An Ising-type formulation of the six-vertex model

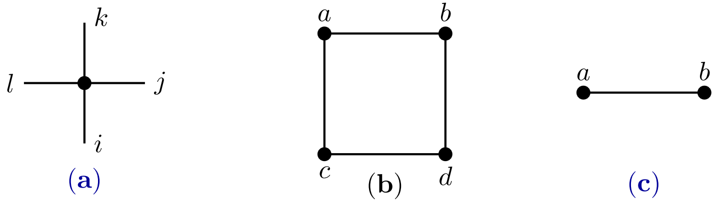
Vladimir Bazhanov and Sergey Sergeev

Australian National University

Integrability in Condensed Matter Physics and Quantum Field Theory,
SwissMAP, Les Diablerets, February 2023

Motivation

- Three types of integrable 2D square lattice models with local Boltzmann weights assigned to (a) **vertices**, (b) **faces** or (c) **edges** of the lattice,



- They are called as (a) **vertex-**, (b) **interaction-round-a-face (IRF)** and (c) **Ising-type** models.

- Integrability: **Yang-Baxter equation (YBE)** (vertex/IRF form), Onsager's **Star-triangle relation (STR)** (Ising type models).

- **Vertex/IRF: Quantum Groups** (Drinfeld86, Jimbo86, Faddeev-Reshetikhin-Takhtajan89). Infinite series of trigonometric R-matrices related to highest weight evaluation reps of affine Lie algebras.

Motivation

- We consider the simplest case of $U_q(\widehat{sl}(2))$. The standard R-matrix is $\mathcal{R}(\lambda|s_1,s_2)$. Here $s_{1,2}$ define highest weights (spins) of two evaluation reps, λ is the spectral parameter.

$$\mathcal{R}(\lambda/\mu\,|s_1,s_2) = (\pi_{s_1}(\lambda)\otimes\pi_{s_2}(\mu))\circ\boldsymbol{\mathcal{R}}$$

- The R-matrix $\mathcal{R}(\lambda|s_1,s_2)$ leads to **higher spin 6-vertex model**. Generic values of s_1 and s_2 correspond to infinite dimensional reps. Finite-dimesional reductions when $2s_1$, $2s_2\in\mathbb{Z}_{\geq 0}$. In particular, $s_1=s_2=\frac{1}{2}$ leads to the usual 6-vertex model (with two spin states).

- The concepts of "higher-spin R-matrices" and "fusion" were developed in the 80's by Stroganov, Fateev, Zamolodchikov, Kulish, Sklyanin, Kirillov, Reshetikhin, VB, Date, Jimbo, Miwa, Okado, ...

Motivation

- **Star-triangle relation:** Not entirely natural in the standard Quantum Groups setup. An algebraic nature of **STR** and related Ising-type models is much less clear in comparison with the vertex model.

- Any Ising-type model can be easily converted into IRF/vertex model, e.g., by "star-square" transformation (Baxter85, Boos-Stroganov98).

- How to construct an Ising-type formulation for **an arbitrary integrable IRF/vertex model** (e.g., the higher spin the 6-vertex model, mentioned above)?

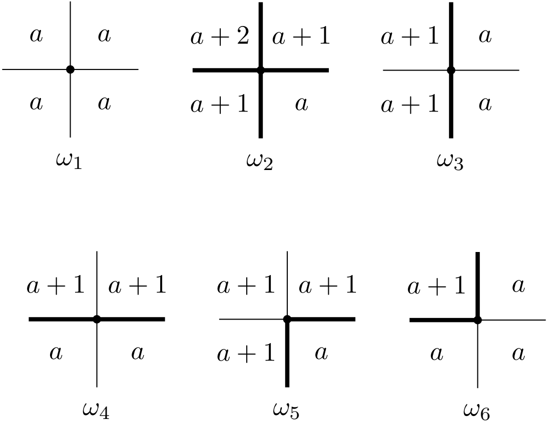
- New algebraic structures are coming from **3D interpretation of the Yang-Baxter equation based on Zamolodchikov tetrahedron equation** (Baxter, VB, Stroganov, Kashaev, Mangazeev, Sergeev, Korepanov, Hietarinta, Maillet, Nijhoff, Kuniba, Okado, ...)

- Our starting point: **Mangazeev'14 formula for $\mathcal{R}(\lambda|s_1, s_2)$ via a terminating basic hypergeometric series ${}_4\varphi_3$.**

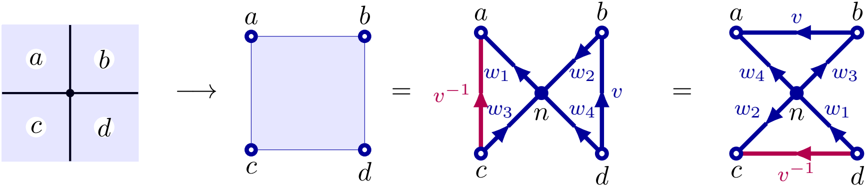
Two-state six-vertex model

Vertex configurations:

1. by the edge arrangements (with bold and thin edges)
2. by integer “heights” arrangements, $a \in \mathbb{Z}$, as in the unrestricted solid-on-solid (SOS) model.



