Pertubative expansion of energy densities in integrable models: the full analytical trans-series

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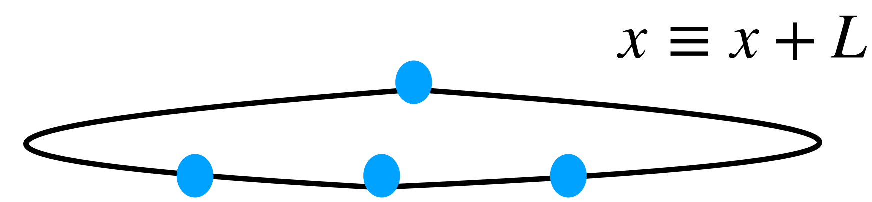
based on 2011.12254, 2011.09897, 2111.15390, 2112.11741, 2204.13365

Motivation: groundstate energy density in integrable models

multiparticle state on the circle
momentum quantization

$$p = m \sinh \theta$$

$$p = m\theta$$

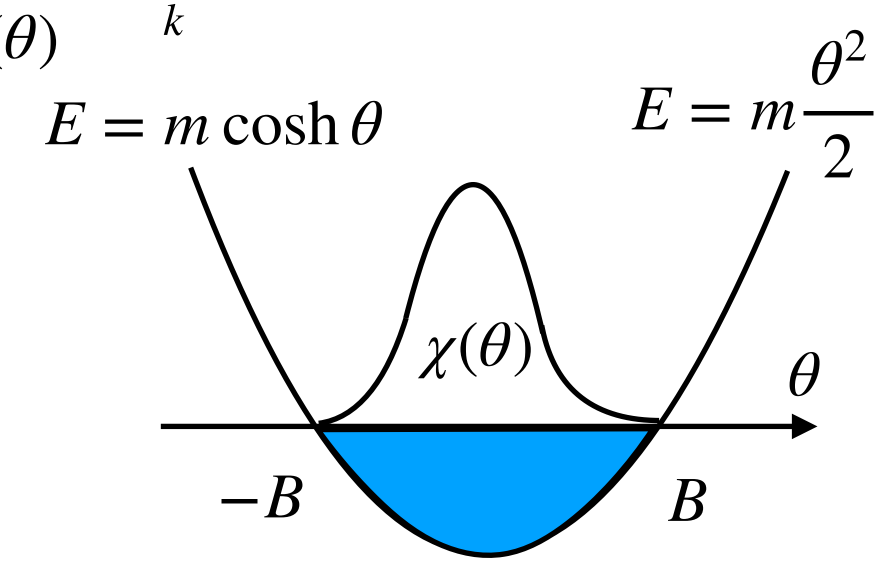


Thermodynamic limit of the Bethe Ansatz: TBA

$$e^{ip_jL} \prod_k S(\theta_j - \theta_k) = 1$$

$$2\pi K(\theta) = -i\partial_\theta \log S(\theta)$$

$$\chi(\theta) - \int_{-B}^{B} K(\theta - \theta')\chi(\theta')d\theta' = m \cosh \theta$$



density

$$\rho(B) = \int_{-B}^{B} \frac{d\theta}{2\pi} \chi(\theta)$$

ground state energy

$$\epsilon(B) = m \int_{-B}^{B} \frac{d\theta}{2\pi} \cosh \theta \chi(\theta)$$

Integrable QFTs in a magnetic field coupled to a conserved charge $\mathscr{H} = \mathscr{H}_0 - hQ$

particles charged under Q **condense into the vacuum** $E_\pm = m \cosh \theta \pm h$

Polyakov, Wiegmann, *Phys.Lett.B* 131 (1983) 121

$$S(\theta) = -\frac{\Gamma(\frac{1}{2} - \frac{i\theta}{2\pi})\Gamma(\Delta - \frac{i\theta}{2\pi})\Gamma(1 + \frac{i\theta}{2\pi})\Gamma(\frac{1}{2} + \Delta + \frac{i\theta}{2\pi})}{\Gamma(\frac{1}{2} + \frac{i\theta}{2\pi})\Gamma(\Delta + \frac{i\theta}{2\pi})\Gamma(1 - \frac{i\theta}{2\pi})\Gamma(\frac{1}{2} + \Delta - \frac{i\theta}{2\pi})}$$

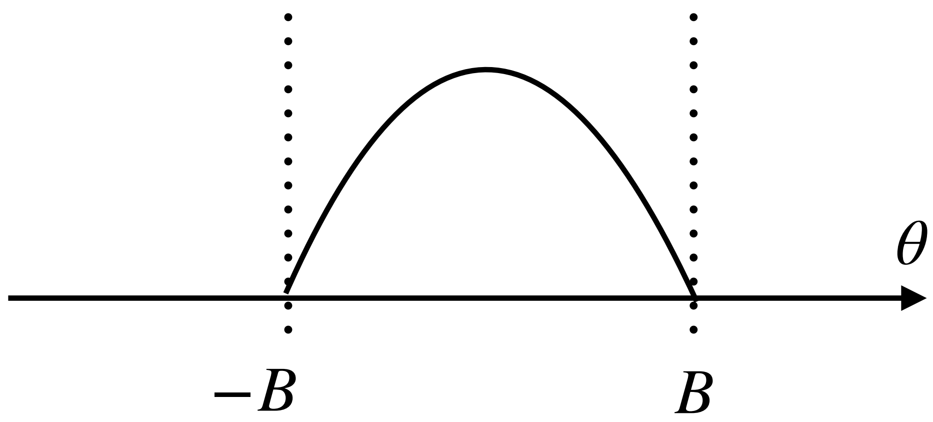
O(N) nonlinear sigma model

Asymptotically free

dynamically generated scale

Hasenfratz, Maggiore, Niedermayer, *Phys.Lett.B* 245 (1990) 522

The mathematical problem



the integral equation for the unknown $\chi_\alpha(\theta, B)$

$$\chi_\alpha(\theta, B) - \int_{-B}^{B} K(\theta-\theta')\, \chi_\alpha(\theta', B) d\theta' = \cosh(\alpha\theta)$$

The observable

$$\mathcal{O}_{\alpha,\beta}(B) = \int_{-B}^{B} \cosh(\beta\theta)\chi_\alpha(\theta, B)\frac{d\theta}{2\pi}$$

the small parameter B^{-1} $\quad e^{-B}$ $\quad (\log B)$

trans-series

$$\mathcal{O}_{\alpha,\beta}(B) = \sum_n e^{-nBA} \sum_m c_{n,m} B^{-m}$$

Aim: calculate $c_{n,m}$

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differential equations

$$\frac{d\mathcal{O}_{\alpha,\beta}}{dB} = \dot{\mathcal{O}}_{\alpha,\beta} = \frac{1}{\pi}\chi_\alpha(B,B)\chi_\beta(B,B)$$

$$\frac{\ddot{\chi}_\alpha(B,B)}{\chi_\alpha(B,B)} - \alpha^2 = f(B)$$

Bajnok, Balog, Vona, *JHEP* 09 (2022) 001

Ristivojevic, *PRL* **130**, 020401

basic observables

$$\mathcal{O}_{1,1} \to \chi_1 \to f \to \chi_\alpha \to \mathcal{O}_{\alpha,\beta}$$

Related problems

$$\chi_\alpha(\theta, B) - \int_{-B}^{B} K(\theta-\theta')\, \chi_\alpha(\theta', B) d\theta' = \cosh(\alpha\theta)$$

$$\mathcal{O}_{\alpha,\beta}(B) = \int_{-B}^{B} \cosh(\beta\theta)\chi_\alpha(\theta, B) d\theta$$

2 dimensional integrable QFTs coupled to a conserved charge $K(\theta) = -\frac{i}{2\pi}\partial_\theta \log S(\theta)$

scattering matrix

Energy density $\mathcal{O}_{1,1}(B) = \epsilon(B)$ **Density** $\mathcal{O}_{1,0}(B) = \rho(B)$

