend Strassen’s support functionals
- Are they complete?
- If complete, then $\omega = 2$
- Characterise slice-rank
- General setting of tensors of order k
- Connect Strassen’s framework to capset

Summary

$$t \geq t' \text{ if } (a \otimes b \otimes c)\, t = t'$$

for some matrices a, b, c



$$t \gtrsim t' \text{ if } t^{\otimes n+o(n)} \geq t'^{\otimes n}$$

$$E_\theta(t) := \max_{\lambda \in \Delta(t)} \{\theta_A H(\lambda_A) + \theta_B H(\lambda_B) + \theta_C H(\lambda_C)\}$$

$$F_\theta(t) := 2^{E_\theta(t)}$$

If all, then $\omega = 2$

Recoupling and Quantum Entropy

Wigner 6j coefficients

SU(2) !

$$H_{j_1} \otimes H_{j_2} \cong \bigoplus_{|j_1-j_2|\leq j_{12}\leq j_1+j_2} H_{j_{12}}$$

Clebsch-Gordan (Wigner 3j) $\langle j_1, m_1 | \langle j_2, m_2 || j_1, j_2, j_{12}, m_{12} \rangle$

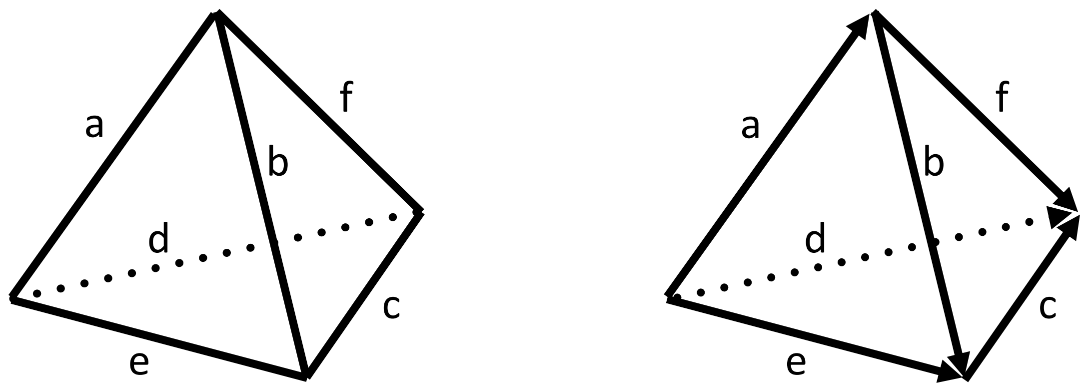
$$\begin{tikzcd}[column sep=tiny]
 & H_{j_1} \otimes H_{j_2} \otimes H_{j_3} \arrow[dl] \arrow[dr] & \\
\displaystyle\bigoplus_{j_{12}=|j_1-j_2|}^{j_1+j_2} H_{j_{12}} \otimes H_{j_3} \arrow[dr] & & \displaystyle\bigoplus_{j_{23}=|j_2-j_3|}^{j_2+j_3} H_{j_1} \otimes H_{j_{23}} \arrow[dl] \\
 & \displaystyle\bigoplus_{j_{123}} \left(\bigoplus_{j_{12}} \right) H_{j_{123}} \cong \bigoplus_{j_{123}} \left(\bigoplus_{j_{23}} \right) H_{j_{123}} &
\end{tikzcd}$$

probability amplitude

Wigner 6j $\langle j_1, j_2, j_{12}, j_{123}, m_{123} || j_1, j_2, j_{23}, j_{123}, m_{123} \rangle$

Semiclassical limit

Wigner, Ponzano & Regge, Roberts...



$$\begin{Bmatrix} ka & kb & kc \\ kd & ke & kf \end{Bmatrix} \sim \begin{cases} \sqrt{\dfrac{2}{3\pi V k^3}} \cos\left\{ \sum (ka+1)\dfrac{\theta_a}{2} + \dfrac{\pi}{4} \right\} & \textit{if } \tau \textit{ is Euclidean} \\ \textit{exponentially decaying} & \textit{if } \tau \textit{ is Minkowskian} \end{cases}$$

