Algebraic Bethe ansatz for the open XXZ spin chain with non-diagonal boundary terms via $U_{\mathfrak{q}}\mathfrak{sl}_2$ symmetry

Dmitry Chernyak

LPENS

Based on arXiv:2207.12772 with A.M. Gainutdinov and H. Saleur and arXiv:2212.09696 with A.M. Gainutdinov, J.L. Jacobsen and H. Saleur

Les Diablerets, February 8, 2023

Motivation

The Hamiltonian of the open XXZ spin chain $(\mathbb{C}^2)^{\otimes N}$ of length N with arbitrary boundary fields is given by

$$H_{\text{n.d.}} := \vec{h}_l \cdot \vec{\sigma}_1 + \vec{h}_r \cdot \vec{\sigma}_N + \frac{1}{2}\sum_{i=1}^{N-1}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \frac{\mathfrak{q}+\mathfrak{q}^{-1}}{2}\sigma_i^z\sigma_{i+1}^z\right)$$

with $\mathfrak{q}$ and $\vec{h}_{l/r}$ 7 parameters and Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Appears in the 6-vertex model, boundary loop models, ASEP...

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Main message : Non-compact spin chains contain a lot of interesting (and unexplored) physics.

Outline

1. Loop models and lattice algebras
2. $U_{\mathfrak{q}}\mathfrak{sl}_2$-invariant realisation
3. Bethe ansatz

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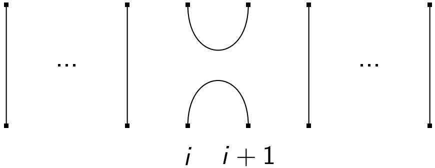
2. $U_q\mathfrak{sl}_2$-invariant realisation

3. Bethe ansatz

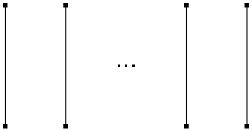
Let N be an integer. For all $1 \leq i \leq N-1$ consider the diagrams

$e_i =$



and

$1 =$



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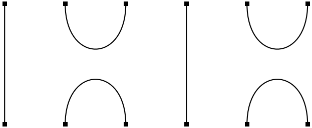
$e_5 e_2 =$ 

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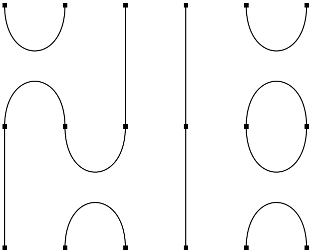

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$e_4 e_2 =$

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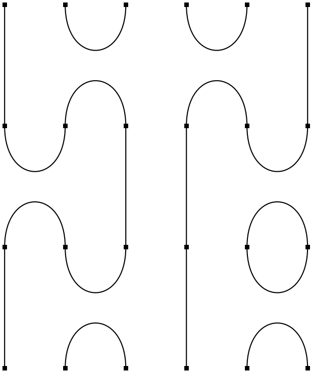

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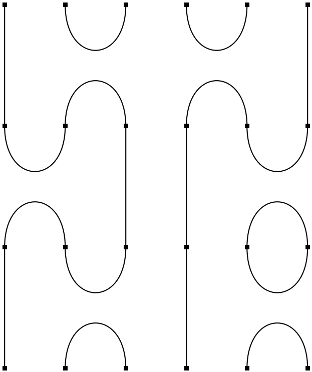
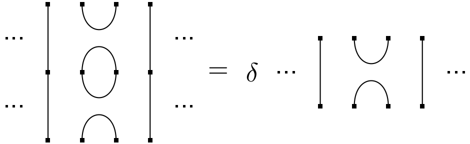


$= e_4 e_2 e_5 e_1 e_5 e_2$

Graphical rules :

$e_i^2 =$  $= \delta e_i$

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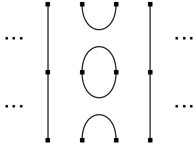
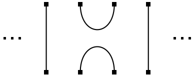
$e_i^2 =$  $= \delta$  $= \delta e_i$

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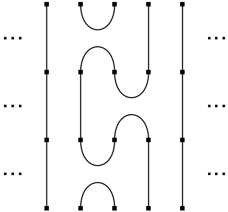
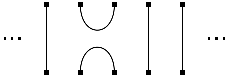
$e_i e_{i+1} e_i =$  $=$  $= e_i$

