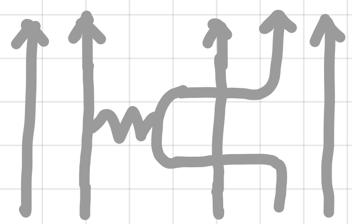
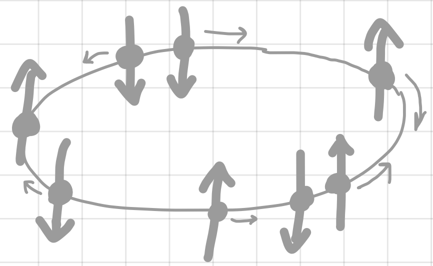
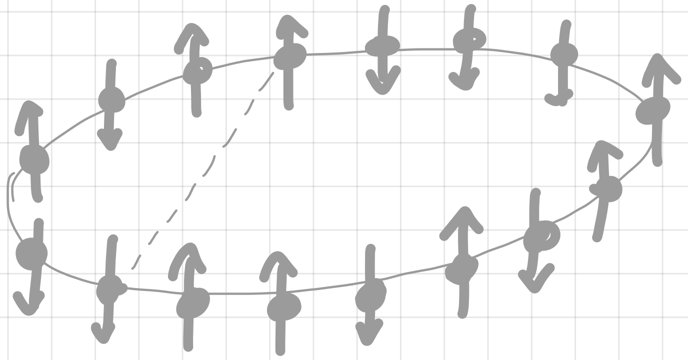
Bethe ansatz
inside
Calogero-Sutherland
models







by

Jules Lamers

Institut de Physique Théorique

based on

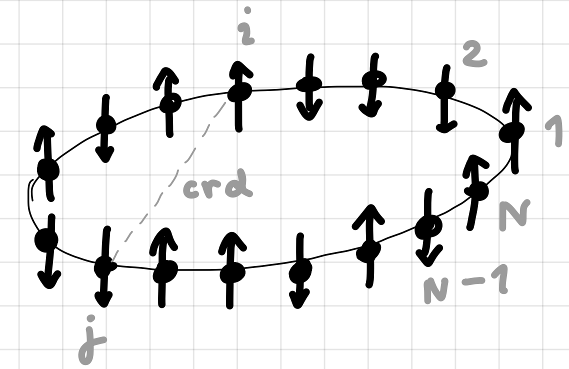
JL, D Serban
arXiv 2212.01373

ongoing with
G. Ferrando, D Serban,
F. Levkovich-Maslyuk

and with
R. Klabbers

Motivation: Heis vs Haldane-Shastry

$$H_{isotr} = \sum_{i<j}^{N} V(i-j)\,\underbrace{(1-P_{ij})}_{(1-\vec{\sigma}_i\cdot\vec{\sigma}_j)/2} \quad \text{on} \quad (\mathbb{C}^2)^{\otimes N}$$



	Heis XXX '28		**Haldane** '88 **– Shastry** '88
	$V_{Heis}(d) = \delta_{d,1}$		$V_{HS}(d) = \dfrac{1}{\sin^2(\frac{\pi}{N}d)} = \dfrac{1}{\mathrm{crd}^2}$
	magnetism		'lattice version' of fract q Hall effect / $su(2)_1$ WZW
higher Ham's	known		known Talstra Haldane '95
↑	↑ transfer matrix		↑
quantum integrab	Yangian structure Faddeev et al, late '70s	≠	Yangian **symmetry** Ha et al '92 degenerate affine Hecke alg Bernard et al '93
↓	↓ alg Bethe ansatz		↓
exact solv (spectrum)	up to solving BAE		in closed form Haldane '91

today: mimic this structure on this side

Plan: inhomog Heis vs spin Cal-Sut

	Heis XXX '28		Haldane '88 – Shastry '88
	↑ homog		↑ 'freezing'
	inhomog XXX		spin Cal-Sut
higher Ham's	known		known
↑	↑ transfer matrix qtr		↑ qdet
quantum integrab	Yangian structure Faddeev et al, late '70s	⟷ 'DAHA dual'	Yangian **symmetry** ↺ affine Schur–Weyl degenerate affine Hecke alg Bernard et al '93
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today: mimic this structure on this side

NB. generalises to XXZ level (nicer)

Plan: inhomog Heis vs spin Cal-Sut

	Heis XXX '28		Haldane '88 – Shastry '88
	↑ homog		↑ 'freezing'
	I inhomog XXX		II spin Cal-Sut
higher Ham's	known		known
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quantum integrab	Yangian structure Faddeev et al, late '70s	⟷ 'DAHA dual'	Yangian symmetry ↻ affine Schur–Weyl degenerate affine Hecke alg Bernard et al '93
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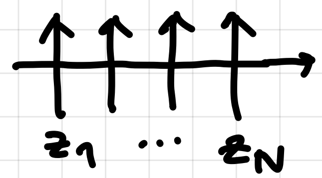
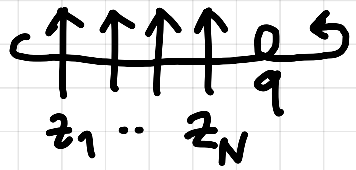
III **today:** mimic this structure on this side