$\updownarrow$

Existence of Eucledian tetrahedron

$\longleftrightarrow$

$$A + B + C = D, \qquad \underbrace{A+B}_{=E}, \qquad \overbrace{B+C}^{=F}$$

Horn's problem
(with Miriam Backens)
SU(d) generalisation

traceless
Hermitian
matrices

Eigenvalues of Quantum States

Which are the possible eigenvalues of quantum states and their reduced density matrices?

$$\mu \quad \rho_A = tr_B \rho_{AB} \qquad\qquad \nu \quad \rho_B = tr_A \rho_{AB}$$

$$\lambda \quad \rho_{AB}$$

- Linear inequalities
  Klyachko 2004, Daftuar & Hayden 2004, Berenstein-Sjamaar 2000, Ressayre 2007
- Probability density
  Christandl, Doran, Kousidis & Walter, 2012
- Multiparticle entanglement
  Walter, Doran, Gross & Christandl, 2013

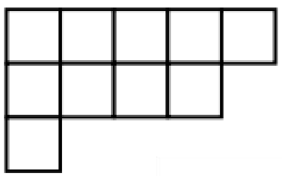
Kronecker coefficients

$$(u_A, u_B) \mapsto u_A \otimes u_B$$

$$[\mu] \otimes [\nu] \cong \bigoplus_{\lambda} \boxed{g_{\mu\nu\lambda}} [\lambda]$$

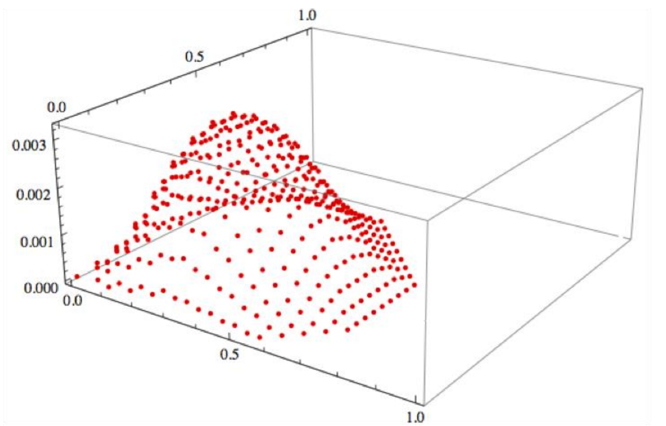
$$V_\lambda \downarrow^{SU(d_A d_B)}_{SU(d_A)\times SU(d_B)} \cong \bigoplus_{\mu,\nu} \boxed{g_{\mu\nu\lambda}} V_\mu \otimes V_\nu$$

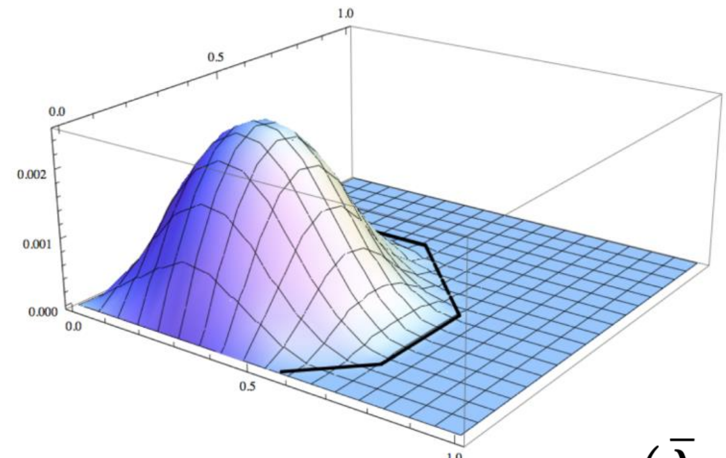
Kronecker coefficient of
symmetric group S_n
via Schur-Weyl duality



boxes =n

Young diagram





$$\bar{\lambda} = \frac{\lambda}{n}$$

$$\lim_{n\to\infty} g_{\mu\nu\lambda}$$ ~probability density for eigenvalues

$$(\bar{\lambda}, \bar{\mu}, \bar{\nu})$$

$$\rho_{AB} \quad \rho_A \quad \rho_B$$

Christandl & Mitchison, Klyachko, Daftuar &Hayden, 2004
Christandl & Harrow, Mitchison 2005, Christandl, Doran, Kousidis & Walter, 2012

P ≠NP?