O(N) sigma models $S(\theta) = -\frac{\Gamma(\frac{1}{2}-\frac{i\theta}{2\pi})\Gamma(\Delta-\frac{i\theta}{2\pi})\Gamma(1+\frac{i\theta}{2\pi})\Gamma(\frac{1}{2}+\Delta+\frac{i\theta}{2\pi})}{\Gamma(\frac{1}{2}+\frac{i\theta}{2\pi})\Gamma(\Delta+\frac{i\theta}{2\pi})\Gamma(1-\frac{i\theta}{2\pi})\Gamma(\frac{1}{2}+\Delta-\frac{i\theta}{2\pi})}$ $\Delta^{-1} = N-2$

N=1 SUSY O(N) sigma models | **O(N) Gross-Neveu model**

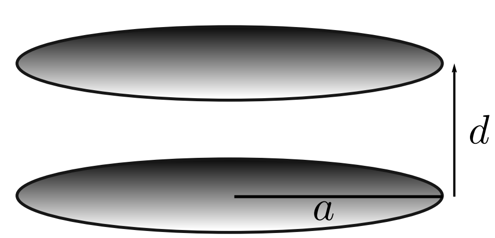
SU(N) principal chiral field, with various charges **SU(2)=O(4) sigma model**

O(3) model $K(\theta) = \frac{1}{\theta^2+\pi^2}$ **Lieb-Liniger model** **Gaudin-Yang model**

coaxial disk capacitor

Q

$\pm Q$



$B = \frac{a}{d}$ $\mathcal{O}_{0,0}(B)$ **capacity**

Plan

- Definition of the 2D O(N) models in a magnetic field

$O(N)$

- Integrable description, Thermodynamic Bethe Ansatz (TBA)
- Expansion of the TBA, perturbative coefficient c_n

- Analytic structure on the Borel plane from asymptotic c_{2000}

$O(4)$

- median resummation, non-perturbative contributions, trans-series
- analytic considerations

$O(3)$

- resurgence and trans-series in the O(3) model, instantons

- Solution based on the Wiener-Hopf and full analytic trans-series

$O(N)$

- Conclusions, outlook

Definition of the O(N) (non-linear) sigma model

N scalar fields in 2D living on the unit sphere $\Phi_1^2 + \ldots + \Phi_N^2 = 1$

magnetic field is coupled the conserved O(N) charge Q_{12}

$$\mathscr{L} = \frac{1}{2\lambda^2}\left\{ \partial_\mu \Phi_i \partial^\mu \Phi_i + 2ih(\Phi_1 \partial_0 \Phi_2 - \Phi_2 \partial_0 \Phi_1) + h^2(\Phi_3^2 + \ldots + \Phi_N^2 - 1) \right\}$$

bare coupling

Euclidean Hamiltonian $\mathscr{H} = \mathscr{H}_0 - hQ_{12}$

Perturbation theory $\Phi_1^2 = 1 - \lambda^2(\varphi_2^2 + \ldots + \varphi_N^2)$ $\lambda\varphi_i = \Phi_i$

$$e^{-V\mathscr{F}(h)} = \int \mathscr{D}^{N-1}[\varphi]\, \mathrm{e}^{-\int d^D x\, \mathscr{L}(x)}$$

$D = 2 - \epsilon$

dimensional regularisation

Legendre transformation

groundstate energy

$$\rho = -\partial\mathscr{F}/\partial h, \qquad \epsilon(\rho) = \mathscr{F}(h) - \mathscr{F}(0) + \rho h$$

density

very hard

Bajnok, Balog, Basso, Korchemsky, Palla, *Nucl.Phys.B* 811 (2009) 438 , 0809.4952

Large B expansion of the TBA

Volin, *Phys.Rev.D* 81 (2010) 105008 • 0904.2744

The resolvent

$$R(\theta) = \int_{-B}^{B} \frac{d\theta'}{2\pi} \frac{\chi(\theta')}{\theta - \theta'}$$

its Laplace transform $z = 2(\theta - B)$

$$\hat{R}(s) = \int_{-i\infty+0}^{i\infty+0} \frac{dz}{2\pi i} e^{sz} R(B + z/2)$$

density: residue at ∞

energy:

$$\frac{\epsilon}{m} = \int_{-B}^{B} \cosh\theta \chi(\theta) \frac{d\theta}{2\pi} = \frac{e^B}{4\pi} \hat{R}(1/2)$$

perturbative expansion in 1/B

O(4)

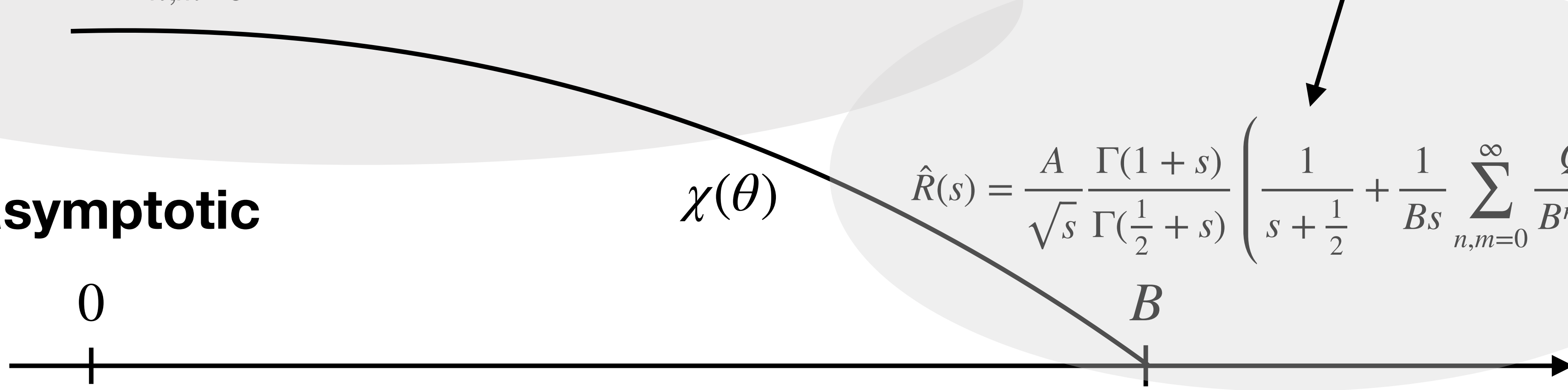
Wiener-Hopf

$$A = \frac{m e^B \sqrt{\pi}}{2\sqrt{2}}$$

$$R(\theta) = 2A\sqrt{B} \sum_{n,m=0}^{\infty} \frac{c_{n,m}}{B^{m-n}(\theta^2 - B^2)^{n+1/2}}$$

$$\hat{R}(s) = \frac{A}{\sqrt{s}} \frac{\Gamma(1+s)}{\Gamma(\frac{1}{2}+s)} \left(\frac{1}{s+\frac{1}{2}} + \frac{1}{Bs} \sum_{n,m=0}^{\infty} \frac{Q_{n,m}}{B^{n+m} s^n} \right)$$

matched asymptotic



$$\rho = A\frac{\sqrt{B}}{\pi}\hat{\rho} = A\frac{\sqrt{B}}{\pi} \sum_{m=0}^{\infty} c_{0,m} B^{-m}$$

$$\epsilon = \frac{m e^B A}{4\sqrt{2\pi}}\left(1 + \sum_{k=0}^{\infty} \epsilon_k B^{-k-1}\right) \qquad \epsilon_k = \sum_{j=0}^{k} 2^{j+1} Q_{j,k-j}$$

Perturbative coefficients

density $\hat{\rho}(B) = 1 + \sum_{n=1}^{\infty} \frac{u_n}{B^n}$

$u_1 = -\frac{3}{8} + \frac{a}{2} \quad u_2 = -\frac{15}{128} + \frac{3a}{16} - \frac{a^2}{8} \quad u_3 = \frac{3\zeta_3}{64} + \frac{a^3}{16} - \frac{9a^2}{64} + \frac{45a}{256} - \frac{105}{1024}$

$a = \ln 2$

odd zeta functions

energy $\hat{\epsilon}(B) = 1 + \sum_{n=1}^{\infty} \frac{\xi_n}{B^n}$

$$\left\{\frac{1}{4}, \frac{9}{32} - \frac{a}{4}, \frac{a^2}{4} - \frac{9a}{16} + \frac{57}{128}, -\frac{a^3}{4} + \frac{27a^2}{32} - \frac{171a}{128} - \frac{27\zeta_3}{256} + \frac{1875}{2048}, \dots\right\}$$