NB. generalises to XXZ level (nicer)

IV bigger picture: long-range spin chains and quantum many-body systems

Inhomogeneous Heis XXX: hamiltonians

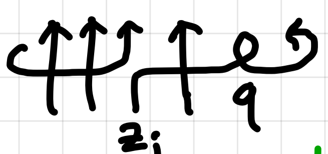
$L_0(u;\underline{z}) = u$  $= R_{0N}(u-z_N)\cdots R_{01}(u-z_1) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}_0$ RLL relations

$t(u;q;\underline{z}) = u$  $= tr_0[q^{\sigma_0^z} L_0(u;\underline{z})] = q\,A + q^{-1} D$ commute $\forall u$

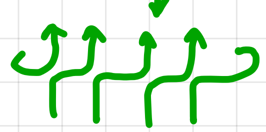
Bethe subalgebra

v1: expand at simple point $u = z_j$

'scattering operators'

$G_j := t(z_j) =$  $= R_{j\,j-1}(z_j - z_{j-1}) \cdots R_{j1}(z_j - z_1)\, q^{\sigma_j^z}$

$\times R_{jN}(z_j - z_N) \cdots R_{j\,j+1}(z_j - z_{j+1})$

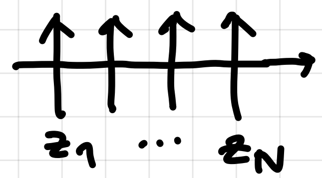


commute $\forall j$, $G_1 \cdots G_N = q^{2S^z}$

homog limit: transl op

semiclass limit: Gaudin

Inhomogeneous Heis XXX: hamiltonians

$L_0(u;\underline{z}) = u$  $= R_{0N}(u-z_N)\cdots R_{01}(u-z_1) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}_0$

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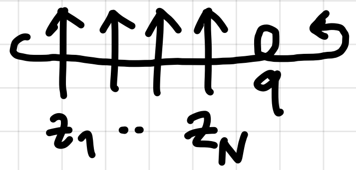
$t(u;q;\underline{z}) = u$  $= \mathrm{tr}_0[q^{\sigma_0^z} L_0(u;\underline{z})] = q\,A + q^{-1} D$

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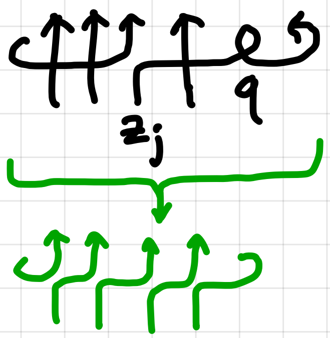
$G_j := t(z_j) =$  $= R_{j\,j-1}(z_j - z_{j-1}) \cdots R_{j1}(z_j - z_1)\, q^{\sigma_j^z} \times R_{jN}(z_j - z_N) \cdots R_{j\,j+1}(z_j - z_{j+1})$

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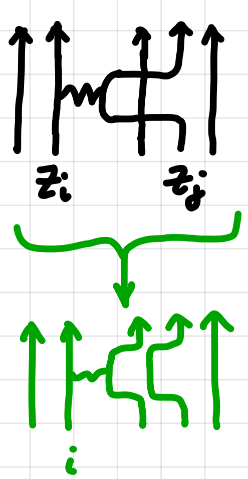
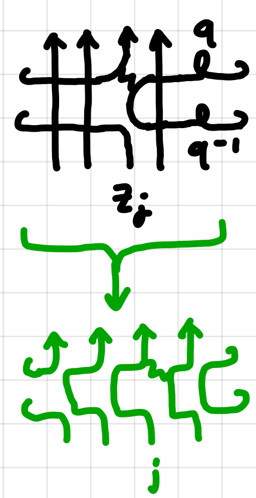
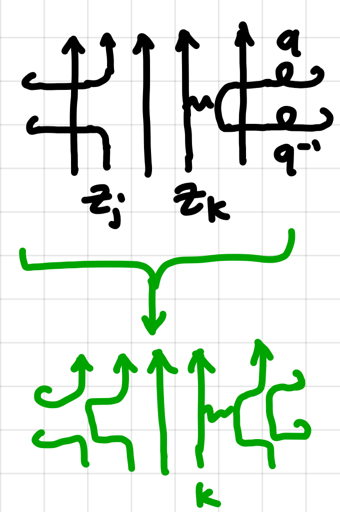
hamiltonians

$H_j := t'(z_j)\, t(z_j)^{-1}$

$\propto \sum_{i(<j)} \frac{1}{1-(z_i-z_j)^2} \times$  $+$  $+ \sum_{k(>j)} \frac{1}{1-(z_j-z_k)^2} \times$ 

$z_j \mapsto 1$
$q \mapsto 1$

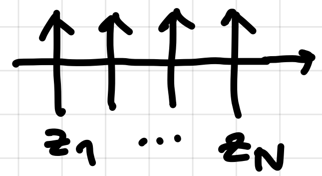
 $= \check{R}'(0) = 1 - P$

commute $\forall j$

note: poles at $z_i - z_j = \pm 1$

homog limit: H_{Heis}

Inhomogeneous Heis XXX: hamiltonians

$L_0(u;\underline{z}) = u$  $= R_{0N}(u-z_N)\cdots R_{01}(u-z_1) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}_0$