Mathematical Structure

$$U(d_A) \times U(d_B) \to U(d_A d_B)$$

$$(u_A, u_B) \mapsto u_A \otimes u_B$$

$$\mathfrak{u}(d_A) \times \mathfrak{u}(d_B) \to \mathfrak{u}(d_A d_B)$$

$$(x_A, x_B) \mapsto x_A \otimes 1_B + 1_A \otimes x_B$$

$$\mathfrak{u}^*(d_A) \times \mathfrak{u}^*(d_B) \leftarrow \mathfrak{u}^*(d_A d_B)$$

$$(\rho_A, \rho_B) \leftarrow \rho_{AB}$$

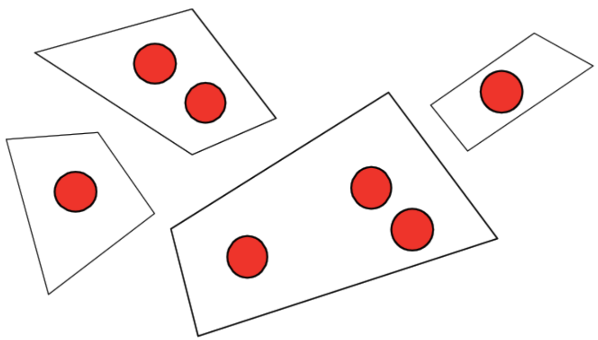
moment map!

Eigenvalues: Intersect image with positive Weyl chamber

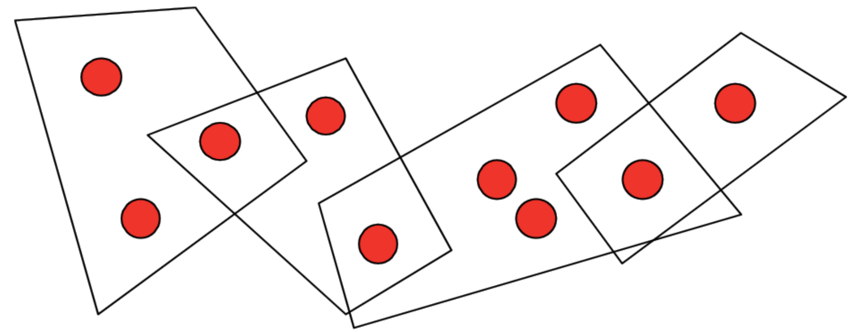
Kirwan's convexity theorem
Duistermaat-Heckman measure
Symplectic Quotient vs GIT quotient

essentially complete solution

Mathematical Structure



Collection of subsets of a set of particles
(non-overlapping)



Collection of subsets of a set of particles
(overlapping)

$$\begin{tikzcd}
\rho_A & & \rho_B \\
& \rho_{AB} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

ok!

$$\begin{tikzcd}
\rho_A & & \rho_B & & \rho_C \\
& \rho_{AB} \arrow[ul] \arrow[ur] & & \rho_{BC} \arrow[ul] \arrow[ur] & \\
& & \rho_{ABC} \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

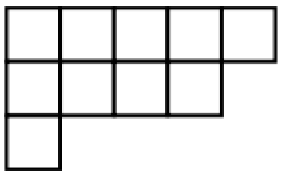
???

Quantum Marginal Problem:
condensed matter physics,
quantum chemistry,
quantum coding theory

Lieb &Ruskai

$$H(\rho_{AB}) + H(\rho_{BC}) \geq H(\rho_B) + H(\rho_{ABC})$$

Recoupling Coefficients of Symmetric Group



$$\begin{tikzcd}[column sep=small]
 & {[\alpha]\otimes[\beta]\otimes[\gamma]} \arrow[dl] \arrow[dr] & \\
\bigoplus_{\mu}[\mu]\otimes H_{\mu}^{\alpha\beta}\otimes[\gamma] \arrow[dr] & & \bigoplus_{\mu}[\alpha]\otimes[\nu]\otimes H_{\nu}^{\beta\gamma} \arrow[dl] \\
 & \bigoplus_{\mu,\lambda}[\lambda]\otimes H_{\mu}^{\alpha\beta}\otimes H_{\lambda}^{\mu\gamma}\cong\bigoplus_{\nu,\lambda}[\lambda]\otimes H_{\lambda}^{\alpha\nu}\otimes H_{\nu}^{\beta\gamma} &
\end{tikzcd}$$

$$\begin{bmatrix}\alpha & \beta & \mu \\ \gamma & \lambda & \nu\end{bmatrix}:\ \mathrm{H}_{\lambda}^{\mu\gamma}\otimes\mathrm{H}_{\mu}^{\alpha\beta}\to\mathrm{H}_{\lambda}^{\alpha\nu}\otimes\mathrm{H}_{\nu}^{\beta\gamma}$$