Volin, *Phys.Rev.D* 81 (2010) 105008 • e-Print: 0904.2744 — 22 coefficients

Marino,Reis, *JHEP* 04 (2020) 160 • e-Print: 1909.12134 — 44 coefficients

50 coefficients

free energy in the running coupling

$$\frac{1}{\alpha} + \frac{1}{2} - B - \frac{1}{2}\log B\alpha = \log 2\hat{\rho}$$

$$f(\alpha) = \frac{\epsilon}{\rho^2} = \frac{\pi}{2}\sum_{n=1}^{\infty} \chi_n \alpha^n \quad = \frac{\pi}{2}\left(1 + \frac{\alpha}{2} + \frac{\alpha^2}{4} + \frac{10 - 3\zeta_3}{32}\alpha^3 + \chi_5\alpha^4 + \dots\right.$$

$$\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{5}{16} - \frac{3\zeta_3}{32}, \frac{53}{96} - \frac{9\zeta_3}{64}, -\frac{189\zeta_3}{512} - \frac{405\zeta_5}{2048} + \frac{487}{384}, \dots\right\}$$

Hasenfratz, Maggiore, Niedermayer, *Phys.Lett.B* 245 (1990) 522

Comparing ordinary perturbation theory in $\frac{h}{\Lambda}$ **to expansion of TBA in** $\frac{h}{m}$

relation between mass and scale can be obtained $m/\Lambda = (8/e)^{\Delta}/\Gamma(1+\Delta)$

Numerical data for O(4)

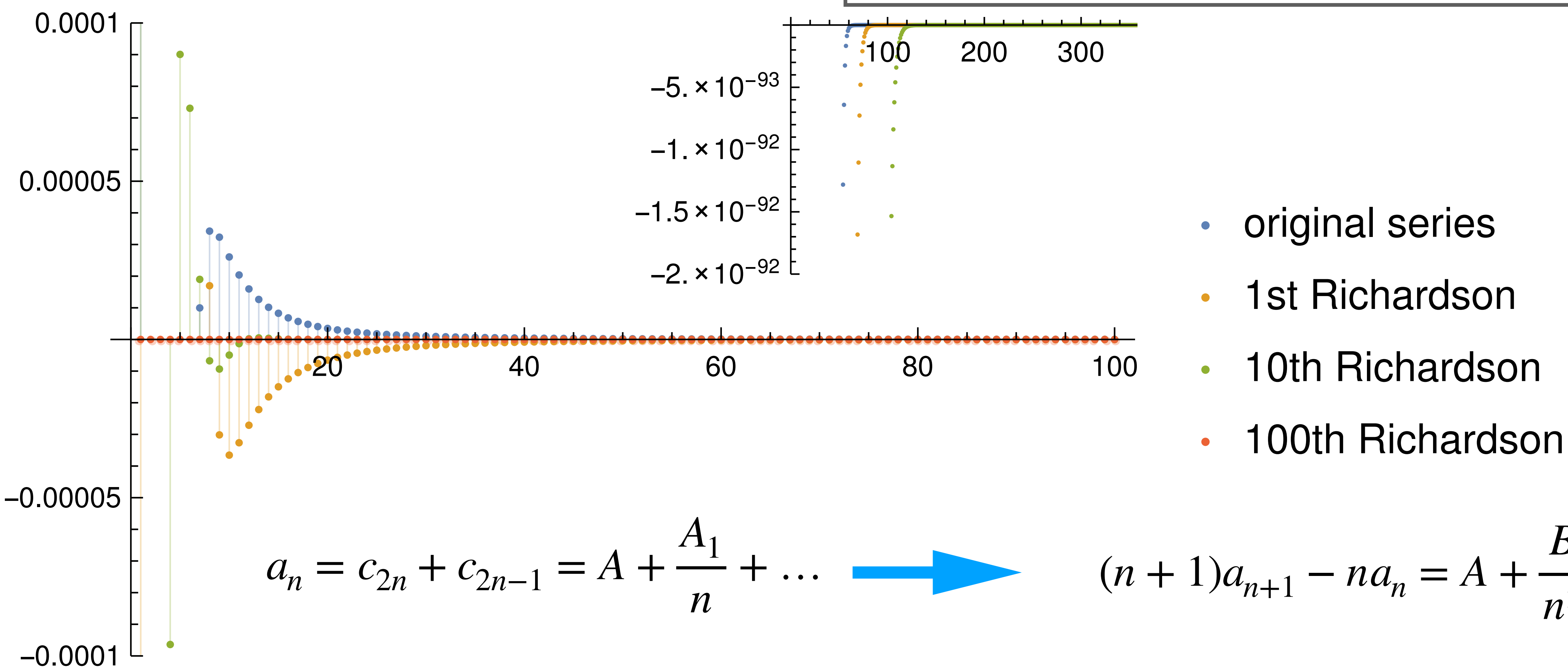
density $\hat{\rho}(B) = 1 + \sum_{n=1}^{\infty} \frac{u_n}{B^n}$ 2000 coefficients for 7000 digits

energy $\hat{\epsilon}(B) = 1 + \sum_{n=1}^{\infty} \frac{\xi_n}{B^n}$ 2000 coefficients for 7000 digits

free energy $\hat{f}(\alpha) = \frac{\hat{\epsilon}}{\hat{\rho}^2} = \sum_{n=1}^{\infty} \chi_n \alpha^n$ 1400 coefficients for 4000 digits

factorial groth:

$$c_n = \frac{\chi_{n+2}\, 2^{n+1}}{\Gamma(n+1)} = p^+ + (-1)^n p^- + \frac{1}{n}(\dots)$$



$$a_n = c_{2n} + c_{2n-1} = A + \frac{A_1}{n} + \dots \quad \longrightarrow \quad (n+1)a_{n+1} - n a_n = A + \frac{B}{n^2} + \dots$$

Asymptotic behaviour

Borel function

$$\Psi(t) = \sum_{n=1}^{\infty} c_n\, t^n$$

perturbative coefficient grow factorially:

$$c_n = \frac{\chi_{n+2}\, 2^{n+1}}{\Gamma(n+1)}$$

how to give meaning to the series?

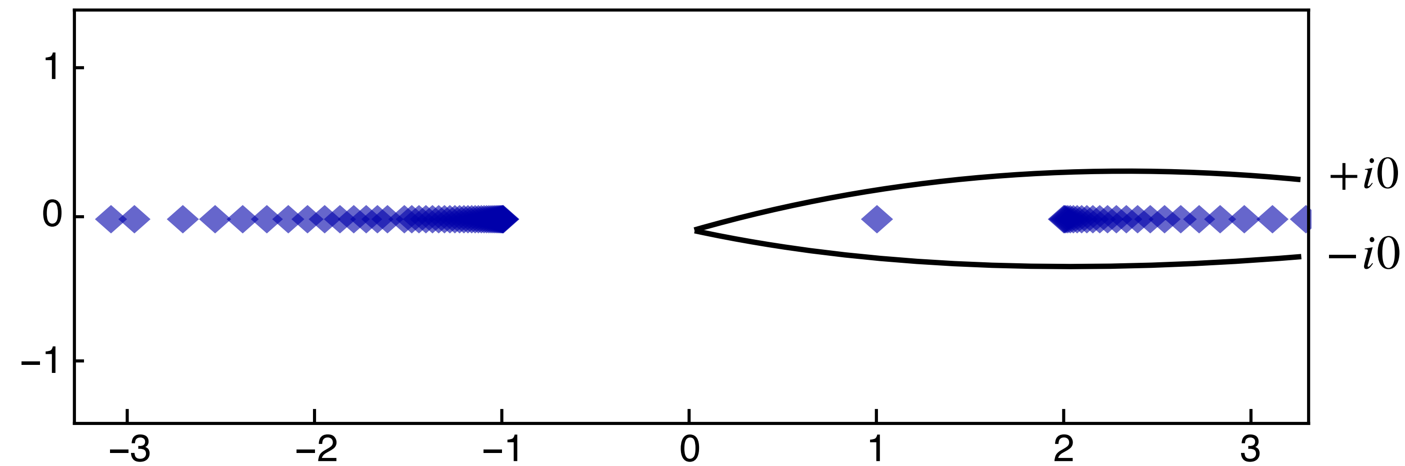
path integral

$$e^{-V\mathscr{F}(h)} = \int \mathscr{D}^3[\varphi]\, \mathrm{e}^{-S[\varphi]} = \int_0^\infty dt e^{-\frac{2t}{\alpha}} \int_{S[\varphi]=\frac{2t}{\alpha}} \mathscr{D}^3[\varphi]\, \mathrm{e}^{-S[\varphi]}$$

convergence radius 1

Pade approximant

$$\Psi(t) \approx \frac{\sum_{i=1}^{n} \beta_i t^i}{1 + \sum_{j=1}^{m} \gamma_j t^j}$$