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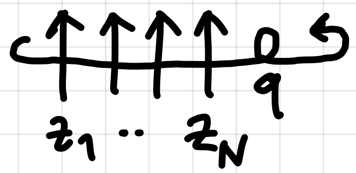
$t(u;q;\underline{z}) = u$  $= tr_0[q^{\sigma_0^z} L_0(u;\underline{z})] = q\,A + q^{-1}D$

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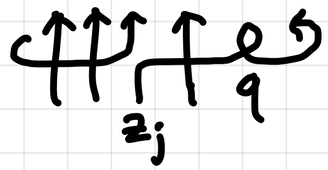
$G_j := t(z_j) =$  $= R_{j\,j-1}(z_j - z_{j-1})\cdots R_{j1}(z_j - z_1)\, q^{\sigma_j^z}$

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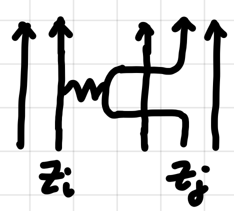
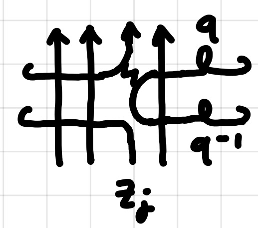
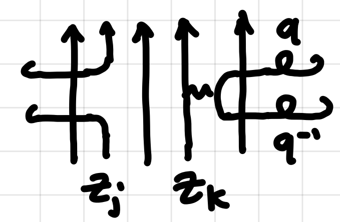
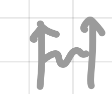
$\propto \sum_{i(<j)} \frac{1}{1-(z_i - z_j)^2} \times$  $+$  $+ \sum_{k(>j)} \frac{1}{1-(z_j - z_k)^2} \times$ 

 $= \check{R}'(0) = 1 - P$

commute $\forall j$

note: poles at $z_i - z_j = \pm 1$

v2: expand at $u = 0$

$$t(u) = \sum_{n=0}^{N} u^{-n}\, t_n(q;\underline{z})$$

$$t_0 = q + q^{-1}$$

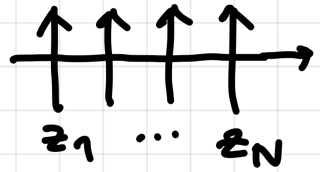
$$t_1 = \sum q^{\sigma_j^z}$$

$$t_2 = \sum_{i<j} q^{\sigma_j^z} P_{ij} - \sum z_j\, q^{\sigma_j^z}$$

$$t_3 = \sum_{i<j<k} q^{\sigma_k^z} P_{ij} P_{ik} - \sum_{i<j} (z_i + z_j)\, q^{\sigma_j^z} P_{ij} + \sum q^{\sigma_j^z} z_j^2$$

$\vdots$

Inhomogeneous Heis XXX: Bethe ansatz

$L_0(u;\underline{z}) = u$ 

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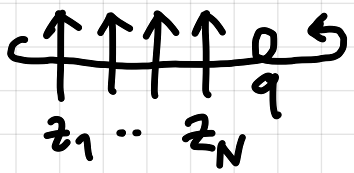
$t(u;q;\underline{z}) = u$ 

$= \mathrm{tr}_0\left[q^{\sigma_0^z} L_0(u;\underline{z})\right] = q\,A + q^{-1} D$

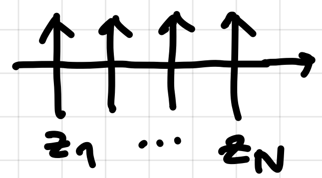
commute $\forall u$
Bethe subalgebra

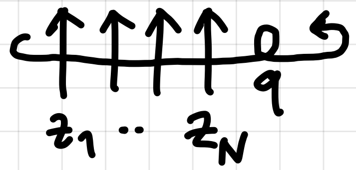
rep th: $\mathbb{C}^2_{z_1} \otimes \cdots \otimes \mathbb{C}^2_{z_N}$ tensor prod of evaluation reps

$$\check{R}_{i\,i+1}(z_i - z_{i+1})\, L_0(u;\ldots z_{i+1}\, z_i \ldots)$$
$$= L_0(u;\ldots z_i\, z_{i+1}\ldots)\, \check{R}_{i\,i+1}(z_i - z_{i+1})$$

$\check{R}(u) := P\,R(u)$
exchange rel^n

Inhomogeneous Heis XXX: Bethe ansatz

$L_0(u;\underline{z}) = u$  $= R_{0N}(u-z_N)\cdots R_{01}(u-z_1) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}_0$ RLL relations

$t(u;q;\underline{z}) = u$  $= \mathrm{tr}_0[q^{\sigma_0^z} L_0(u;\underline{z})] = q\,A + q^{-1} D$ commute $\forall u$