Recoupling & Eigenvalues

Theorem 1. *If there exists a quantum state ρ_{ABC} with eigenvalues r_A, r_B, r_C, r_{AB}, r_{BC}, r_{ABC} then there exist Young diagrams $\alpha, \beta, \gamma, \mu, \nu, \lambda \vdash k$ with $k \to \infty$ such that*

$$\lim_{k\to\infty} (\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\mu}, \overline{\nu}, \overline{\lambda}) = (r_A, r_B, r_C, r_{AB}, r_{BC}, r_{ABC}) \quad (8)$$

and

$$\| \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix} \|_{\mathrm{HS}} \geq \frac{1}{\mathrm{poly}(k)}. \quad (9)$$

Conversely, if $(r_A, r_B, r_C, r_{AB}, r_{BC}, r_{ABC})$ *is not associated to a tripartite density matrix then for every sequence of Young diagrams satisfying* (8) *we have*

$$\| \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix} \|_{\mathrm{HS}} \leq \exp(-\Omega(k)). \quad (10)$$

Proof

$$\left(\mathbf{C}^d\right)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} [\lambda] \otimes V_\lambda^d$$

Schur-Weyl

$$\mathrm{tr}(P_\lambda^d \rho^{\otimes k}) \leq \mathrm{poly}(k) \exp(-k||\bar{\lambda} - r||_1^2/2)$$

Keyl-Werner

$$\begin{tikzcd}[column sep=tiny]
& \displaystyle\bigoplus_{\alpha,\beta,\gamma} [\alpha] \otimes [\beta] \otimes [\gamma] \otimes V_\alpha^a \otimes V_\beta^b \otimes V_\gamma^c \arrow[dl] \arrow[dr] & \\
\displaystyle\bigoplus_{\alpha,\beta,\gamma,\mu} [\mu] \otimes H_\mu^{\alpha\beta} \otimes [\gamma] \otimes V_\alpha^a \otimes V_\beta^b \otimes V_\gamma^c \arrow[dr] & & \displaystyle\bigoplus_{\alpha,\beta,\gamma,\nu} [\alpha] \otimes [\nu] \otimes H_\nu^{\beta\gamma} \otimes V_\alpha^a \otimes V_\beta^b \otimes V_\gamma^c \arrow[dl] \\
& \displaystyle\bigoplus_{\alpha,\beta,\gamma,\mu,\lambda} [\lambda] \otimes H_\mu^{\alpha\beta} \otimes H_\lambda^{\mu\gamma} \otimes V_\alpha^a \otimes V_\beta^b \otimes V_\gamma^c \cong \bigoplus_{\alpha,\beta,\gamma,\nu,\lambda} [\lambda] \otimes H_\lambda^{\alpha\nu} \otimes H_\nu^{\beta\gamma} \otimes V_\alpha^a \otimes V_\beta^b \otimes V_\gamma^c &
\end{tikzcd}$$

$$\begin{array}{c}\exists\ \rho_{ABC} \\ \text{with } (r_A, r_B, r_{AB}, r_{BC}, r_{ABC})\end{array} \longleftrightarrow tr P_\lambda^{ABC} P_\gamma^b P_\mu^{ab} P_\nu^{bc} \rho_{ABC}^{\otimes k} \geq 1/poly(k) \longleftrightarrow \left\| \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix} \right\|_{\mathrm{HS}} \geq \frac{1}{\mathrm{poly}(k)}$$

$$(r_A, r_B, r_{AB}, r_{BC}, r_{ABC}) \approx (\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\mu}, \bar{\nu}, \bar{\lambda})$$

Formulation as Semiclassical Limit

subgroup chain

$$\begin{tikzcd}[column sep=tiny]
 & U(abc) \arrow[dl] \arrow[dr] & \\
U(ab)\times U(c) \arrow[dr] & & U(a)\times U(bc) \arrow[dl] \\
 & U(a)\times U(b)\times U(c) &
\end{tikzcd}$$