SOS $\to$ IRF $\to$ "Ising-type star", $\qquad w_i(n) = 0, \quad n < 0,$



$$\mathcal{W}(a,b,c,d) = \omega_1 \frac{v(b-d)}{v(a-c)} \sum_{n=\max(b,c)}^{a} w_1(a-n)\, w_2(n-b)\, w_3(n-c)\, w_4(n-d)$$

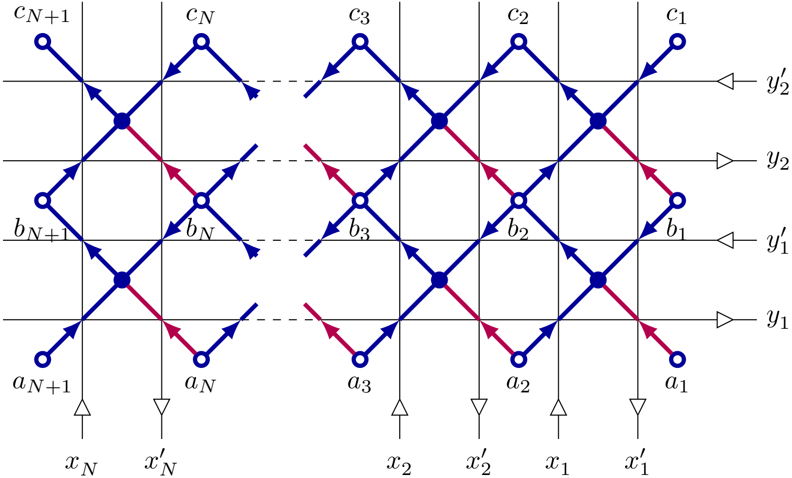
$$= \omega_1 \frac{v(a-b)}{v(c-d)} \sum_{n=d}^{\min(b,c)} w_4(a-n)\, w_3(b-n)\, w_2(c-n)\, w_1(n-d)\,,$$

$$w_1(0) = w_2(0) = w_3(0) = w_4(0) = 1\,, \qquad w_1(1) = \epsilon\,(\omega_5^2 - \omega_3\omega_4)/(\omega_1\omega_5)\,,$$

$$w_2(1) = \epsilon\,\omega_3/\omega_5\,, \quad w_3(1) = \epsilon\,\omega_4/\omega_5\,, \quad w_4(1) = \omega_5/(\epsilon\,\omega_1)\,,$$

$$w_4(2) = \omega_5^2\,(\omega_1\omega_2 + \omega_3\omega_4 - \omega_5^2)/(\epsilon^2\,\omega_1^2\,\omega_3\,\omega_4)\,, \qquad \epsilon = v(1)/v(0)\,.$$

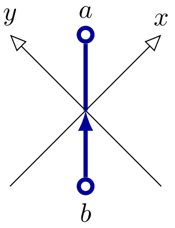
A new Ising-type model



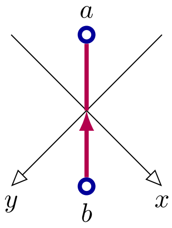
Diagonal square lattice with directed edges. Integer spins at "black" and "white" sites. Directed thin lines carrying spectral variables $x_1, x'_1, x_2, x'_2, \ldots$, $y_1, y'_1, y_2, y'_2, \ldots$.

Two types of edges distinguished by relative orientation of the edge and thin lines, passing through the edge.

(i)



$= V_{x/y}(a-b)\,,$ (ii)



$= \dfrac{1}{V_{x/y}(a-b)}\,.$

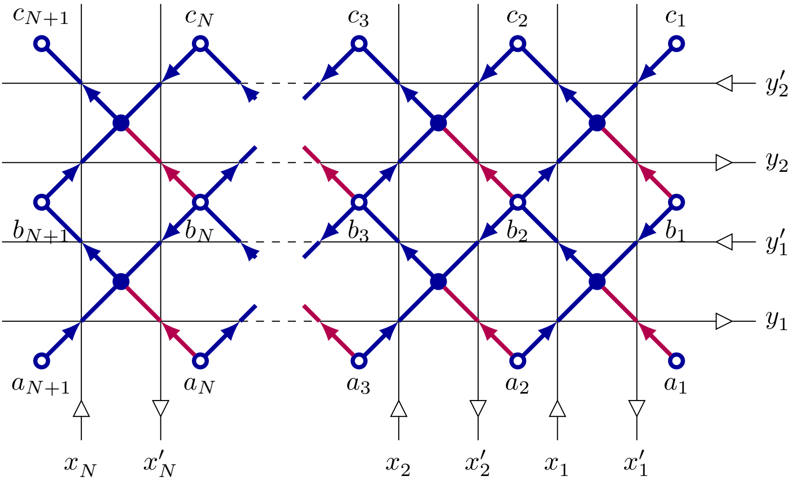
$$V_x(n) = \left(\frac{q}{x}\right)^n \frac{(x^2;q^2)_n}{(q^2;q^2)_n}\,, \qquad n\in\mathbb{Z}\,,$$