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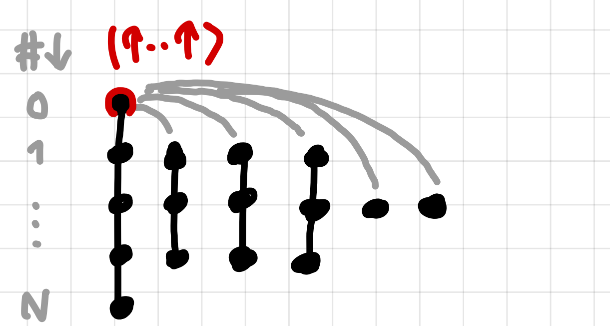
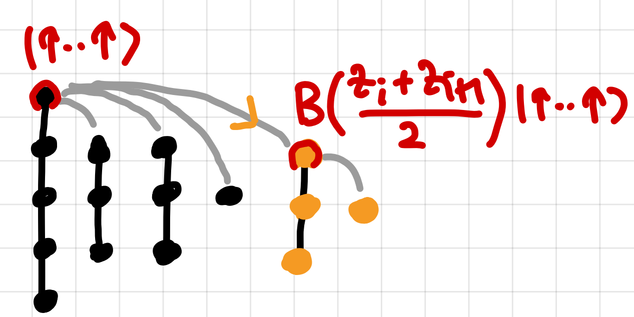
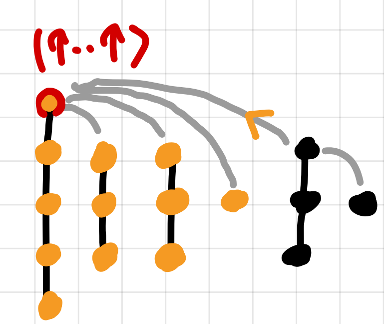
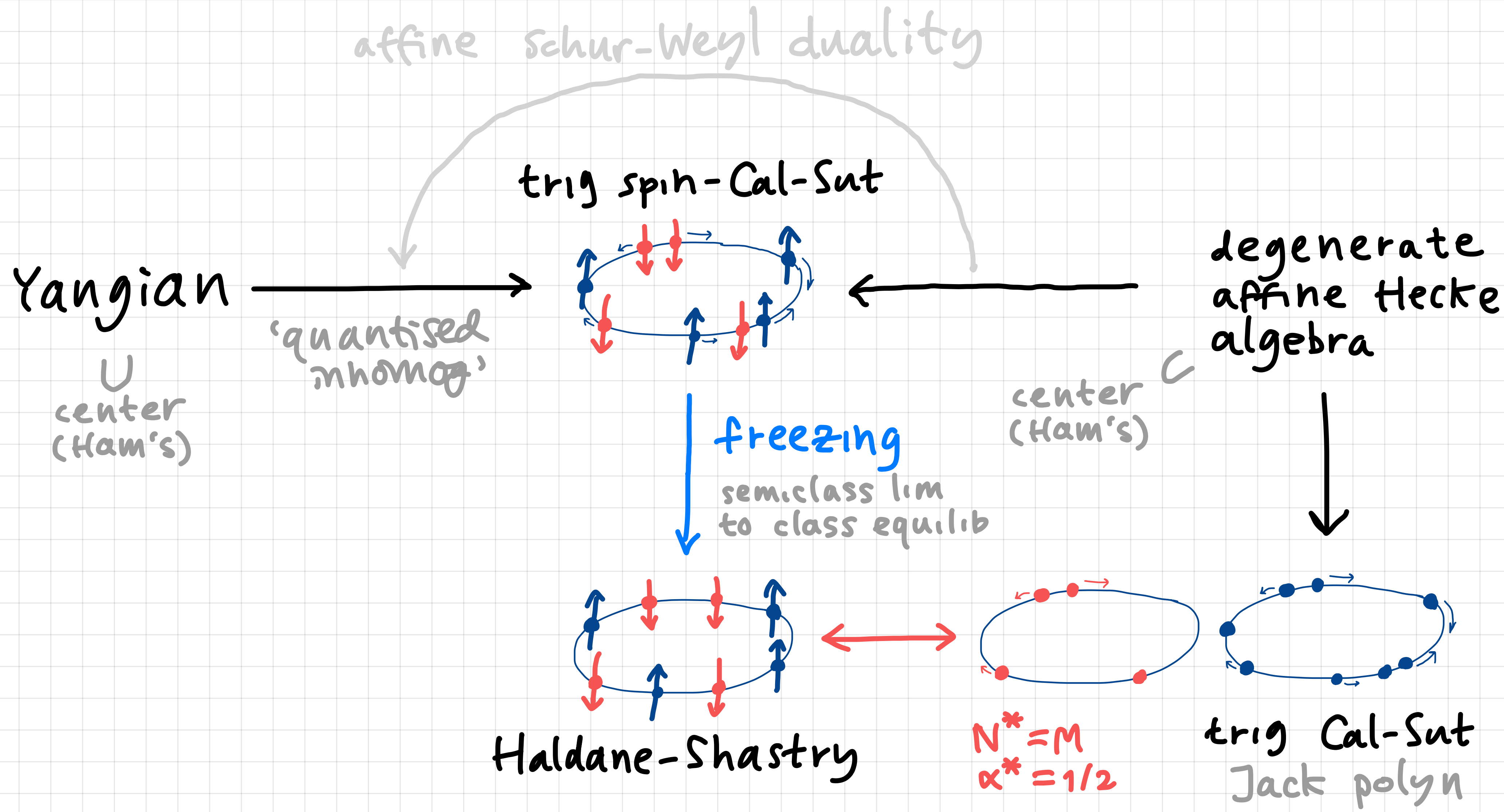
exchange rel^n