Ambiguity from the pole

$$\frac{i\pi^2\alpha\,\mathrm{res}_1\Psi(t)}{4}\mathrm{e}^{-\frac{2}{\alpha}}$$

$$f^{(\pm)}(\alpha) = \frac{\pi}{2}\left[\chi_1\,\alpha + \chi_2\,\alpha^2 + \alpha \int_0^{\infty \pm i0} e^{-\frac{2t}{\alpha}}\,\Psi(t)\,dt\right]$$

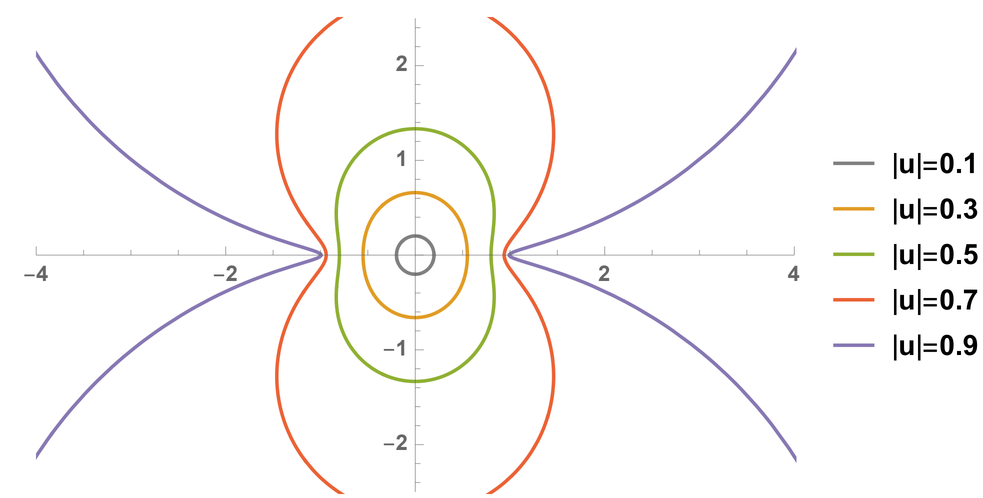
Better approximation ⟶ **Conformal mapping**

Conformal mapping vs numerical solution

Conformal mapping maps the t-plane to the unit u-disk

$$u(t) = \frac{1-\sqrt{1-t^2}}{t}$$

$$\tilde{\Psi}(u) = \sum_{n=1}^{\infty} b_n u^n \qquad \tilde{\Psi}(t) = \sum_{n=1}^{\infty} b_n u(t)^n$$



$$f^{(\pm)}(\alpha) = \frac{\pi}{2}\left[\chi_1\,\alpha + \chi_2\,\alpha^2 + \alpha \int_0^{\infty \pm i0} e^{-\frac{2t}{\alpha}}\,\tilde{\Psi}(t)\,dt\right]$$

Compare to numerical solution in a Chebyshev basis

$$\chi(\theta) = \sum_{j=1}^{(n_c+1)/2} s_j\, T_{2j-2}(\theta/B), \qquad T_n(x) = \cos(n \arccos x)$$

precision=30 digits

$$\mathrm{Im}\left(f^{(+)}(\alpha)\right) = \alpha c_0 \mathrm{e}^{-2/\alpha} + \alpha \mathrm{e}^{-4/\alpha}(c_1 + c_2\alpha + \ldots)$$

$c_0 = 1.70067333(1)$

$c_1 = -1.70067333(1)$

$c_2 = 0.637752(1)$

$c_3 = -0.1727(1)$

$$\mathrm{Re}\left(f^{(+)}(\alpha)\right) = f_{\mathrm{TBA}}(\alpha) + \alpha \mathrm{e}^{-8/\alpha}(d_1 + d_2\alpha + \ldots)$$

$d_1 = -0.9206(1)$

$d_2 = 0.575(3)$

Asymptotic analysis

$\Phi(z) = 1 + \sum_{n=1}^{\infty} s_n / z^n$ $\qquad z = 2B \qquad$ $c_n = s_{n+1}/n!$ $\qquad \Psi(t) = \sum_{n=0}^{\infty} c_n t^n$

Asymptotic large n behaviour

$$c_n = (-1)^n \left(p^- + \frac{p_0^-}{n} + \frac{p_1^-}{n(n-1)} + \dots \right) \quad + \left(p^+ + \frac{p_0^+}{n} + \frac{p_1^+}{n(n-1)} + \dots \right)$$

$$+ 2^{-n} \left(q^+ + \frac{q_0^+}{n} + \frac{q_1^+}{n(n-1)} + \dots \right)$$

$\log(1+t)\left[-p_0^- + p_1^-(1+t) + \dots\right]$ $\quad + \frac{p^-}{1+t}$ $\qquad \frac{p^+}{1-t}$

$\log(1+t)\bar{\Psi}(1+t)$

alien derivative $\qquad \Delta_{\pm 1} \Psi(z) = \mp i 2\pi \left\{ p^{\pm} \pm \sum_{m=0}^{\infty} \frac{(\pm 1)^m p_m^{\pm}}{z^{m+1}} \right\}$

Dorigoni, *Annals Phys.* 409 (2019) 167914 • e-Print: 1411.3585

Aniceto, Basar, Schiappa, *Phys.Rept.* 809 (2019) 1-135 • e-Print: 1802.10441

Asymptotics for $\hat{f}(\alpha)$

numerical fitting 2.80308535473939142809960724226717498614747943851074832268840733301275 730867946963527968381041400288 7

$$p^- = -\frac{e}{8\pi}$$ for 150 digits

$$p_0^- = 0$$ for 147 digits

$$p_1^- = \frac{e}{4\pi}$$ for 144 digits

$$p_2^- = \frac{\mathrm{e}}{4\pi}\left(-\frac{1}{2} - \frac{3}{4}\zeta_3\right)$$

http://wayback.cecm.sfu.ca/projects/EZFace/

$\Delta_{-1}\hat{f}$

$$p^+ = \frac{8}{e\pi}$$

$$p_0^+ = 0$$

$$p_1^+ = 0$$

$\Delta_1\hat{f}$

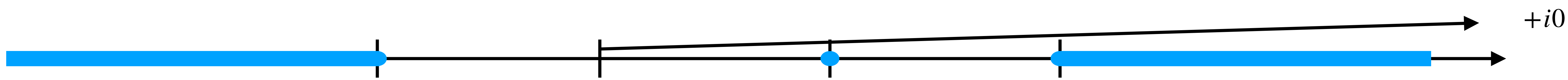
$$q^- = \frac{16}{e^2\pi}$$ for 80 digits

$$q_0^- = \frac{16}{e^2\pi}\left(-\frac{3}{4}\right)$$

$$q_0^- = \frac{16}{e^2\pi}\left(\frac{13}{32}\right)$$

$$q_0^- = \frac{16}{e^2\pi}\left(-\frac{99}{256} + \frac{3}{8}\zeta_3\right)$$

$\Delta_2\hat{f}$



imaginary ambiguity

$$f^{(+)}(\alpha) = \frac{\pi}{2}\left[\alpha \int_0^{\infty+i0} e^{-\frac{2t}{\alpha}} B(t)\,dt\right]$$

$$\Im\mathrm{m}(f^{(+)}(\alpha)) = \frac{4\pi}{\mathrm{e}^2}\alpha \mathrm{e}^{-2/\alpha} + \alpha \mathrm{e}^{-4/\alpha}\left(-\frac{4\pi}{\mathrm{e}^2} + \frac{3\pi}{2\mathrm{e}^2}\alpha - \frac{13\pi}{32\mathrm{e}^2}\alpha^2 + \ldots\right)$$

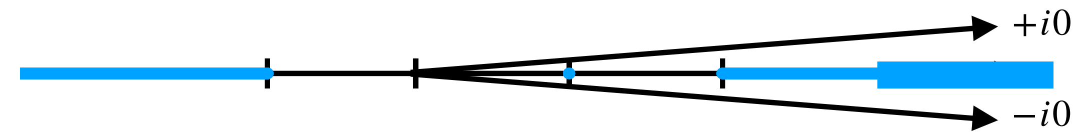
$c_0 = 1.70067333$ $c_1 = -1.70067333(1)$ $c_2 = 0.637752(1)$ $c_3 = -0.1727(1)$

real ambiguity???