$$(x;q^2)_n = \prod_{j=0}^{n-1}(1-x\,q^{2j})\,, \qquad (x;q^2)_{-n} = \frac{1}{(xq^{-2n};q^2)_n}\,,$$

and q is an arbitrary parameter of the model. Note that

$$V_x(n)\equiv 0\,, \qquad \text{for } n<0\,,$$

Boundary conditions



$$a_1 \le a_2 \le a_3 \le \dots \qquad a_1 \le b_1 \le c_1 \le \dots$$

$$b_{N+1} - a_{N+1} = b_1 - a_1 \,, \qquad c_{N+1} - b_{N+1} = c_1 - b_1 \,, \qquad \text{etc.}$$

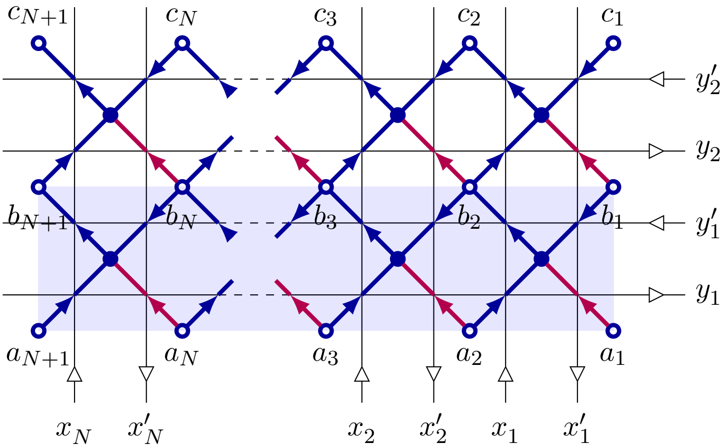
Partition function

The lattice spin configuration is called *admissible* if all the spin differences appearing in the edge weight functions are non-negative (i.e., for admissible configurations the values of spins cannot decrease along the edge directions).

$$Z = \sum_{\text{(admissible spins)}} w_{\text{fields}} \prod_{\text{(edges)}} \text{(edge weights)}\,.$$

For the above boundary conditions the admissible configurations exhaust all lattice spin configurations with non-zero Boltzmann weight.

Transfer matrices



$$\left(\mathbb{T}(y, y' \,|\, x_N, x'_N, \ldots, x_1, x'_1)\right)^{b_{N+1}, b_N, \ldots, b_1}_{a_{N+1}, a_N, \ldots, a_1}$$

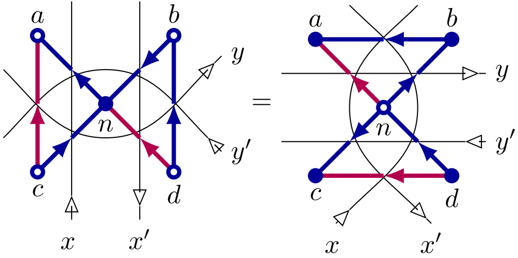
form 2-parameter commuting family.

Integrability condition: the "star-star relation"

$$\mathcal{W}(a,b,c,d\,|\,x,x',y,y') = \overline{\mathcal{W}}(a,b,c,d\,|\,x,x',y,y')\,.$$

$$\mathcal{W}(a,b,c,d) = \frac{V_{y/y'}(b-d)}{V_{y/y'}(a-c)} \sum_{n=\max(b,c)}^{a} \frac{V_{x/y'}(a-n)V_{y'/x'}(n-b)V_{y/x}(n-c)}{V_{y/x'}(n-d)}\,.$$

$$\overline{\mathcal{W}}(a,b,c,d) = \frac{V_{x/x'}(a-b)}{V_{x/x'}(c-d)} \sum_{n=d}^{\min(b,c)} \frac{V_{x/y'}(n-d)V_{y'/x'}(c-n)V_{y/x}(b-n)}{V_{y/x'}(a-n)}\,.$$



Follows from two application of the second Sears's transformation formula for basic hypergeometric series (1951).

Mangazeev's higher spin R-matrix (2014)

$$\mathcal{R}(x,x',y,y')_{i_1,i_2}^{j_1,j_2} = \delta_{i_1+i_2,j_1+j_2}\, \mathcal{W}(i_1+i_2,j_2,i_1,0\,|x,x',y,y')\,.$$

$$i_1 = c-d\,,\qquad i_2 = a-c\,,\qquad j_1 = a-b\,,\qquad j_2 = b-d\,,$$

$$x = \lambda_1\, q^{-s_1}\,,\quad x' = \lambda_1\, q^{s_1}\,,\quad y = \lambda_2\, q^{-s_2}\,,\quad y' = \lambda_2\, q^{s_2}\,,$$

$$\mathcal{R}(\lambda_1/\lambda_2\,|\,s_1,s_2) = \mathcal{R}(x,x',y,y')\,,$$

$$\mathcal{R}_{12}(\lambda_1/\lambda_2\,|\,s_1,s_2)\;\; \mathcal{R}_{13}(\lambda_1/\lambda_3\,|\,s_1,s_3)\;\; \mathcal{R}_{23}(\lambda_2/\lambda_3\,|\,s_2,s_3)$$