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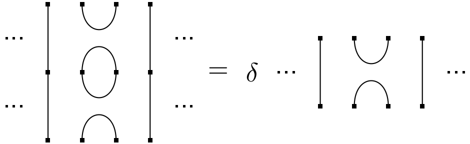
$e_i^2 =$  $= \delta e_i$

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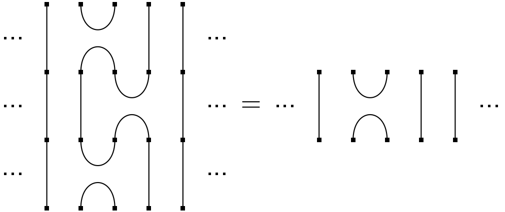
$e_i e_{i+1} e_i =$  $= e_i$

The resulting algebra is called the **Temperley-Lieb (TL) algebra** and denoted $\mathrm{TL}_{\delta,N}$.

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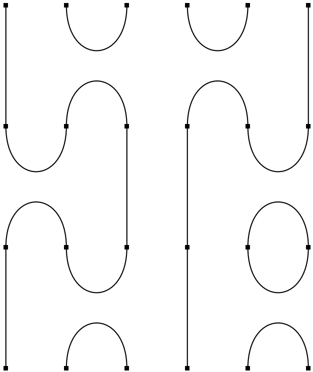
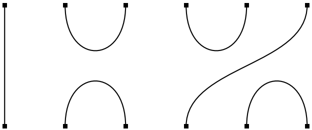


$$= \quad \delta$$



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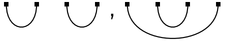
$\mathcal{W}_0 = \mathbb{C}\langle$  $\rangle$

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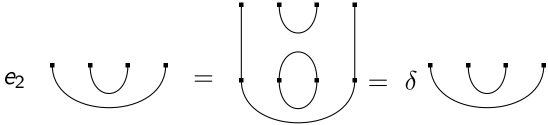

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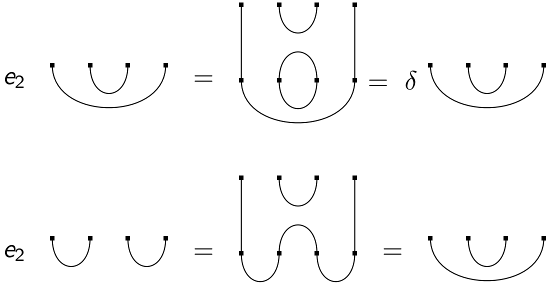

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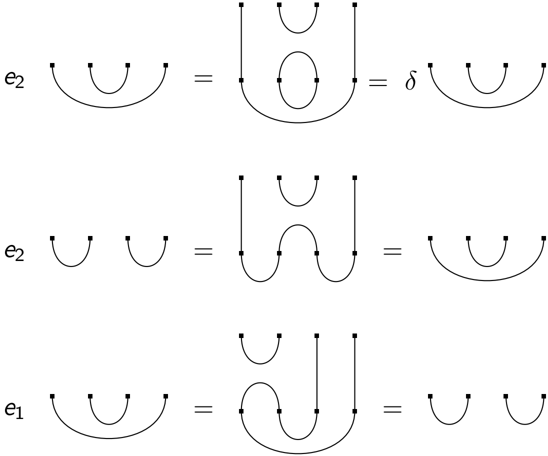

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Introduce an additional generator b_l satisfying

$$b_l^2 = b_l\,, \qquad e_1 b_l e_1 = y_l e_1\,, \qquad [b_l, e_i] = 0 \quad \text{for} \quad 2 \leq i \leq N-1$$

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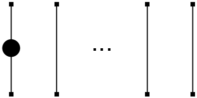
$b_l =$ 

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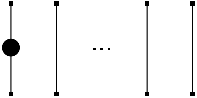
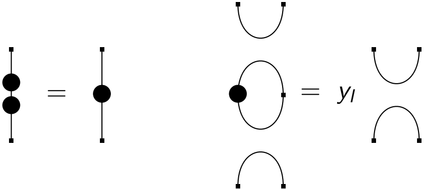
$b_l =$ 



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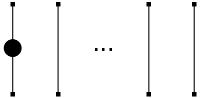
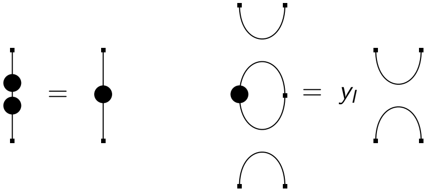
$b_l =$ 



This defines the **Blob algebra** $\mathrm{B}_{\delta,y_l,N}$.