Median resummation and Stokes automorphism

$x = \frac{2}{\alpha}$

$$S_\pm(f) = f^{(\pm)} = \chi_1 + \alpha\chi_2 + \int_0^{\infty \pm i0} e^{-tx}\, \Psi(t)dt$$



cut of the cut of the cut…

$$S_+(f) - S_-(f) = -S_+\left(e^{-x}\Delta_1 f + e^{-2x}\Delta_2 f + \ldots + \frac{e^{-2x}}{2}\Delta_1^2 f + \ldots\right)$$

$$S_+(f) = S_-(\mathfrak{S}f) \quad ; \quad S_-(f) = S_+(\mathfrak{S}^{-1}f)$$

$$\mathfrak{S} = \exp\left\{-\sum_{n=1}^{\infty} e^{-nx}\Delta_n\right\}$$

Median resummation

$$S_{\mathrm{med}}(f) = S_-(\mathfrak{S}^{\frac{1}{2}}f) = S_+(\mathfrak{S}^{-\frac{1}{2}}f) = S_+(e^{\frac{1}{2}\sum e^{-nx}\Delta_n}f)$$

$$S_{\mathrm{med}}(f) = S_+\left(f + \frac{e^{-x}}{2}\Delta_1 f + \frac{e^{-2x}}{2}\Delta_2 f + \ldots + \frac{e^{-4x}}{8}\Delta_1\Delta_3 f + \frac{e^{-4x}}{8}\Delta_2^2 f + \ldots\right)$$

$$\Delta_1 f = -\frac{16i}{\mathrm{e}^2} \qquad \Delta_2 f = \frac{16i}{\mathrm{e}^2}\left(1 - \frac{3}{4x} + \frac{13}{32x^2} - \left(\frac{99}{256} - \frac{3}{8}\zeta_3\right)\frac{1}{x^3} + \ldots\right)$$

We need $\Delta_2\Delta_2 f$

need the asymptotics of $\Delta_2 f$

$$S_+(\Delta_2 f) - S_-(\Delta_2 f) = -S_+(e^{-2x}\Delta_2\Delta_2 f) + \ldots$$

we found $S_{\mathrm{med}}(f) = \mathfrak{Re}(S_+(f)) + \frac{32}{\mathrm{e}^4}\mathrm{e}^{-8/\alpha}(1 - \frac{5\alpha}{8} + \ldots)$ $\quad d_1 = 0.58607 \quad \frac{d_2}{d_1} = -0.6246$

Comparison with TBA

Median resummation

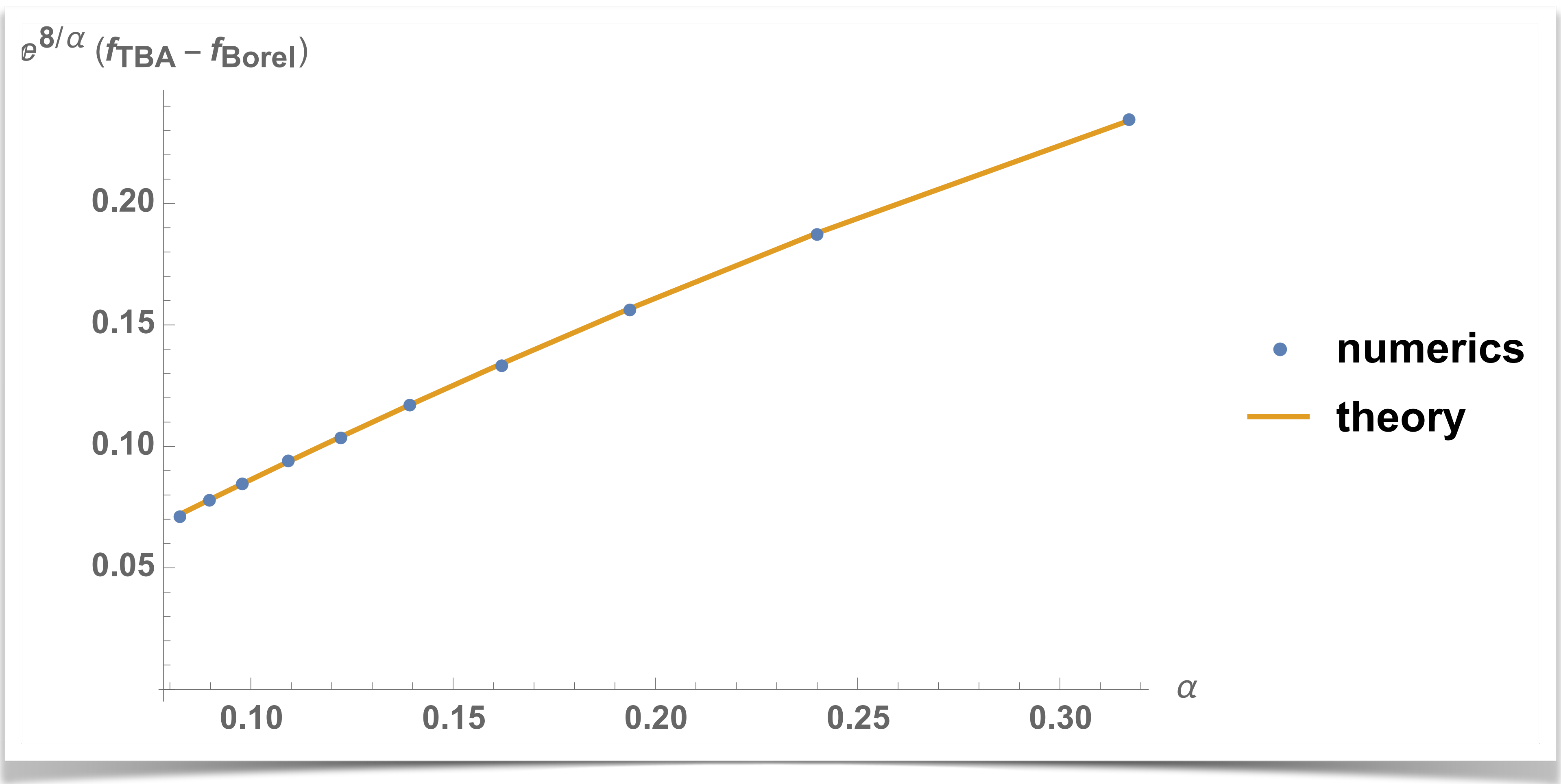
$$S_{\text{med}}(f) = S_+\Big(f + \frac{e^{-x}}{2}\Delta_1 f + \frac{e^{-2x}}{2}\Delta_2 f + \ldots + \frac{e^{-4x}}{8}\Delta_1\Delta_3 f + \frac{e^{-4x}}{8}\Delta_2^2 f + \ldots\Big)$$

$$f_{\text{Borel}} = S_{\text{med}}(f) = \mathfrak{Re}(S_+(f)) + \frac{32}{e^4}e^{-8/\alpha}(1 - \frac{5\alpha}{8} + \ldots)$$

$$d_1 = 0.58607 \qquad \frac{d_2}{d_1} = -0.6246$$

$$\frac{1}{\alpha} + \frac{1}{2} - B - \frac{1}{2}\log B\alpha = \log 2\hat{\rho}$$