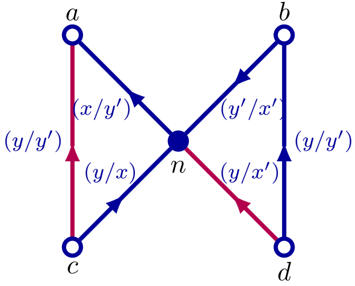
$$= \mathcal{R}_{23}(\lambda_2/\lambda_3\,|\,s_2,s_3)\;\; \mathcal{R}_{13}(\lambda_1/\lambda_3\,|\,s_1,s_3)\;\; \mathcal{R}_{12}(\lambda_1/\lambda_2\,|\,s_1,s_2)$$

Integrability condition: the "star-star relation"

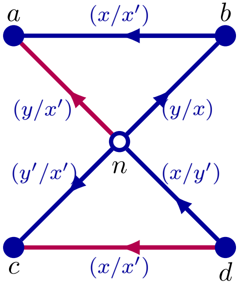
$$\mathcal{W}(a,b,c,d\,|\,x,x',y,y') = \overline{\mathcal{W}}(a,b,c,d\,|\,x,x',y,y')\,.$$

$$\mathcal{W}(a,b,c,d) = \frac{V_{y/y'}(b-d)}{V_{y/y'}(a-c)} \sum_{n=\max(b,c)}^{a} \frac{V_{x/y'}(a-n)V_{y'/x'}(n-b)V_{y/x}(n-c)}{V_{y/x'}(n-d)}\,.$$

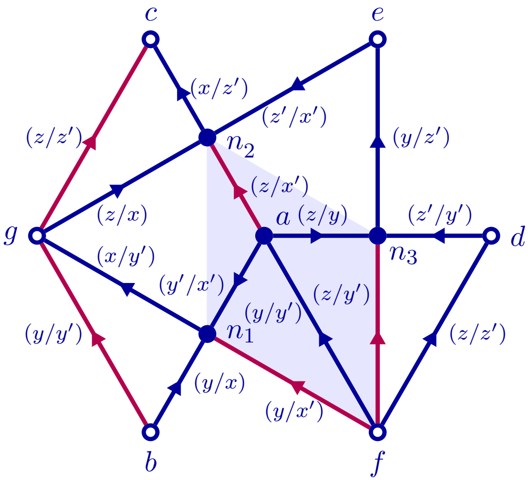
$$\overline{\mathcal{W}}(a,b,c,d) = \frac{V_{x/x'}(a-b)}{V_{x/x'}(c-d)} \sum_{n=d}^{\min(b,c)} \frac{V_{x/y'}(n-d)V_{y'/x'}(c-n)V_{y/x}(b-n)}{V_{y/x'}(a-n)}\,.$$

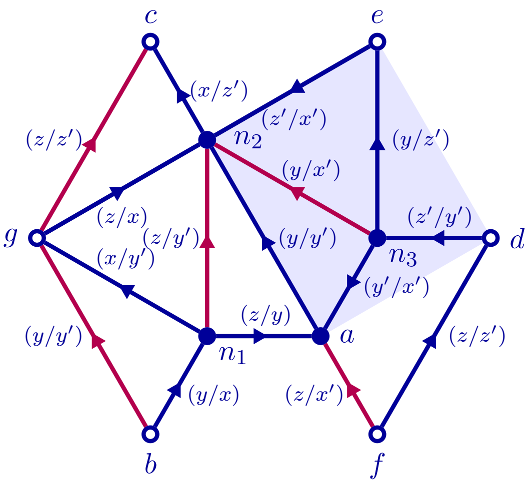


=

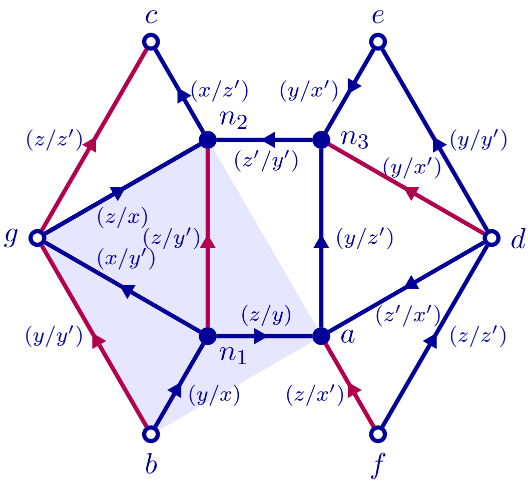


$$\sum_{a\in\mathbb{Z}} \mathcal{W}(g,a,b,f\,|\,\boldsymbol{x},\boldsymbol{y})\;\mathcal{W}(c,e,g,a\,|\,\boldsymbol{x},\boldsymbol{z})\;\mathcal{W}(e,d,a,f\,|\,\boldsymbol{y},\boldsymbol{z}) \;=$$