	generically $z_i - z_{i+1} \neq \pm 1$	fusion $z_i - z_{i+1} = \pm 1$	
$\check{R}_{i\,i+1}(z_i - z_{i+1})$	invertible	$\propto 1 \pm P$ not invertible	
Yangian rep	irred	reducible, indecomposable	
order of z_i, z_{i+1}	not important (irreps isom)	determines invariant subspace	
structure of Hilb space	 $\underline{5} \oplus 3\cdot\underline{3} \oplus 2\cdot\underline{1} = \underline{2}^{\otimes 4}$	 $\mathrm{im}(1-P)$	 $\mathrm{im}(1+P)$
alg Bethe ansatz	✓	✓ mod ↗ ✓	✓ ✗

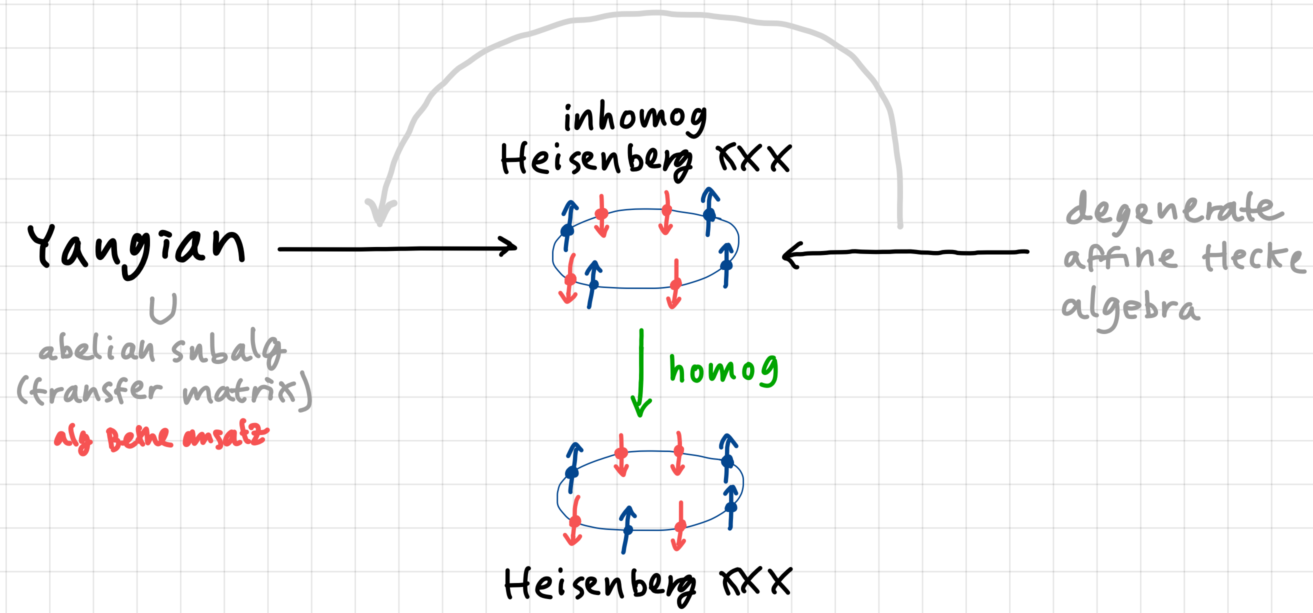
Spin-Calogero-Sutherland: overview

Drinfeld '86

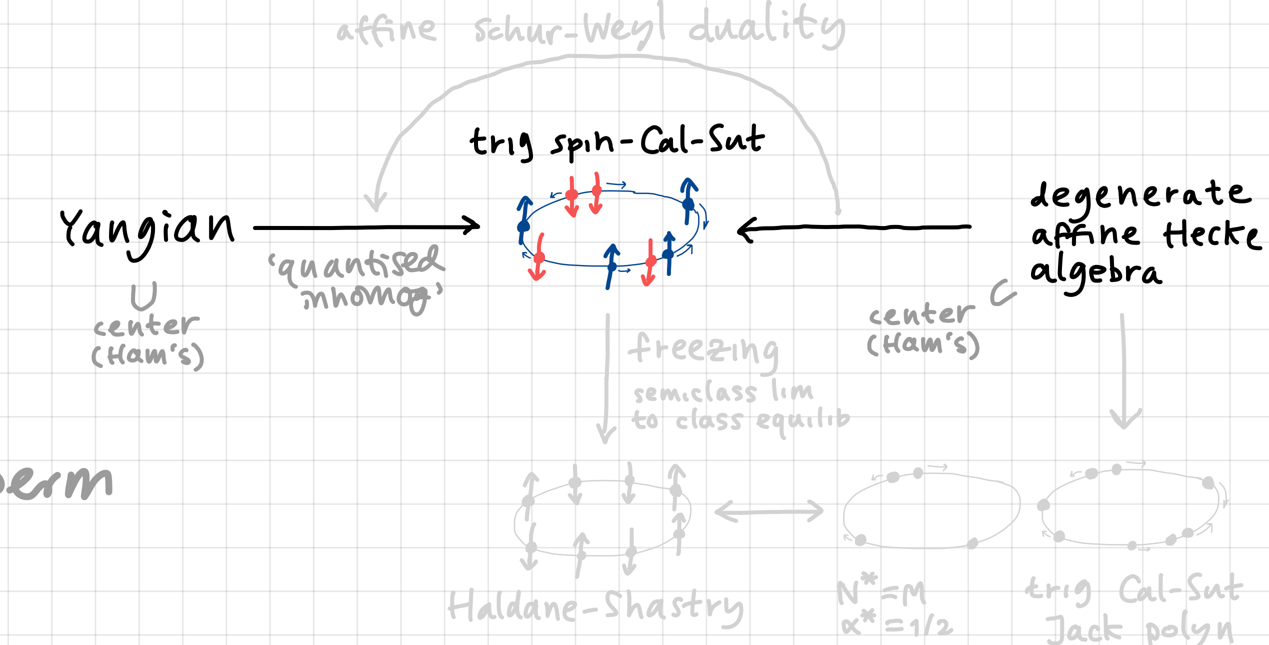
Bernard et al '93



cf inh XXX:
('DAHA dual')



Spin-Calogero-Sutherland: hamiltonians



view evaluation rep $\mathbb{C}^2_z \cong \mathbb{C}^2[z]$