We compare to the numerical solution of TBA



Trans-series

Analytic structure of the free energy on the Borel plane

The expansion of the physical observable is a trans-series

$$f_{\mathrm{TBA}} = \sum_{m=0}^{\infty} e^{-\frac{2}{\alpha}m} \sum_{n=1}^{\infty} \chi_n^{(m)} \alpha^{n-1}$$

$\chi_n^{(0)} = \chi_n$

perturbative coefficients

$\mathrm{e}^{-2/\alpha}\chi_n^{(1)}$

$$\Delta_1 f = -\frac{16i}{\mathrm{e}^2}$$

$\mathrm{e}^{-4/\alpha}\chi_n^{(2)}$

$$\Delta_2 f = \frac{16i}{\mathrm{e}^2}\left(1 - \frac{3}{4x} + \frac{13}{32x^2} - \left(\frac{99}{256} - \frac{3}{8}\zeta_3\right)\frac{1}{x^3} + \ldots\right)$$

$\mathrm{e}^{-8/\alpha}\chi_n^{(4)}$

$$\Delta_2^2 f = \frac{32}{\mathrm{e}^4}(1 - \frac{5\alpha}{8} + \ldots)$$

large n asymptotics

large n asymptotics

What is the full trans-series? Is it fixed by the perturbative part?

Asymptotic behaviour in O(3)

perturbative coefficients grow factorially:

$$c_n = \frac{\chi_n \, 2^{n-1}}{\Gamma(n)}$$

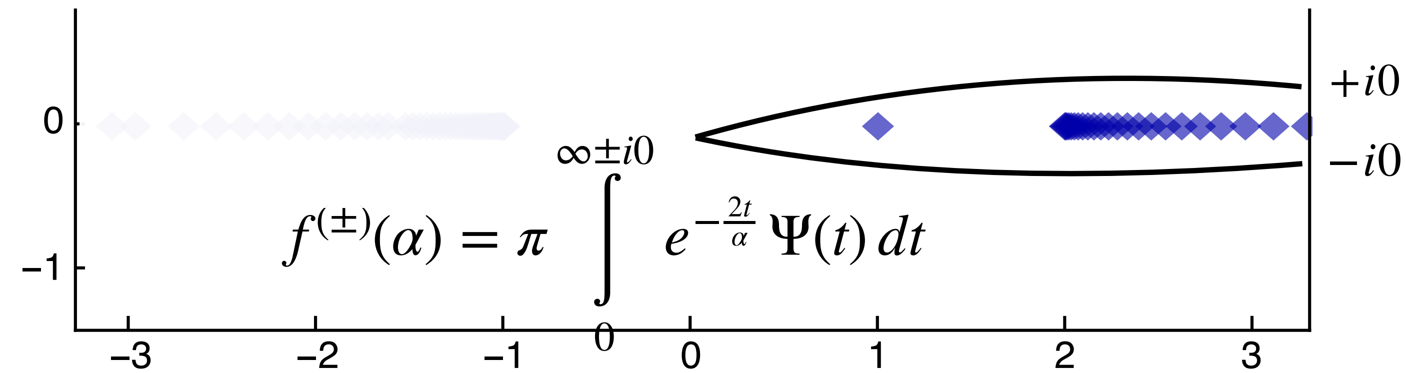
Borel function

$$\Psi(t) = \sum_{n=1}^{\infty} c_n \, t^n$$

Pade approximant

$$\Psi(t) \approx \frac{\sum_{i=1}^{n} \beta_i t^i}{1 + \sum_{j=1}^{m} \gamma_j t^j}$$

improved with conformal mapping



$$f^{(\pm)}(\alpha) = \pi \int_0^{\infty \pm i0} e^{-\frac{2t}{\alpha}} \Psi(t)\, dt$$

Lateral Borel resummation

Asymptotic analysis for 336 terms

$$c_n = \frac{8}{e^2} - 2^{-n} a_0 \left(n + a_1 + \frac{a_2}{(n-1)} + \dots \right)$$

$$a_0 = \frac{64}{\pi e^4} \qquad \frac{a_1}{a_0} = -\frac{3}{2} \qquad \frac{a_2}{a_0} = -\frac{1}{8} \qquad \frac{a_n}{a_0} = \Gamma(n+2)(b_0 + \frac{b_1}{n} + \dots)$$

comparison to numerical solution

$$\Im m(f^+) = \frac{16\pi}{e^2} e^{-\frac{2}{\alpha}} + \frac{e^{-\frac{4}{\alpha}}}{\alpha} \pi \sum_{n=0} a_n 4^{2-n} \alpha^n$$

expected real deviation

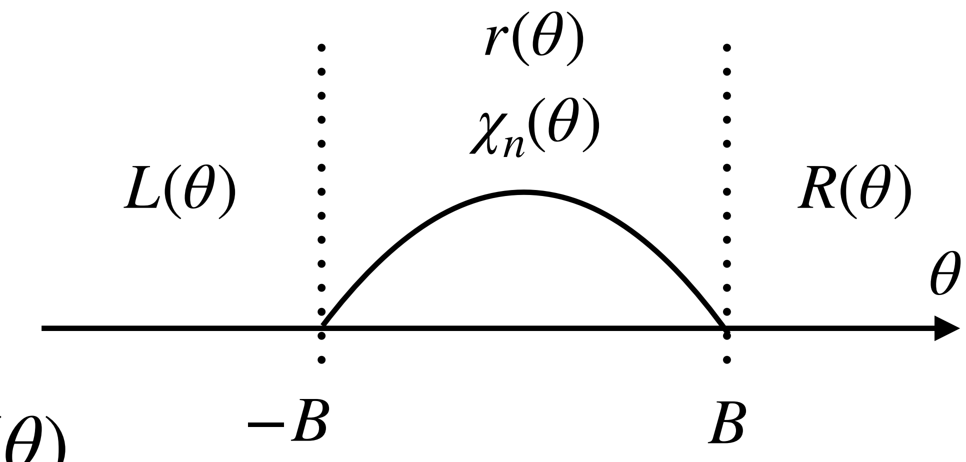
$$f_{\text{TBA}} - \Re e(f^+) = e^{-\frac{8}{\alpha}} (b_0 + b_1 \alpha + \dots)$$

real deviation observed

$$f_{\text{TBA}} - \Re e(f^+) = e^{-\frac{2}{\alpha}} A_0 (\frac{2}{\alpha} + A_1 + A_2 \log \alpha + A_3 \alpha + \dots)$$

Instantons???

Wiener-Hopf solution



1. Extend the integral equation for the whole line

$$\chi_n(\theta) - \int_{-\infty}^{\infty} d\theta' K(\theta-\theta')\chi_n(\theta') = r(\theta) + L(\theta) + R(\theta)$$

$$r(\theta) = \cosh(n\theta)$$

2. Use Fourier transformation and invert the kernel $\quad (1-\tilde{K})\tilde{\chi}_n = \tilde{r} + \tilde{L} + \tilde{R}$

$$\frac{1}{1-\tilde{K}(\omega)} = G_+(\omega)G_-(\omega)$$

$G_\pm(\omega)$ are analytical on the UHP/LHP $\qquad f_\pm = e^{\pm i\omega B} f$