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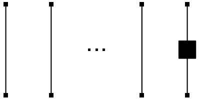
$b_r =$ 

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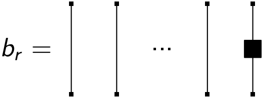
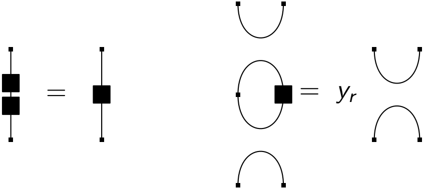


satisfying



with some weight $y_r \in \mathbb{C}$, that is

$$b_r^2 = b_r\,, \qquad e_{N-1} b_r e_{N-1} = y_r e_{N-1}\,, \qquad [b_r, e_i] = 0 \quad \text{for} \quad 1 \leq i \leq N-2\,.$$

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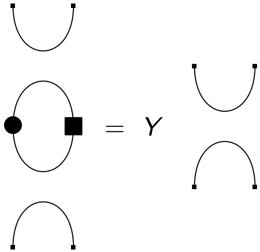

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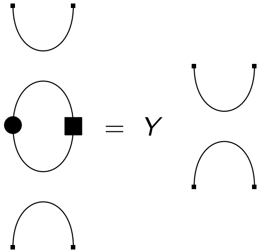

with some weight $Y \in \mathbb{C}$.

This defines the **two-boundary Temperley-Lieb algebra** $2B_{\delta,y_{l/r},Y,N}$.

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For example, for $N = 2$

$\mathcal{W} = \mathbb{C}\Big\langle$  $\Big\rangle$

Introduce

$$\mathbf{H} := -\mu_l b_l - \mu_r b_r - \sum_{i=1}^{N-1} e_i \in 2\mathrm{B}_{\delta, y_{l/r}, Y, N}$$

for some $\mu_{l/r} \in \mathbb{C}$.

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Theorem (J. de Gier, A. Nichols '09)

For some explicit mapping of parameters $(\mathfrak{q}, \overrightarrow{h}_{l/r}) \leftrightarrow (\delta, y_{l/r}, Y, \mu_{l/r})$

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Idea : Find a different realisation of $\mathcal{W}$ to diagonalise $H_{\mathrm{n.d.}}$!

Definition

$U_{\mathfrak{q}}\mathfrak{sl}_2$ is generated by E, F, K and K^{-1} with relations

$$KEK^{-1} = \mathfrak{q}^2 E\,, \quad KFK^{-1} = \mathfrak{q}^{-2} F\,, \quad [E, F] = \frac{K - K^{-1}}{\mathfrak{q} - \mathfrak{q}^{-1}}\,.$$

It is a $\mathfrak{q}$-deformation of the Lie algebra $\mathfrak{sl}_2$: in the limit $\mathfrak{q} \to 1$ we recover the commutation relations of the $\mathfrak{sl}_2$ triple (E, F, H) with $K^{\pm 1} = \mathfrak{q}^{\pm H}$.

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Representations

Very similar to $\mathfrak{sl}_2$. For example, the spin-$\frac{1}{2}$ representation in the basis $\{|\uparrow\rangle\,, |\downarrow\rangle\}$ is given by

$$\mathrm{E}_{\mathbb{C}^2} = \sigma^+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathrm{F}_{\mathbb{C}^2} = \sigma^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\mathrm{K}^{\pm 1}_{\mathbb{C}^2} = \mathfrak{q}^{\pm\sigma^z} = \begin{pmatrix} \mathfrak{q}^{\pm 1} & 0 \\ 0 & \mathfrak{q}^{\mp 1} \end{pmatrix}.$$

Using the coproduct of $U_\mathfrak{q}\mathfrak{sl}_2$ it can be extended to an action on $(\mathbb{C}^2)^{\otimes N}$.

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- For special boundary conditions

$$H_{\rm sym.} := \frac{\mathfrak{q}-\mathfrak{q}^{-1}}{4}(\sigma^z_N-\sigma^z_1)+\frac{1}{2}\sum_{i=1}^{N-1}\left(\sigma^x_i\sigma^x_{i+1}+\sigma^y_i\sigma^y_{i+1}+\frac{\mathfrak{q}+\mathfrak{q}^{-1}}{2}\sigma^z_i\sigma^z_{i+1}\right)$$

is invariant under its $\mathfrak{q}$-deformation $U_{\mathfrak{q}}\mathfrak{sl}_2$.