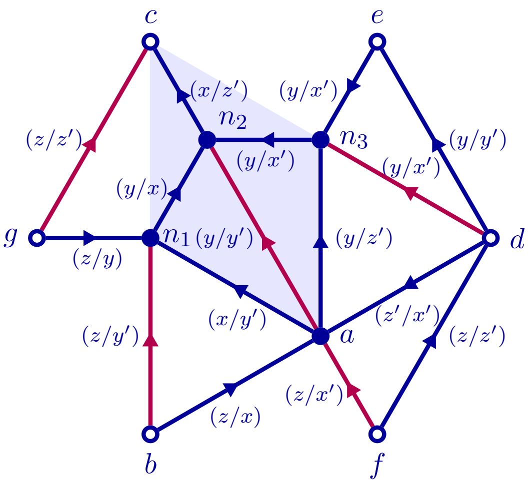


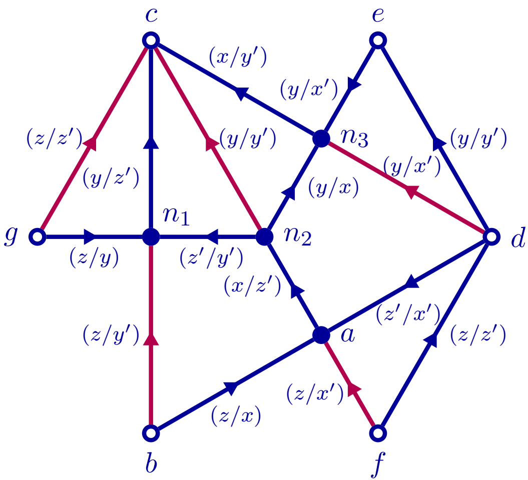


c
e
(x/z')
(z'/x')
(z/z')
n_2
(y/z')
(y/x')
(z/x)
(z'/y')
g
(z/y')
(y/y')
d
(x/y')
n_3
(y'/x')
(z/y)
a
(y/y')
(z/z')
n_1
(y/x)
(z/x')
b
f



c
e
(x/z')
(y/x')
n_2
(z/z')
n_3
(y/y')
(z'/y')
(y/x')
(z/x)
g
(z/y')
(y/z')
d
(x/y')
(z/y)
(z'/x')
(y/y')
a
(z/z')
n_1
(y/x)
(z/x')
b
f

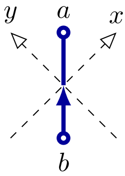

c
e
(x/z')
(y/x')
n_2
(z/z')
n_3
(y/y')
(y/x')
(y/x')
(y/x)
g
n_1
(y/y')
(y/z')
d
(z/y)
(x/y')
(z'/x')
(z/y')
a
(z/z')
(z/x')
(z/x)
b
f

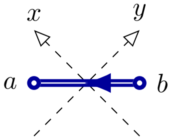


The star-star relation implies the YBE and commutativity of transfer matrices

Star-triangle relation (STR)

STR involves the edges of the type (i) and type (iii)

(i) : 

$= V_{x/y}(a-b)$, (iii) : 

$= W_{x,y}(a-b)$.

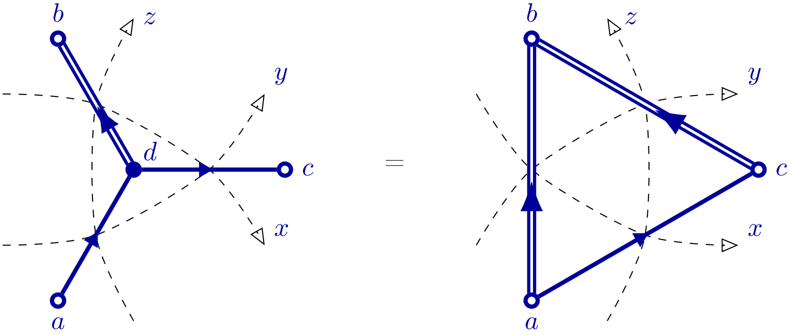
$$W_{x,y}(n) = \left(\frac{y}{x}\right)^n \frac{(x^2;q^2)_n}{(y^2;q^2)_n}, \qquad n \in \mathbb{Z}. \tag{1}$$

The new W weights possess the following symmetries

$$W_{x,y}(n) = W_{q/y,q/x}(-n) = \frac{1}{W_{y,x}(n)} = \frac{1}{W_{q/x,q/y}(-n)}, \qquad n \in \mathbb{Z}. \tag{2}$$

$$W_{x,q}(n) = V_x(n), \qquad W_{q,x}(n) = \frac{1}{V_x(n)}.$$

Note that W does not have a "difference property"!