3. By shifting the functions separate the equations into analytic on the UHP/LHP

$$\frac{\chi_+}{G_+} = G_- r_+ + \alpha(G_+ X_+) + G_- X_-$$

$$\alpha(\omega) = e^{2i\omega B}\frac{G_-(\omega)}{G_+(\omega)}$$

$$f^{(\pm)}(\omega) = \mp\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\frac{f(\omega')}{\omega-\omega'\mp i\epsilon}$$

$$X_+(\omega) = e^{-i\omega B}\tilde{R}(\omega) = X_-(-\omega)$$

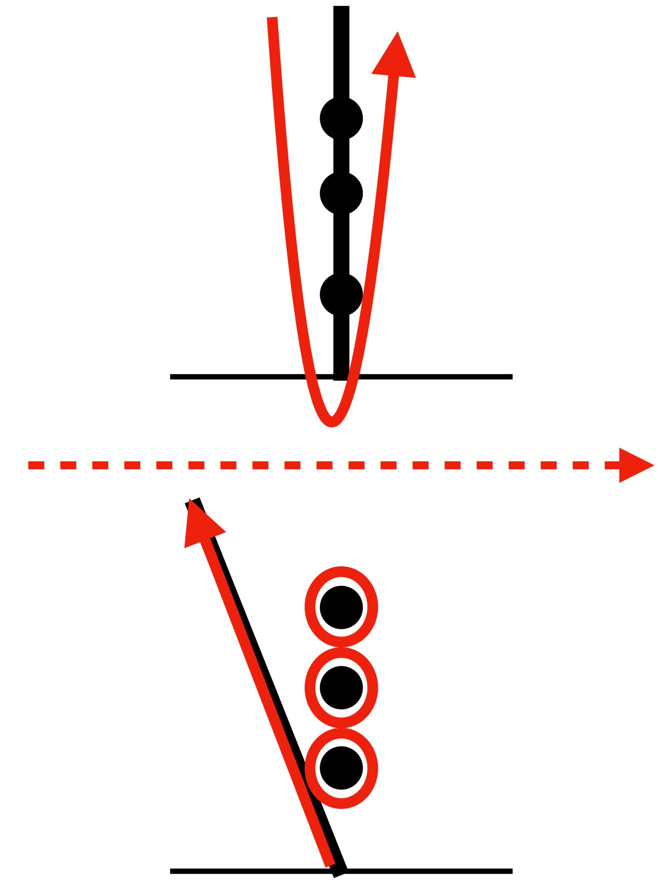
$$0 = (G_- r_+)^{(-)} + (\alpha(G_+ X_+))^{(-)} + G_- X_-$$

$$\frac{\chi_+}{G_+} = (G_- r_+)^{(+)} + (\alpha(G_+ X_+))^{(+)} \longrightarrow \mathcal{O}_{1,1} = \frac{e^B}{2\pi}\chi_+(i)$$

but where is the trans-series ?

Trans-series from Wiener-Hopf

after field redefinition and contour deformation ($N > 3$)



$$q_n(2i\xi) + \frac{i}{2\pi}\int_C \frac{\alpha(2i\xi')q_n(2i\xi')}{\xi+\xi'}d\xi' = \frac{1}{n-2\xi}$$

$$q_n(ivx) = Q_n(x)$$

- Marino, Miravitllas, Reis, *JHEP* 08 (2022) 279 • 2111.11951

$$Q_n(x) + 2i\sum_{l=1}^{\infty}\frac{H_l q_{n,l}}{vx+2l\xi_0}e^{-4l\xi_0 B} + \frac{1}{\pi}\int \frac{e^{-2Bvy}\beta(vy/2)Q_n(y)}{x+y}dy = \frac{1}{n-2vx}$$

$$q_{n,l} = q_n(2il\xi_0)$$

$$e^{-2Bvy}\beta(vy/2) = e^{-y}\mathscr{A}(y)$$

running coupling $2B = \frac{1}{v} + (2\Delta - 1)\ln v + L$ **power series in** v **no** $\log v$

$$q_{n,s} - 2i\sum_{l=1}^{\infty}H_l q_{n,l}e^{-4Bl\xi_0}A_{-2l,-2s} = A_{n,-2s}$$

$$\mathcal{O}_{nm} = \frac{e^{(n+m)B}}{4\pi}G_+(im)G_+(in)W_{n,m}$$

$$W_{n,m} = \frac{1}{n+m} + 2i\sum_{l=1}^{\infty}\frac{H_l q_{n,\kappa_l}e^{-4B\xi_0 l}}{m-\kappa_l} + \frac{v}{\pi}\int_{C_+}\frac{e^{-x}\mathscr{A}(x)Q_n(x)}{m-vx}dx = A_{n,m} + \ldots$$

$$P_\alpha(x) + \int_{C_+}\frac{e^{-y}\mathscr{A}(y)P_\alpha(y)}{x+y}\frac{dy}{\pi} = \frac{1}{\alpha - vx}$$

$$A_{\alpha,\beta} = \frac{1}{\alpha+\beta} + \langle P_\alpha\rangle_\beta$$

$$\langle Q\rangle_\beta = \int_{C_+}\frac{e^{-x}\mathscr{A}(x)Q(x)}{\beta - vx}\frac{vdx}{\pi}$$

Full trans-series solution in O(4)

$$S_l = \frac{((2l-1)!!)^2}{2^{2l-1} l!(l-1)!}$$

$$W_{1,1} = A_{1,1} + Me^{-2B} + \sum_{l_1,l_2,\ldots} e^{-4(l_1+l_2+\ldots)B} iS_{l_1} iS_{l_2} \ldots A_{1,-2l_1} A_{-2l_1,-2l_2} \ldots A_{-l_k,1}$$

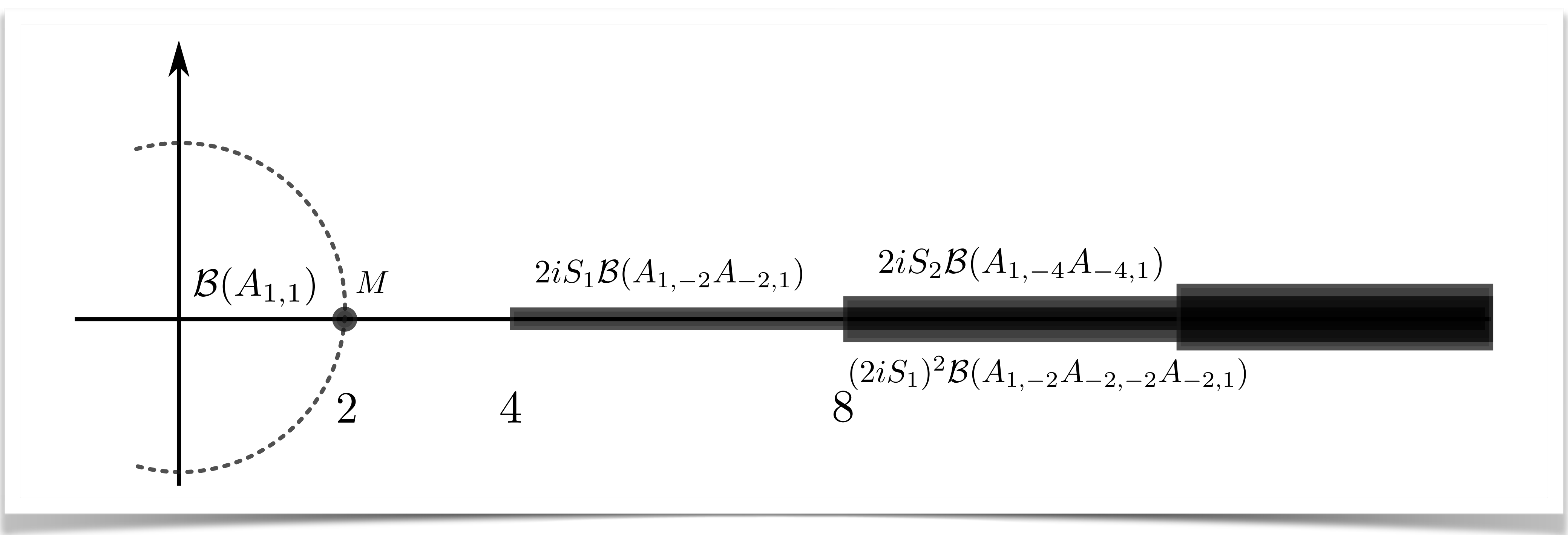
$$M = -2i$$

from the differential equations $A_{1,1} \to A_{n,m}$

$$A_{n,m} = \frac{1}{m+n} + \frac{v}{4mn} + \frac{v^2(20\gamma mn + 9m + 9n)}{32m^2n^2} + \frac{v^3\left(m^2\left(640\gamma^2n^2 + 636\gamma n + 225\right) + 6mn(106\gamma n + 39) + 225n^2\right)}{384m^3n^3} + O\left(v^4\right)$$