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  is invariant under its $\mathfrak{q}$-deformation $U_{\mathfrak{q}}\mathfrak{sl}_2$.
- More generally, the hamiltonian densities

$$e_i = -\frac{1}{2}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \frac{\mathfrak{q}+\mathfrak{q}^{-1}}{2}(\sigma_i^z\sigma_{i+1}^z - 1)\right) - \frac{\mathfrak{q}-\mathfrak{q}^{-1}}{4}(\sigma_{i+1}^z - \sigma_i^z)$$

  such that

$$H_{\text{sym.}} = \frac{\mathfrak{q}+\mathfrak{q}^{-1}}{4}(N-1) - \sum_{i=1}^{N-1} e_i$$

  also commute with $U_{\mathfrak{q}}\mathfrak{sl}_2$...

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is invariant under its $\mathfrak{q}$-deformation $U_{\mathfrak{q}}\mathfrak{sl}_2$.

- More generally, the hamiltonian densities

$$e_i = -\frac{1}{2}\left(\sigma_i^x\sigma_{i+1}^x+\sigma_i^y\sigma_{i+1}^y+\frac{\mathfrak{q}+\mathfrak{q}^{-1}}{2}(\sigma_i^z\sigma_{i+1}^z-1)\right)-\frac{\mathfrak{q}-\mathfrak{q}^{-1}}{4}(\sigma_{i+1}^z-\sigma_i^z)$$

such that

$$H_{\text{sym.}} = \frac{\mathfrak{q}+\mathfrak{q}^{-1}}{4}(N-1)-\sum_{i=1}^{N-1}e_i$$

also commute with $U_{\mathfrak{q}}\mathfrak{sl}_2$...

- ... and generate a representation of $\mathrm{TL}_{\delta,N}$ with $\delta=\mathfrak{q}+\mathfrak{q}^{-1}$!

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For $\mathcal{X}_{l/r}$ we will take infinite-dimensional **Verma modules** of $U_{\mathfrak{q}}\mathfrak{sl}_2$.

Definition

Take $\alpha \in \mathbb{C}$ and set $\mathcal{V}_\alpha := \bigoplus_{0\leq n} \mathbb{C}\,|n\rangle$. Then $U_{\mathfrak{q}}\mathfrak{sl}_2$ acts on $\mathcal{V}_\alpha$ as

$$\mathrm{E}_{\mathcal{V}_\alpha}|n\rangle = [n]_{\mathfrak{q}}[\alpha-n]_{\mathfrak{q}}\,|n-1\rangle \ ,$$

$$\mathrm{F}_{\mathcal{V}_\alpha}|n\rangle = |n+1\rangle \ , \qquad [x]_{\mathfrak{q}} := \frac{\mathfrak{q}^x-\mathfrak{q}^{-x}}{\mathfrak{q}-\mathfrak{q}^{-1}}$$

$$\mathrm{K}^{\pm1}_{\mathcal{V}_\alpha}|n\rangle = \mathfrak{q}^{\pm(\alpha-1-2n)}\,|n\rangle \ .$$

One can show that we have the $U_{\mathfrak{q}}\mathfrak{sl}_2$ irrep decomposition

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- Take b_r the projector on $\mathcal{V}_{\alpha_r+1}$ acting on $\mathbb{C}^2 \otimes \mathcal{V}_{\alpha_r}$.

Then

$$b_l^2 = b_l \,, \qquad e_1 b_l e_1 = y_l e_1 \,, \qquad [b_l, e_i] = 0 \quad \text{for} \quad 2 \leq i \leq N-1$$

$$b_r^2 = b_r \,, \quad e_{N-1} b_r e_{N-1} = y_r e_{N-1} \,, \quad [b_r, e_i] = 0 \quad \text{for} \quad 1 \leq i \leq N-2$$

with

$$y_{l/r} = \frac{[\alpha_{l/r} + 1]_\mathfrak{q}}{[\alpha_{l/r}]_\mathfrak{q}} \,.$$

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- With

$$Y = \frac{\mathfrak{q}^{\alpha_l+\alpha_r+1} + \mathfrak{q}^{-\alpha_l-\alpha_r-1} - \mathsf{C}}{(\mathfrak{q}^{\alpha_l}-\mathfrak{q}^{-\alpha_l})(\mathfrak{q}^{\alpha_r}-\mathfrak{q}^{-\alpha_r})}$$

e_i, b_l and b_r define a representation of the (universal) two-boundary Temperley-Lieb algebra.