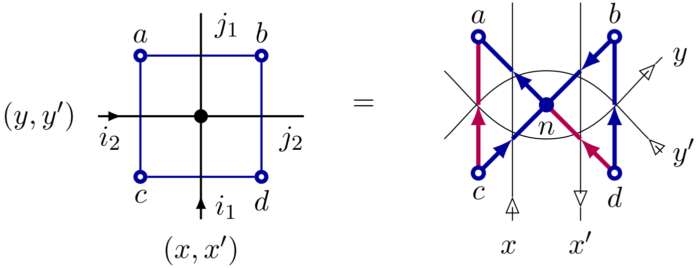


$$\sum_d V_{y/z}(d-a)\, W_{z,x}(b-d)\, V_{x/y}(c-d) \;=\; W_{z,y}(b-c)\, V_{x/z}(c-a)\, W_{y,x}(b-a)\,,$$

$$\sum_d V_{y/z}(a-d)\, W_{z,x}(d-b)\, V_{x/y}(d-c) \;=\; W_{z,y}(c-b)\, V_{x/z}(a-c)\, W_{y,x}(a-b)\,.$$

Pfaff-Saalshchütz-Jackson summation formula (1797-1890-1910)

$${}_3\phi_2\left(\begin{array}{ccc} q^{-2n}, & \mathsf{a}, & \mathsf{b} \\ & \mathsf{c}, & q^{2-2n}\,\mathsf{a}\,\mathsf{b}/\mathsf{c} \end{array}\middle|\, q^2, q^2\right) = \frac{(\mathsf{c}/\mathsf{a},\, \mathsf{c}/\mathsf{b}\,;\, q^2)_n}{(\mathsf{c},\; \mathsf{c}/\mathsf{a}\,\mathsf{b}\,;\, q^2)_n}$$



$$\mathcal{R}(x,x',y,y')^{j_1,j_2}_{i_1,i_2} = \delta_{i_1+i_2,j_1+j_2}\,\mathcal{W}(i_1+i_2,j_2,i_1,0\,|x,x',y,y')\,.$$

$$i_1 = c-d\,,\qquad i_2 = a-c\,,\qquad j_1 = a-b\,,\qquad j_2 = b-d\,,$$

$$x' = q^{2s_1}x\,,\qquad y' = q^{2s_2}y\,,\qquad 2s_2\in\mathbb{Z}_{\geq 0}$$

Reduction in the second space:

$$2s_2\in\mathbb{Z}_{\geq 0}\;:\quad \mathcal{R}(\lambda\,|\,s_1,s_2)^{j_1,j_2}_{i_1,i_2} = 0\,,\quad 0\leq i_2\leq 2s_2,\quad j_2>2s_2\,,\quad \forall\, i_1,j_1\,.$$

The LLR relations

$$\boldsymbol{\mathcal{L}}_2(y,y',z)\;\boldsymbol{\mathcal{L}}_1(x,x',z)\;\mathcal{R}_{12}(x,x',y,y') = \mathcal{R}_{12}(x,x',y,y')\;\boldsymbol{\mathcal{L}}_1(x,x',z)\;\boldsymbol{\mathcal{L}}_2(y,y',z)\,.$$

$$\boldsymbol{\mathcal{L}}(x,x',y) = \begin{pmatrix} \mu\, q^{\frac{H}{2}} - \mu^{-1}\, q^{-\frac{H}{2}} & F \\ E & \mu\, q^{-\frac{H}{2}} - \mu^{-1}\, q^{\frac{H}{2}} \end{pmatrix}, \quad \mu = (xx')^{\frac{1}{2}}/y\,.$$

Algebra $U_q(sl_2)$:

$$[H,E] = 2E\,, \qquad [H,F] = -2F\,, \qquad [E,F] = (q-q^{-1})\,(q^H - q^{-H})\,.$$

Infinite-dimensional highest weight representation $\pi^+_{s_1}$ of this algebra (the parameter s_1 is defined by $x'/x = q^{2s_1}$),

$$\pi^+_{s_1}[H]\,|j\rangle = (2s_1-2j)\,|j\rangle\,,\ \pi^+_{s_1}[E]\,|j\rangle = [q^{2s_1+1-j}]\,|j-1\rangle\,,\ \pi^+_{s_1}[F]\,|j\rangle = [q^{j+1}]\,|j+1\rangle\,,$$

basis vectors $|j\rangle \in \mathbb{C}^{(\infty)}$, $j = 0,1,2,\ldots,\infty$.

Symmetry transformation: change the frame!

Multiply by $q^{\frac{H}{2}}$

$$\widetilde{\mathcal{L}} = \begin{pmatrix} q^{-s_1}\left(\mu\, q^H - \mu^{-1}\right) & \mu^{-1}\, F\, q^{\frac{H}{2}} \\ \mu\, E\, q^{\frac{H}{2}} & q^{+s_1}\left(\mu - \mu^{-1}\, q^H\right) \end{pmatrix}$$

The Weyl algebra

$$\boldsymbol{u}\,\boldsymbol{v} = q^2\,\boldsymbol{v}\,\boldsymbol{u}\,.$$

Choosing the representation

$$\boldsymbol{u}\,|\,j\,\rangle = |\,j\,\rangle\, q^{2s_1-2j}\,, \quad \boldsymbol{v}\,|\,j\,\rangle = |\,j-1\,\rangle\,,$$