'ambient space' $\mathbb{C}^2[z]^{\otimes N} \cong (\mathbb{C}^2)^{\otimes N} \otimes \mathbb{C}[z_1, \dots, z_N]$

rep of deg AHA: $1 \otimes s_{ij}$ $\quad z_i \leftrightarrow z_j$

$1 \otimes d_j$ $\quad$ Dunkl $\quad d_j = \frac{1}{\beta} z_j \partial_{z_j} + \mathrm{rat}(\underline{z}) \cdot \mathrm{perm}$

$d_i s_{i\,i+1} = s_{i\,i+1} d_{i+1} + 1$

$[d_i, d_j] = 0$

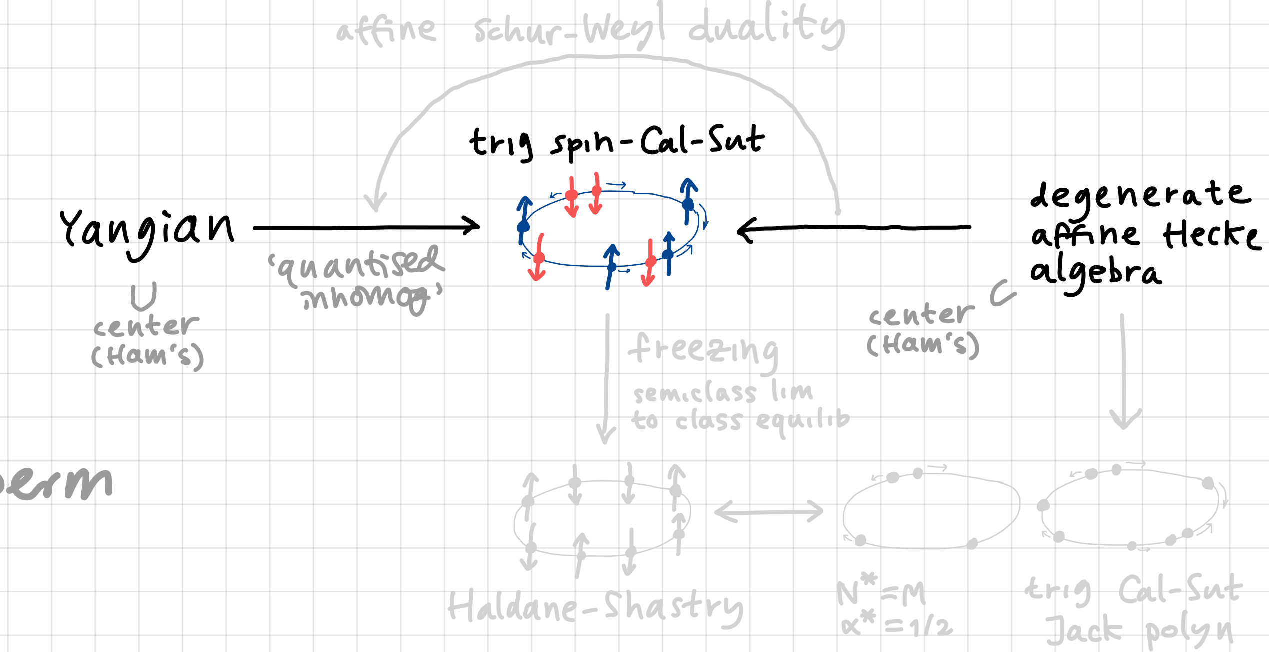
'physical subspaces' $\widetilde{\mathcal{H}}^{\pm} := \{ |\psi\rangle : s_{i\,i+1} P_{i\,i+1} |\psi\rangle = \pm |\psi\rangle \}$ bosons / fermions

rep of Yangian (by deg AHA rel^{ns})

$\widetilde{L}_0^{\pm}(u) := L_0(u; \mp \underline{d}) = u$ $\mp d_1 \cdots \mp d_N$ 'quantised' operator-valued inhomogeneities