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Using the fusion rules

$$\mathcal{V}_\alpha \otimes \mathcal{V}_\beta = \bigoplus_{n \geq 0} \mathcal{V}_{\alpha+\beta-1-2n} \qquad \text{and} \qquad \mathcal{V}_\alpha \otimes \mathbb{C}^2 = \mathcal{V}_{\alpha+1} \oplus \mathcal{V}_{\alpha-1}$$

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we obtain

$$\mathcal{V}_{\alpha_l} \otimes (\mathbb{C}^2)^{\otimes N} \otimes \mathcal{V}_{\alpha_r} = \bigoplus_{M\geq 0} \mathcal{V}_{\alpha_l+\alpha_r-1+N-2M} \otimes \mathcal{Z}_M$$

where the $\mathcal{Z}_M$ are some multiplicity spaces of dimension

$$d_M := \dim \mathcal{Z}_M = \begin{cases} \displaystyle\sum_{k=0}^{M} \binom{N}{k} & \text{for} \quad 0 \leq M \leq N \\ 2^N & \text{for} \quad M \geq N \end{cases}$$

Since $\mathsf{C}_{\mathcal{V}_\alpha} = \mathfrak{q}^{\alpha} + \mathfrak{q}^{-\alpha}$,

$$Y_{\mathcal{Z}_M} = \frac{\left[M + 1 - \frac{N}{2}\right]_{\mathfrak{q}} \left[\alpha_l + \alpha_r - M + \frac{N}{2}\right]_{\mathfrak{q}}}{[\alpha_l]_{\mathfrak{q}}[\alpha_r]_{\mathfrak{q}}} := Y_M\,.$$

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Theorem (D.C., J.L. Jacobsen, A.M. Gainutdinov, H. Saleur '22)

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ii) For $M \geq N$, $\mathcal{Z}_M \cong \mathcal{W}_M$ and is irreducible.

Corollary

Define

$$H_{2b} := -\mu_l b_l - \mu_r b_r - \sum_{i=1}^{N-1} e_i$$

acting on $\mathcal{V}_{\alpha_l} \otimes (\mathbb{C}^2)^{\otimes N} \otimes \mathcal{V}_{\alpha_r}$ *and denote* $H^{(M)}_{\mathrm{n.d.}} := H_{\mathrm{n.d.}}(Y = Y_M)$.

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- The "quantization condition" on Y is precisely the Nepomechie condition.
- We have shown that it originates from $U_{\mathfrak{q}}\mathfrak{sl}_2$-fusion rules.

1. Loop models and lattice algebras

2. $U_q\mathfrak{sl}_2$-invariant realisation

3. Bethe ansatz

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- No need for a K-matrix: just use the affine R-matrix in Verma representation !
- The reference state is just the highest-weight vector $|\Uparrow\rangle := |0\rangle \otimes |\uparrow\rangle^{\otimes N} \otimes |0\rangle$.

$U_{\mathfrak{q}}\mathfrak{sl}_2$ admits a universal R-matrix

$$\mathsf{R} = \mathfrak{q}^{\frac{\mathsf{H}\otimes\mathsf{H}}{2}} \sum_{k\geq 0} \frac{(\mathfrak{q}-\mathfrak{q}^{-1})^{2k}}{\prod_{n=1}^{k}(\mathfrak{q}^{n}-\mathfrak{q}^{-n})} \mathfrak{q}^{k(k-1)/2} \mathsf{E}^{k} \otimes \mathsf{F}^{k} .$$

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For two representations $\mathcal{X}$ and $\mathcal{Y}$, the operators

$$\begin{aligned} P_{\mathcal{X},\mathcal{Y}} \circ \mathsf{R}_{\mathcal{X},\mathcal{Y}} \;:\; \mathcal{X}\otimes\mathcal{Y} &\to \mathcal{Y}\otimes\mathcal{X} \,, \\ \mathsf{R}^{-1}_{\mathcal{Y},\mathcal{X}} \circ P_{\mathcal{X},\mathcal{Y}} \;:\; \mathcal{X}\otimes\mathcal{Y} &\to \mathcal{Y}\otimes\mathcal{X} \end{aligned}$$

where

$$\begin{aligned} P_{\mathcal{X},\mathcal{Y}} \;:\; \mathcal{X}\otimes\mathcal{Y} &\to \mathcal{Y}\otimes\mathcal{X} \\ x\otimes y &\mapsto y \otimes x \end{aligned}$$

are $U_{\mathfrak{q}}\mathfrak{sl}_2$-intertwiners.