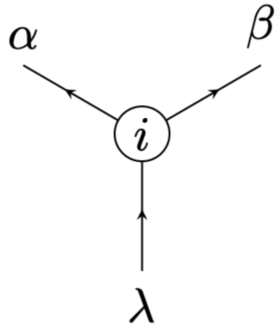
$$\begin{tikzcd}[column sep=tiny]
 & \bigoplus_{\lambda}[\lambda]\otimes V_{\lambda}^{abc} \arrow[dl] \arrow[dr] & \\
\bigoplus_{\gamma,\mu,\lambda}[\lambda]\otimes H_{\lambda}^{\mu\gamma}\otimes V_{\mu}^{ab}\otimes V_{\gamma}^{c} \arrow[dr] & & \bigoplus_{\alpha,\nu,\lambda}[\lambda]\otimes H_{\lambda}^{\alpha\nu}\otimes V_{\alpha}^{a}\otimes V_{\nu}^{bc} \arrow[dl] \\
 & \bigoplus_{\alpha,\beta,\gamma,\mu\lambda}[\lambda]\otimes H_{\mu}^{\alpha\beta}\otimes H_{\lambda}^{\mu\gamma}\otimes V_{\alpha}^{a}\otimes V_{\beta}^{b}\otimes V_{\gamma}^{c}\cong\bigoplus_{\alpha,\beta,\gamma,\mu\lambda}[\lambda]\otimes H_{\nu}^{\beta\gamma}\otimes H_{\lambda}^{\alpha\nu}\otimes V_{\alpha}^{a}\otimes V_{\beta}^{b}\otimes V_{\gamma}^{c} &
\end{tikzcd}$$

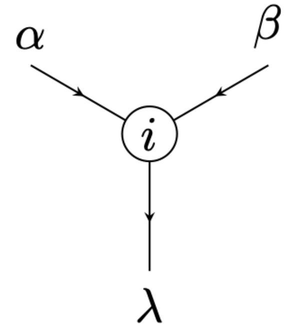
6j symbols for subgroup chain = 6j symbols of S_n

Grand unifying theories $U(1)\times SU(2)\times SU(3)\to SU(6)$

Graphical calculus

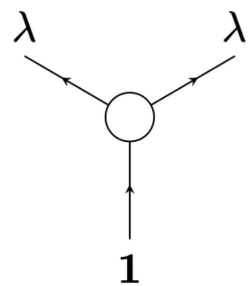
basis

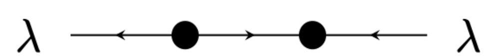
$\phi_i =$ 

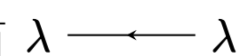
and $\phi_i^\dagger =$ 

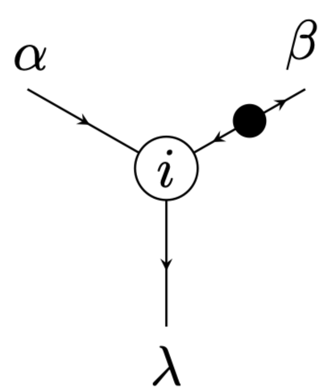
$$\mathrm{tr}\, \phi_j^\dagger \phi_i = \dim[\lambda]\delta_{ij}$$