Drinfeld '86
Bernard et al '93

Spin-Calogero-Sutherland: hamiltonians



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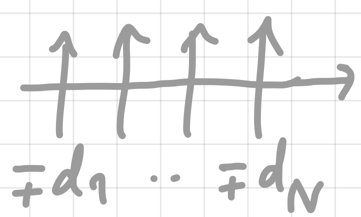
$\tilde{L}_0^{\pm}(u) := L_0(u; \mp \underline{d}) = u$ 

'quantised' operator-valued inhomogeneities

Drinfeld '86
Bernard et al '93

center

$$\mathrm{qdet}\, \tilde{L}_0^{\pm}(u) = \prod_{j=1}^{N} \frac{u \mp d_j + 1}{u \mp d_j}$$

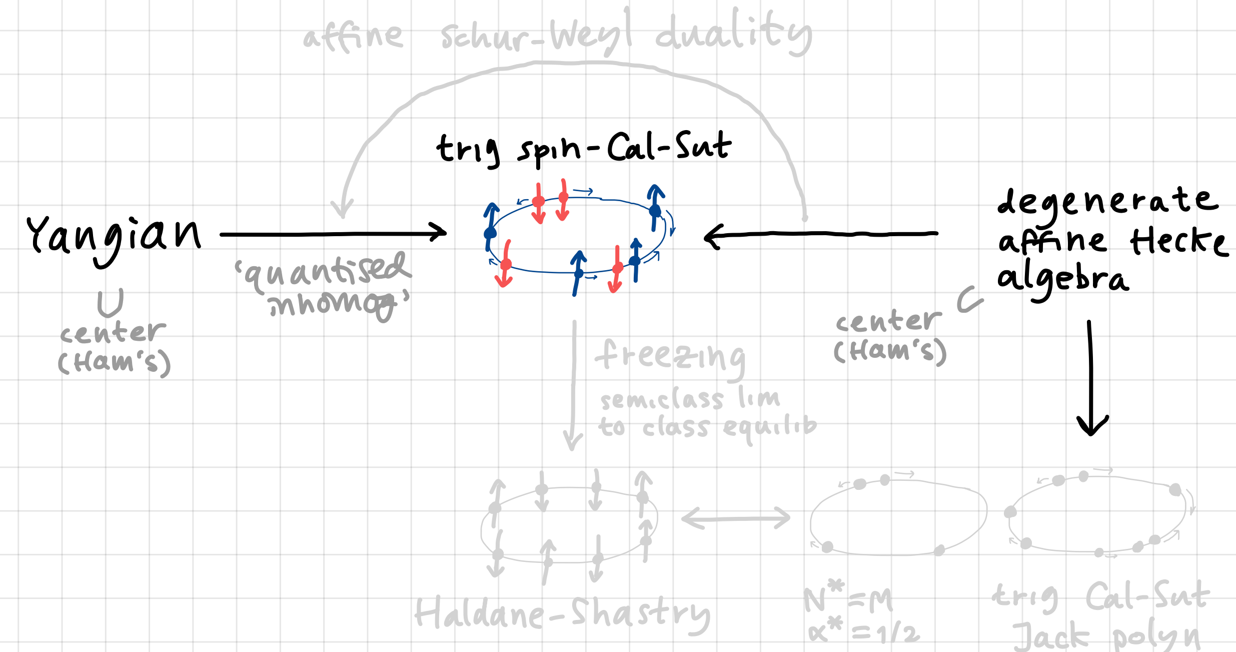
$\rightsquigarrow$ symm polyn in Dunkls (center of deg AHA)

$$\beta \sum d_j = \tilde{P}_{CS} = -i \sum_{j=1}^{N} \partial_{x_j} \qquad z_j = e^{i x_j} \qquad \text{momentum}$$

$$\beta^2 \sum d_j^2 = \tilde{H}^{\pm}_{CS} = -\frac{1}{2} \sum_{j=1}^{N} \partial^2_{x_j} + \sum_{i<j}^{N} \frac{\beta(\beta \mp P_{ij})}{4 \sin((x_i - x_j)/2)^2} \qquad \text{Hamiltonian}$$

$\vdots$

Spin-Calogero-Sutherland: spectrum



deg AHA: $1\otimes s_{ij}$ $\quad z_i \leftrightarrow z_j$

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$[d_i, d_j] = 0$

'physical subspaces' bosons/fermions

$$\tilde{\mathcal{H}}^\pm := \bigcap_{i=1}^{N-1} \ker(s_{i\,i+1} P_{i\,i+1} \mp 1)$$

$$\tilde{L}_0^\pm(u) := L_0(u; \mp \underline{d})$$

structure

$$\tilde{\mathcal{H}}^\pm \cong \bigoplus_\lambda \tilde{\mathcal{H}}^\pm_\lambda \,, \qquad \tilde{\mathcal{H}}^\pm_\lambda \begin{cases} \tilde{P}(\lambda) = \sum \lambda_j \\ \tilde{E}(\lambda) = \frac{1}{2}\sum \lambda_j^2 + \frac{\beta}{2}\sum (N-2j+1)\lambda_j \\ \text{Yangian irrep} \end{cases}$$