Introduce, for any representation $\mathcal{X}$ of $U_\mathfrak{q}\mathfrak{sl}_2$,

$$R_{\mathcal{X},\mathbb{C}^2}(u) := e^u \mathrm{R}_{\mathcal{X},\mathbb{C}^2} - e^{-u} P_{\mathbb{C}^2,\mathcal{X}} \circ \mathrm{R}^{-1}_{\mathbb{C}^2,\mathcal{X}} \circ P_{\mathcal{X},\mathbb{C}^2},$$
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Then for any three representations $\mathcal{X}_{1,2,3}$ of $U_{\mathfrak{q}}\mathfrak{sl}_2$ with at least two of them isomorphic to $\mathbb{C}^2$ the Yang-Baxter equation

$$R_{\mathcal{X}_1,\mathcal{X}_2}(u-v)R_{\mathcal{X}_1,\mathcal{X}_3}(u)R_{\mathcal{X}_2,\mathcal{X}_3}(v) = R_{\mathcal{X}_2,\mathcal{X}_3}(v)R_{\mathcal{X}_1,\mathcal{X}_3}(u)R_{\mathcal{X}_1,\mathcal{X}_2}(u-v)$$

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is satisfied.

This sufficient to build the monodromy and transfer matrix !

Define the monodromy

$$\mathcal{T}(u) := T(u)\hat{T}(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix},$$

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By construction $t(u)$ is $U_{\mathfrak{q}}\mathfrak{sl}_2$-invariant and

$$H_{2b} = c_1 + c_2 \frac{\mathrm{d}}{\mathrm{d}u}\Big|_{u=\hbar/2} t(u)$$

with $\mathfrak{q} = e^{\hbar}$, c_1, c_2 some explicit constants and $\mu_{l/r}$ related to $\zeta_{l/r}$.

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iff the rapidities $\{v_m\}_{1\le m\le M}$ satisfy the **Bethe ansatz equations (BAE)**

$$\frac{\Delta_l(v_m)\Delta_r(v_m)}{\Delta_l(-v_m)\Delta_r(-v_m)}\left(\frac{\sinh\big(v_m+\hbar/2\big)}{\sinh\big(v_m-\hbar/2\big)}\right)^{2N} = \prod_{\substack{k=1\\k\neq m}}^{M}\frac{\sinh(v_m-v_k+\hbar)\sinh(v_m+v_k+\hbar)}{\sinh(v_m-v_k-\hbar)\sinh(v_m+v_k-\hbar)}$$

$$\Delta_{l/r}(u) := 1 - \mu_{l/r}\frac{\sinh\big(u-\hbar/2\big)\sinh\Big(u+\hbar(\alpha_{l/r}-1/2)\Big)}{\sinh(\hbar)\sinh\Big(\hbar\alpha_{l/r}\Big)}$$

$$\propto \sinh\left(u+\hbar\frac{\alpha_{l/r}-1}{2}-\zeta_{l/r}\right)\sinh\left(u+\hbar\frac{\alpha_{l/r}-1}{2}+\zeta_{l/r}\right).$$

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- One needs to transform $\alpha_{l/r}, \mu_{l/r} \to -\alpha_{l/r}, -\mu_{l/r}$ and $M \to N-M-1$ in the BAE.
- We can even reach "negative" values of M.

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Denote E_j the ground state of $H_b|_{\mathcal{H}_{N/2-j}}$. Then

$$E_j = Ne_{\mathrm{b}} + E_{\mathrm{s}} + \frac{\pi v_{\mathrm{F}}}{N}\left(-\frac{c}{24} + h_{\alpha,\alpha+2j}\right) + o(1/N^2)\,,$$

where

- e_{b} is the bulk energy per site,
- E_{s} is the surface energy,
- $v_{\mathrm{F}} = p \sin\frac{\pi}{p}$ is the Fermi velocity,
- $c = 1 - \frac{6}{p(p-1)}$ is the central charge,
- $h_{r,s} = \frac{(pr-(p-1)s)^2-1}{4p(p-1)}$ are conformal weights.