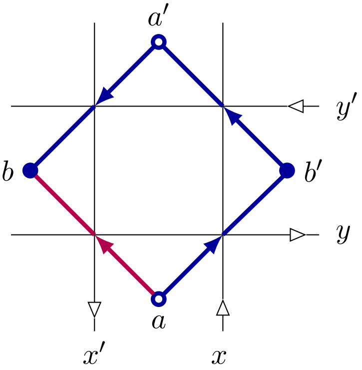
$$\widetilde{\mathcal{L}} = \begin{pmatrix} \mu\, q^{-s_1}\,\boldsymbol{u} - \mu^{-1}\, q^{-s_1} & \mu^{-1}\, q^{s_1}\,\boldsymbol{v}^{-1}\,(\boldsymbol{u} - q^{-2s_1}) \\ \mu\, q^{s_1}\,\boldsymbol{v}\,(1 - q^{-2s_1}\,\boldsymbol{u}) & \mu\, q^{s_1} - \mu^{-1}\, q^{s_1}\,\boldsymbol{u} \end{pmatrix}.$$

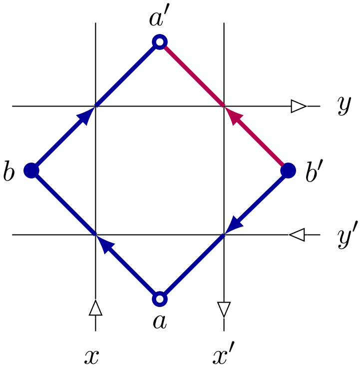
Smart choice:

$$\boldsymbol{u}\,|\,a\,\rangle = |\,a-1\,\rangle\,, \quad \boldsymbol{v}\,|\,a\,\rangle = |\,a\,\rangle\, q^{2a}\,, \quad a \in \mathbb{Z}\,.$$

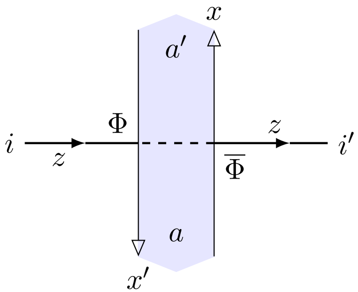
Factorized R-matrices

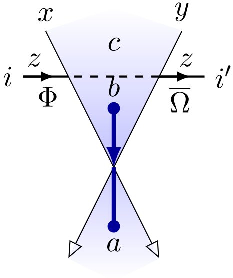
$$\mathbb{R}(\boldsymbol{x},\boldsymbol{y})_{a\,,\,b}^{a'\,,\,b'} \;=\; \frac{V_{y'/x'}(b-a')V_{x/y'}(a'-b')V_{y/x}(b'-a)}{V_{y/x'}(b-a)}\,,$$



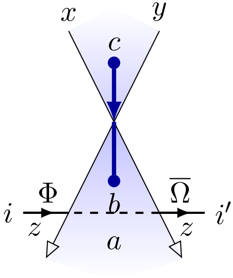


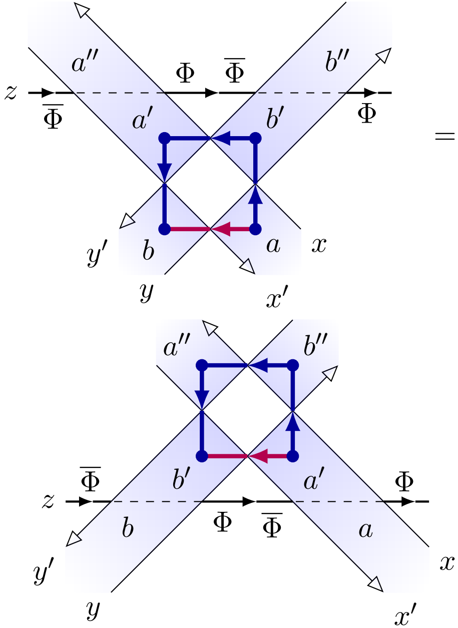
$$\left(\mathbb{L}(x,x',z)_a^{a'}\right)_{i,i'} = \Phi(x',z)_{a,i}^{a'}\,\overline{\Phi}(x,z)_{a,i'}^{a'} =$$