$$v = \frac{1}{2B}$$

$$W_{1,1} = A_{1,1} + Me^{-2B} + ie^{-4B}S_2 A_{1,-2}^2 + e^{-8B}((iS_2)^2 A_{1,-2}^2 A_{-2,-2} + iS_4 A_{1,-4}^2) + \ldots$$



$$\mathcal{O}_{11} = \frac{e^{2B}}{4\pi} G_+(i)^2 W_{1,1}$$

uniquely defined and free of ambiguities, agrees with TBA

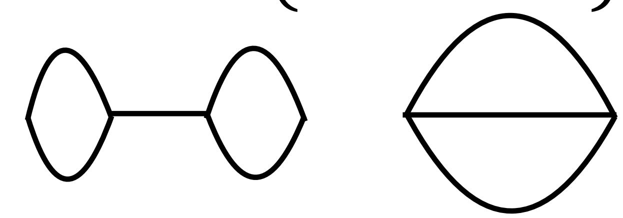
Conclusions

- The integrable description enabled to calculate high number of perturbative coefficient with high precision in the O(3) and O(4) models
- The asymptotic analysis of the perturbative coefficients revealed the analytic structure on the Borel plane with poles and cuts.
- The various alien derivatives with the median resummation provided a trans-series ansatz, whose leading terms matched perfectly with the numerical solution of the TBA equation in O(4)
- However, it failed to describe the leading real deviation from TBA in the O(3). This might be related to instantons!
- By expanding the integral equation using the Wiener-Hopf method, a trans-series form can be derived and systematically calculated, which matches in the O(3) model with the numerical solutions of the TBA equation
- The full trans-series solution is determined in terms of the perturbative $A_{n,m}$ basis, which can be explicitly calculated.
- The perturbative part completely determines all the non-perturbative corrections in the O(N>3) models but not in O(3), which might be related to an instanton saddle point

Standard perturbation theory

$$\mathcal{F}(h)-\mathcal{F}(0) = -\frac{h^2}{2\lambda^2}+\frac{N-2}{4\pi}h^{2-\varepsilon}\left\{\frac{1}{\varepsilon}+\frac{\gamma}{2}+\frac{1}{2}\right\}+\lambda^2\frac{N-2}{16\pi^2}h^{2-2\varepsilon}\left\{\frac{1}{\varepsilon}+\gamma+\frac{1}{2}\right\}$$

Bajnok, Balog, Basso, Korchemsky, Palla, *Nucl.Phys.B* 811 (2009) 438 , 0809.4952



renormalized coupling

$$\lambda^2 = (\mu e^{\frac{\gamma}{2}})^{\varepsilon} Z_1 \tilde{g}^2 \qquad \mu\frac{d\tilde{g}}{d\mu} = \beta(\tilde{g}) = -\beta_0\tilde{g}^3-\beta_1\tilde{g}^5+\ldots$$

$$\mathcal{F}(h)-\mathcal{F}(0) = -\frac{h^2}{2}\left\{\frac{1}{\tilde{g}^2}-2\beta_0\left(\ln\frac{\mu}{h}+\frac{1}{2}\right)-2\beta_1\tilde{g}^2\left(\ln\frac{\mu}{h}+\frac{1}{4}\right)+O(\tilde{g}^4)\right\}$$

RG invariant dynamically generated scale

$$\Lambda = \mu e^{-\int^{\tilde{g}}\frac{dg}{\beta(g)}} = \mu e^{-\frac{1}{2\beta_0\tilde{g}^2}}\tilde{g}^{-\beta_1/\beta_0^2}\left[1+\frac{1}{2\beta_0}\left(\frac{\beta_1^2}{\beta_0^2}-\frac{\beta_2}{\beta_0}\right)\tilde{g}^2+\ldots\right] \qquad \Delta = \frac{1}{N-2}$$

running coupling

$$\frac{1}{\tilde{\alpha}}+\Delta\ln\tilde{\alpha} = \ln\frac{h}{\Lambda_{\overline{MS}}} \qquad \mathcal{F}(h)-\mathcal{F}(0) = -\beta_0 h^2\left\{\frac{1}{\tilde{\alpha}}-\frac{1}{2}-\frac{\Delta\tilde{\alpha}}{2}+O(\tilde{\alpha}^2)\right\}$$

After Legendre transformation

$$\frac{1}{\alpha}+(\Delta-1)\ln\alpha = \ln\frac{\rho}{2\beta_0\Lambda_{\overline{MS}}} \qquad \epsilon(\rho) = \rho^2\pi\Delta\left\{\alpha+\frac{\alpha^2}{2}+\Delta\frac{\alpha^3}{2}+O(\alpha^4)\right\}$$

Analytical resurgence in O(4)

recall: matched asymptotic

the resolvent

$$R(\theta) = 2A\sqrt{B}\sum_{n,m=0}^{\infty}\frac{c_{n,m}}{B^{m-n}(\theta^2 - B^2)^{n+1/2}}$$

Wiener-Hopf $A = \frac{m e^{B}\sqrt{\pi}}{2\sqrt{2}}$

Laplace transform

$$\hat{R}(s) = \frac{A}{\sqrt{s}}\frac{\Gamma(1+s)}{\Gamma(\frac{1}{2}+s)}\left(\frac{1}{s+\frac{1}{2}} + \frac{1}{Bs}\sum_{n,m=0}^{\infty}\frac{Q_{n,m}}{B^{n+m}s^n}\right)$$

$$\sum_{k=r}^{m} E_{k-r,k}c_{k-r,m-k} = \sum_{k=r}^{m}(a_{k-r} + a_{k-r-1})Q_{k-1,m-k}$$

Closed equations for the Q-s

$$\mathcal{O}_{1,\alpha} \propto e^{(1+\alpha)B}\sum_{k=0}^{\infty} c_k (2B)^{-k-1}$$

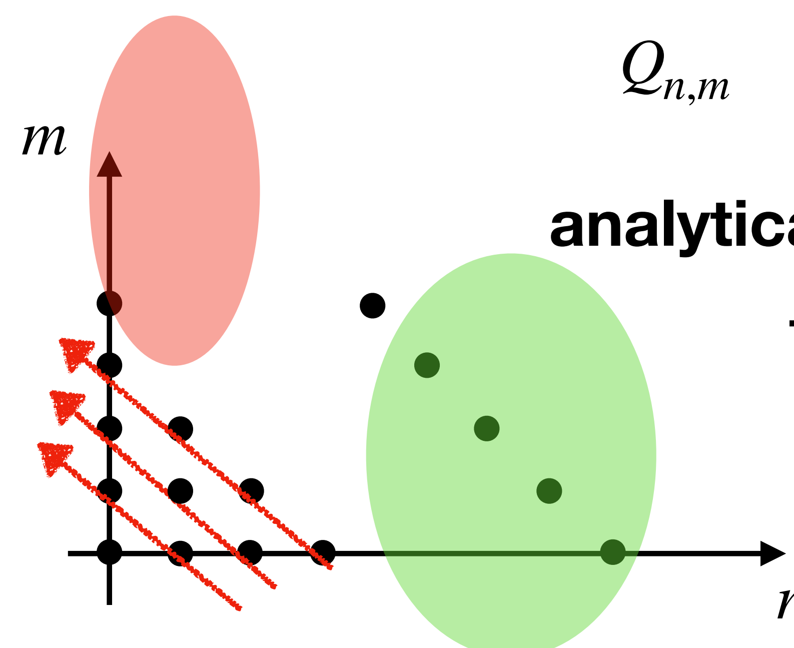
$$c_k = \frac{\alpha+1}{\alpha}\sum_{j=0}^{k} Q_{j,k-j}\left(\frac{2}{\alpha}\right)^j$$

leading large k asymptotic

$$\Delta_{-1}\mathcal{O}_{1,\alpha} = i\,\mathcal{O}_{1,1}\mathcal{O}_{1,\alpha}$$

ansatz+solution

$Q_{n,m}$



analytical solution

for large n

finite m