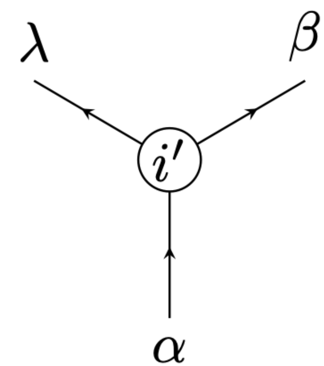
self-duality

$:=$ 



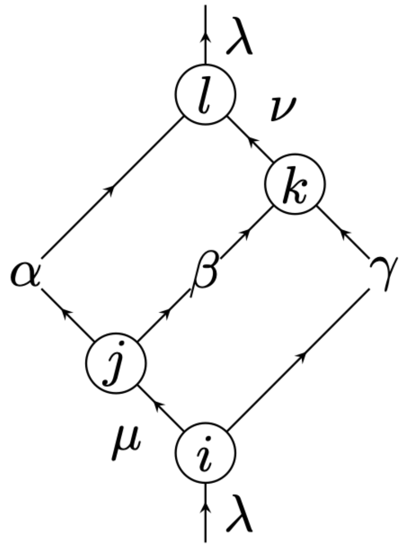
$= \dfrac{1}{\dim[\lambda]}$ 

$\sqrt{\dfrac{\dim[\alpha]\,\dim[\beta]}{\dim[\lambda]}}$ 

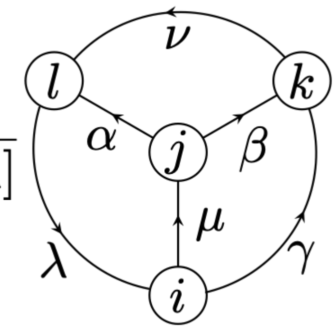
$= \sum_{i'} U_{ii'}$ 

Recoupling

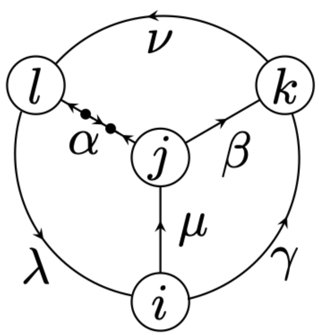
$$\mathbf{1}_\lambda \otimes \begin{bmatrix}\alpha & \beta & \mu \\ \gamma & \lambda & \nu\end{bmatrix}^{kl}_{ij} =$$



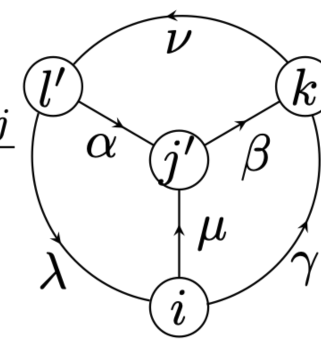
$$\begin{bmatrix}\alpha & \beta & \mu \\ \gamma & \lambda & \nu\end{bmatrix}^{kl}_{ij} = \frac{1}{\dim[\lambda]}$$



$$= \frac{\dim[\alpha]}{\dim[\lambda]}$$



$$= \frac{\sqrt{\dim[\mu]\dim[\nu]}}{\sqrt{\dim[\beta]\dim[\lambda]}} \sum_{j'l'} \frac{U_{ll'}\overline{V}_{j'j}}{\dim[\nu]}$$



$$\Rightarrow \quad \boxed{\left\| \begin{bmatrix}\alpha & \beta & \mu \\ \gamma & \lambda & \nu\end{bmatrix} \right\|_{\mathrm{HS}} = \sqrt{\frac{\dim[\mu]\dim[\nu]}{\dim[\beta]\dim[\lambda]}} \left\| \begin{bmatrix}\alpha & \mu & \beta \\ \gamma & \nu & \lambda\end{bmatrix} \right\|_{\mathrm{HS}}}$$

Strong subadditivity

$$\rho_{ABC}$$

$\Big\downarrow$ Theorem

exists

$$\lim_{k\to\infty}(\overline{\alpha},\overline{\beta},\overline{\gamma},\overline{\mu},\overline{\nu},\overline{\lambda}) = (r_A, r_B, r_C, r_{AB}, r_{BC}, r_{ABC})$$

$$\left\|\begin{bmatrix}\alpha & \beta & \mu\\ \gamma & \lambda & \nu\end{bmatrix}\right\|_{\mathrm{HS}} \geq \frac{1}{\mathrm{poly}(k)}$$

$\Big\downarrow$ Symmetry

$$\left\|\begin{bmatrix}\alpha & \beta & \mu\\ \gamma & \lambda & \nu\end{bmatrix}\right\|_{\mathrm{HS}} = \sqrt{\frac{\dim[\mu]\dim[\nu]}{\dim[\beta]\dim[\lambda]}}\left\|\begin{bmatrix}\alpha & \mu & \beta\\ \gamma & \nu & \lambda\end{bmatrix}\right\|_{\mathrm{HS}}$$

$$\frac{\dim[\mu]\dim[\nu]}{\dim[\beta]\dim[\lambda]} \geq \frac{1}{\mathrm{poly}(k)}$$

$\Big\downarrow$ Dimension vs Entropy

$$\frac{1}{k}\log_2\dim[\lambda] \;\to\; H(r)$$

$$H(\rho_{AB}) + H(\rho_{BC}) \geq H(\rho_B) + H(\rho_{ABC})$$

....Wigner?