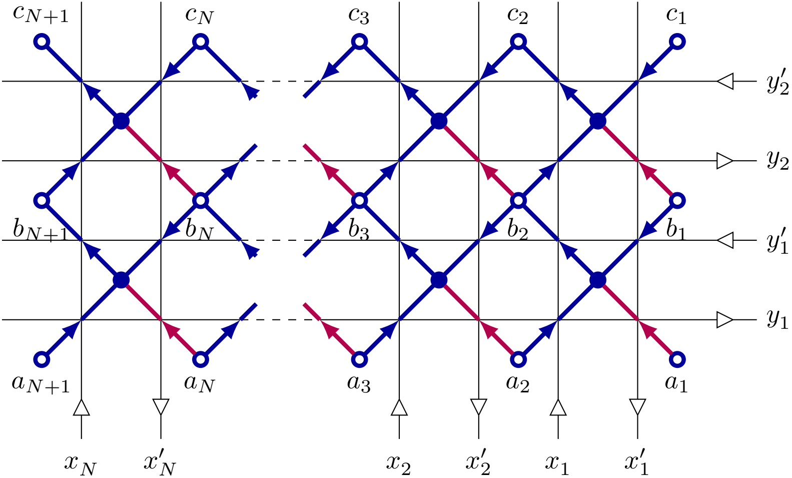
=





a''
b''
z
Φ
$\overline{\Phi}$
$\overline{\Phi}$
Φ
a'
b'
$=$
y'
b
a
x
y
x'
a''
b''
$\overline{\Phi}$
b'
a'
Φ
z
b
Φ
$\overline{\Phi}$
a
x
y'
y
x'

A new Ising-type model?



Square lattice can be formed by either 4-edge "stars" (two types) or by two types of "diamonds" (box diagrams). For R-matrices it is like going from momentum to coordinate representation.

Coordinate space one-loop 4-point Green functions in QFT, (for R-matrix: shift operator Cartan elements)

$$W(x_1,x_2,x_3,x_4) = G(x_1-x_2)\,G(x_2-x_3)\,G(x_3-x_4)\,G(x_4-x_1)$$

where $G(x)=G(-x)$ is the 2-point function in a scalar QFT. The Fourier transform

$$\int \Big(\prod_{i=1}^{4} e^{ik_i x_i}\, d^n x_i\Big) W(x_1,x_2,x_3,x_4) \;= \delta^{(n)}(k_1+k_2+k_3+k_4)\, \widetilde{W}(k_a,k_b,k_c,k_d)$$

The momentum space four-point function (for R-matrix: diagonal Cartan elements)

$$\widetilde{W}(k_a,k_b,k_c,k_d) = \int d^n k\, \widetilde{G}(k-k_a)\, \widetilde{G}(k-k_b)\, \widetilde{G}(k-k_c)\, \widetilde{G}(k-k_d)\,.$$

where the new variables k_a,k_b,k_c,k_d are chosen such that

$$k_1 = k_c - k_d\,,\quad k_2 = k_a - k_c\,,\quad k_3 = k_b - k_a\,,\quad k_4 = k_d - k_b\,,$$

$$G(x) = \int \frac{d^n k}{(2\pi)^n}\, e^{ikx}\, \widetilde{G}(k)\,.$$

The relation between two R-matrices

$$\sum_{a',b'\in\mathbb{Z}} \mathbb{R}(x,x',y,y')_{a\ ,\ b}^{a',b'} \left(\frac{x}{x'}q^{2j_1}\right)^{a'} \left(\frac{y}{y'}q^{2j_2}\right)^{b'} = \sum_{i_1,i_2\geq 0} \left(\frac{x}{x'}q^{2i_1}\right)^{a} \left(\frac{y}{y'}q^{2i_2}\right)^{b} \left(\frac{y'^{j_1}x^{j_2}}{y^{i_1}x'^{i_2}}\right) \mathcal{R}(x,x',y,y')_{j_1,j_2}^{i_1,i_2}$$

Relations of this type commonly appear as (self-) duality transformations in the Ising-type lattice model (Baxter:85,Wu:82,Boos-Stroganov:98).

Further contributions into algebraic theory of 6-vertex model:

- Construction of Baxter's $\mathbf{Q}$-operators

$$\varphi(y,y')\mathbf{T}(y,y') = \mathbf{Q}(y')\,\overline{\mathbf{Q}}(y)\,.$$

(Baxter:72,VB-Lukyanov-Zamolodchikov:98, Fabricius-McCoy:03,Boos-Göhmann-Klümper-Nirov-Razumov:14,Mangazeev:14)

- Bethe Ansatz (coordinate & algebraic)

$$\mathbf{\Psi} = \sum_{1\le r_1\le r_2\le\cdots\le r_M\le N} \Psi(r_1,r_2,\ldots,r_M)\;\widehat{\sigma}^+_{r_M}\;\cdots\;\widehat{\sigma}^+_{r_1}\,\mathbf{\Psi}_0\,, \qquad M\ge 0\,,$$

(Baxter:71, Gaudin-McCoy-Wu:81, Korepin:82, Takhtajan-Faddeev:79, Slavnov:2007, Kotousov-Lukyanov:20)

- Connection to chiral Potts model (six-vertex model as a descendant of the six-vertex model)

  (Au-Yang-McCoy-Perk-Tang-Yan-Sah-Baxter-87, VB-Kashaev-Mangazeev-Stroganov:89-90, Date-Jimbo-Miki-Miwa:90)

- Generalizations to other affine algebras. Already developed: $U_q(sl(n))$ (reproduce symmetric tensor R-matrices Bosnjak-Mangazeev:16)

- Hidden 3D structure of Yang-Baxter equation for other affine algebras and superalgebras (Sergeev:99-09,Kuniba-Okado-Sergeev:2015,...)

- **Plausibly, all vertex models associated with quantized affine Lie algebras and superalgebras can be reformulated as Ising-type models.**