By the Cardy formula these corrections provide the CFT spectrum in the continuum

$$\lim_{N\to\infty} \mathrm{tr}_{\mathcal{H}_{N/2-j}}\, q^{\frac{N}{\pi v_F}(H_b - Ne_b - E_s)} = \frac{q^{-\frac{c}{24}+h_{\alpha,\alpha+2j}}}{\prod_{n=1}^{+\infty}(1-q^n)}\,.$$

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Application

Loop model partition function on a cylinder of parameter $\tau = M/N$:

$$Z_\tau(\delta, y) = \sum_{j\in\mathbb{Z}} \frac{\sin\frac{\pi(\alpha+1)}{p}}{\sin\frac{\pi\alpha}{p}} \frac{q^{-\frac{c}{24}+h_{\alpha,\alpha+2j}}}{\prod_{n=1}^{+\infty}(1-q^n)}$$

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- Does not depend on the coupling constant μ.
- Related to spanning forests and (η, ξ) ghost CFT for $p = 2$.

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Coulomb gas/loop model prediction:

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- Additional symmetry in the continuum ?

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- Relation to loop models, 2D random geometry, ASEP...

Backstage : "Dual" BAE

$$\frac{\overline{\Delta}_l(v_m)\overline{\Delta}_r(v_m)}{\overline{\Delta}_l(-v_m)\overline{\Delta}_r(-v_m)}\left(\frac{\sinh\big(v_m+\hbar/2\big)}{\sinh\big(v_m-\hbar/2\big)}\right)^{2N}=\prod_{\substack{k=1\\k\neq m}}^{\overline{M}}\frac{\sinh(v_m-v_k+\hbar)\sinh(v_m+v_k}{\sinh(v_m-v_k-\hbar)\sinh(v_m+v_k}$$

with $\overline{M}=N-M-1$ and

$$\bar{\Delta}_{l/r}(u)=:1-\mu_{l/r}\frac{\sinh\big(u-\hbar/2\big)\sinh\Big(u-\hbar(\alpha_{l/r}+1/2)\Big)}{\sinh(\hbar)\sinh\Big(\hbar\alpha_{l/r}\Big)}$$

$$=\frac{\sinh\Big(u-\hbar\frac{\alpha_{l/r}+1}{2}-\zeta_{l/r}\Big)\sinh\Big(u-\hbar\frac{\alpha_{l/r}+1}{2}+\zeta_{l/r}\Big)}{\sinh\Big(\frac{\hbar\alpha_{l/r}}{2}-\zeta_{l/r}\Big)\sinh\Big(\frac{\hbar\alpha_{l/r}}{2}+\zeta_{l/r}\Big)}\,.$$

They come from the isomorphism

$$2\mathrm{B}_{\delta,y_{l/r},Y_M,N}\cong 2\mathrm{B}_{\delta,\delta-y_{l/r},\delta-y_l-y_r+Y_{\overline{M}},N}\,,\qquad b_{l/r}\to 1-b_{l/r}\,.$$

Backstage : Explicit expressions of $b_{l/r}$

We have

$$b_l = \frac{1}{[\alpha_l]_q}\begin{pmatrix} \frac{q^{\alpha_l}-q^{-1}K^{-1}}{q-q^{-1}} & F \\ qK^{-1}E & \frac{qK^{-1}-q^{-\alpha_l}}{q-q^{-1}} \end{pmatrix}, \quad b_r = \frac{1}{[\alpha_r]_q}\begin{pmatrix} \frac{qK-q^{-\alpha_r}}{q-q^{-1}} & qKF \\ E & \frac{q^{\alpha_r}-q^{-1}K}{q-q^{-1}} \end{pmatrix}$$

written as 2×2 matrices with elements in $\mathrm{End}(\mathcal{V}_{\alpha_{l/r}})$.

$b_{l/r}$ is the projector on the $\mathcal{V}_{\alpha_{l/r}+1}$ factor of $\mathcal{V}_{\alpha_l}\otimes\mathbb{C}^2$ or $\mathbb{C}^2\otimes\mathcal{V}_{\alpha_r}$.