Can be embedded into quantum marginal problem!

extended Horn‘s problem

$$\underbrace{pP + \overbrace{qQ}^{}}_{=?}\!\!\!\!\!\!\!\!\!\!\!\!\overbrace{ + rR}^{=?} = ?$$

$$\begin{aligned} |\phi\rangle_{ABCD} =& \sqrt{p}\sum_i id_{ABC} \otimes \sqrt{P}|i\rangle_A|0\rangle_B|0\rangle_C|i\rangle_D \\ &+ \sqrt{q}\sum_i id_{ABC} \otimes \sqrt{Q}|0\rangle_A|i\rangle_B|0\rangle_C|i\rangle_D \\ &+ \sqrt{r}\sum_i id_{ABC} \otimes \sqrt{R}|0\rangle_A|0\rangle_B|i\rangle_C|i\rangle_D \end{aligned}$$

$$\begin{aligned} \rho_A &= pP + |0\rangle\langle 0|(q+r) \\ \rho_B &= qQ + |0\rangle\langle 0|(p+r) \\ \rho_C &= rR + |0\rangle\langle 0|(p+q) \\ \rho_{AB} &= pP + qQ + r|00\rangle\langle 00| \\ \rho_{BC} &= p|00\rangle\langle 00| + qQ + rR \\ \rho_{ABC} &= pP + qQ + rR \end{aligned}$$

there is also a representation-theory relation (cf Murnaghan)

Fun Application

$$H(\rho_{AB}) + H(\rho_{BC}) \geq H(\rho_B) + H(\rho_{ABC})$$

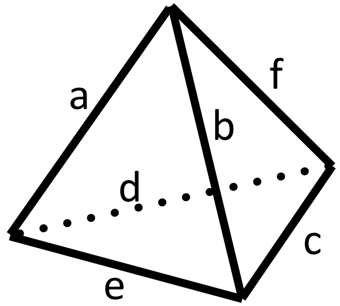
$$\Downarrow$$

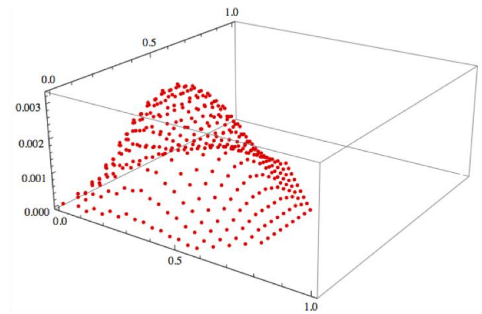
$$h(r) + (p+q)H(\frac{pP+qQ}{p+q}) + h(p) + (q+r)H(\frac{qQ+rR}{q+r})$$
$$\geq h(q) + qH(Q) + H(pP+qQ+rR)$$

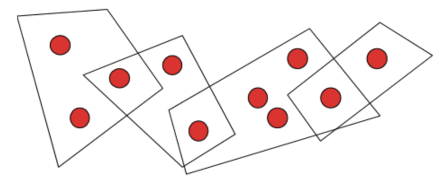
new (?) concavity-like inequality for entropy

Summary

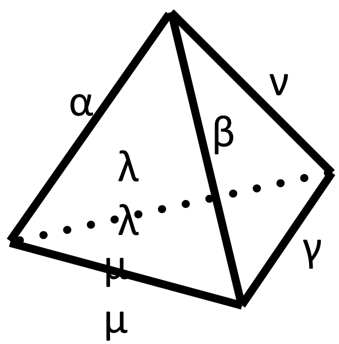
- Motivation







- Result



$\sim$

$$\begin{tikzcd}[column sep=small] \rho_A & & \rho_B & & \rho_C \\ & \rho_{AB} \arrow[ul] \arrow[ur] & & \rho_{BC} \arrow[ul] \arrow[ur] & \\ & & \rho_{ABC} \arrow[ul] \arrow[ur] & & \end{tikzcd}$$

- Application

$$H(\rho_{AB}) + H(\rho_{BC}) \geq H(\rho_B) + H(\rho_{ABC})$$

$$h(r) + (p+q)H(\frac{pP+qQ}{p+q}) + h(p) + (q+r)H(\frac{qQ+rR}{q+r})$$
$$\geq h(r) + rH(R) + H(pP+qQ+rR)$$

- Future

Spin foams, spin networks, entanglement, ...