} as in scalar case

} degeneracies

partitions $\ell(\lambda)\leq N$

$(\mathbb{C}^2)^{\otimes N} \supset$ 'effective inh Heis spin chain' $\xrightarrow{\;?\;} \tilde{\mathcal{H}}^\pm_\lambda$

inhomog $\tilde{z}_j := \mp \delta_j(\lambda)$

$$\delta_j(\lambda) = \frac{1}{\beta}\lambda_j + \frac{N-2j+1}{2}$$

nonsymm Jack $\quad d_j E_\lambda = \delta_j(\lambda)\, E_\lambda \quad (\lambda = \lambda^+)$

Takemura Uglov '97

Spin-Calogero-Sutherland: spectrum

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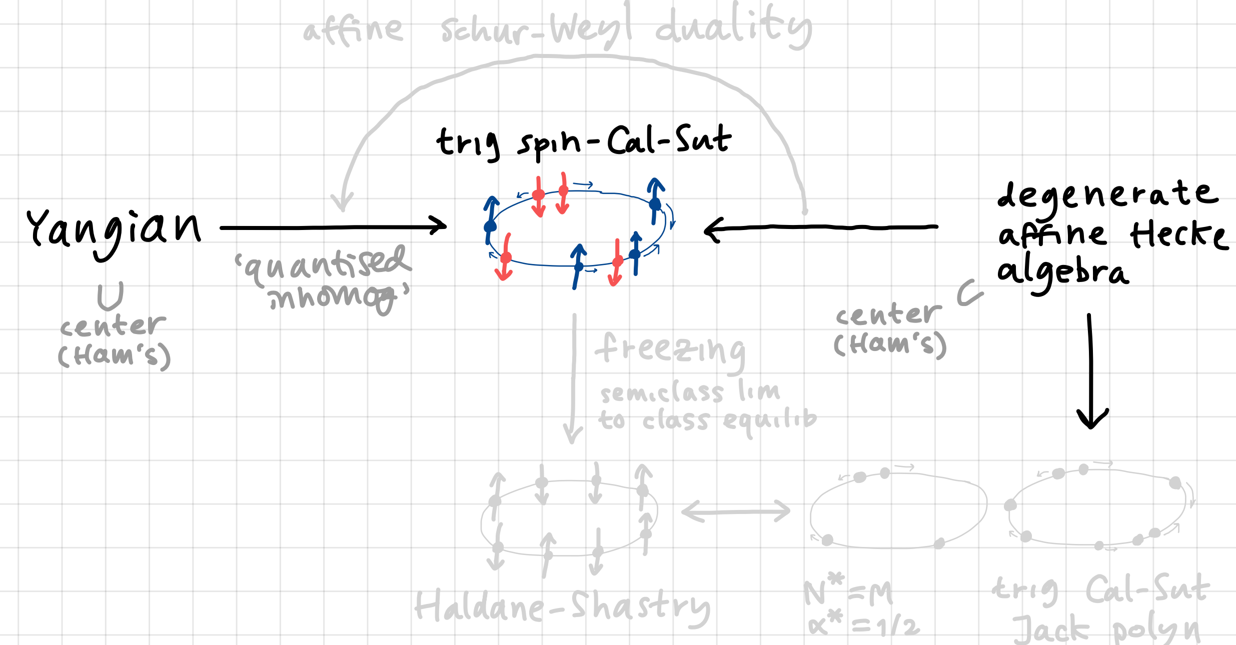
$d_i\, s_{i\,i+1} = s_{i\,i+1}\, d_{i+1} + 1$

$[d_i, d_j] = 0$



'physical subspaces' $\quad \tilde{\mathcal{H}}^{\pm} := \bigcap_{i=1}^{N-1} \ker(s_{i\,i+1} P_{i\,i+1} \mp 1)$

bosons/fermions $\quad \tilde{L}_0^{\pm}(u) := L_0(u; \mp \underline{d})$

structure

$\tilde{\mathcal{H}}^{\pm} \cong \bigoplus_{\lambda} \tilde{\mathcal{H}}^{\pm}_{\lambda}$, $\quad \tilde{\mathcal{H}}^{\pm}_{\lambda} \begin{cases} \tilde{P}(\lambda) = \sum \lambda_j \\ \tilde{E}(\lambda) = \frac{1}{2}\sum \lambda_j^2 + \frac{\beta}{2}\sum (N-2j+1)\lambda_j \\ \text{Yangian irrep} \end{cases}$ } as in scalar case } degeneracies

partitions $\ell(\lambda)\le N$

$(\mathbb{C}^2)^{\otimes N} \supset$ 'effective inh Heis spin chain' $\xrightarrow{\ \wr\ }$ $\tilde{\mathcal{H}}^{\pm}_{\lambda}$

inhomog $\tilde{z}_j := \mp \delta_j(\lambda)$

$\delta_j(\lambda) = \frac{1}{\beta}\lambda_j + \frac{N-2j+1}{2}$

nonsymm Jack $\quad d_j E_\lambda = \delta_j(\lambda) E_\lambda$ $\quad (\lambda = \lambda^+)$

Takemura Uglov '97

note: fusion iff $\lambda_j = \lambda_{j+1}$ $\quad \delta_j(\lambda) - \delta_{j+1}(\lambda) = 1$

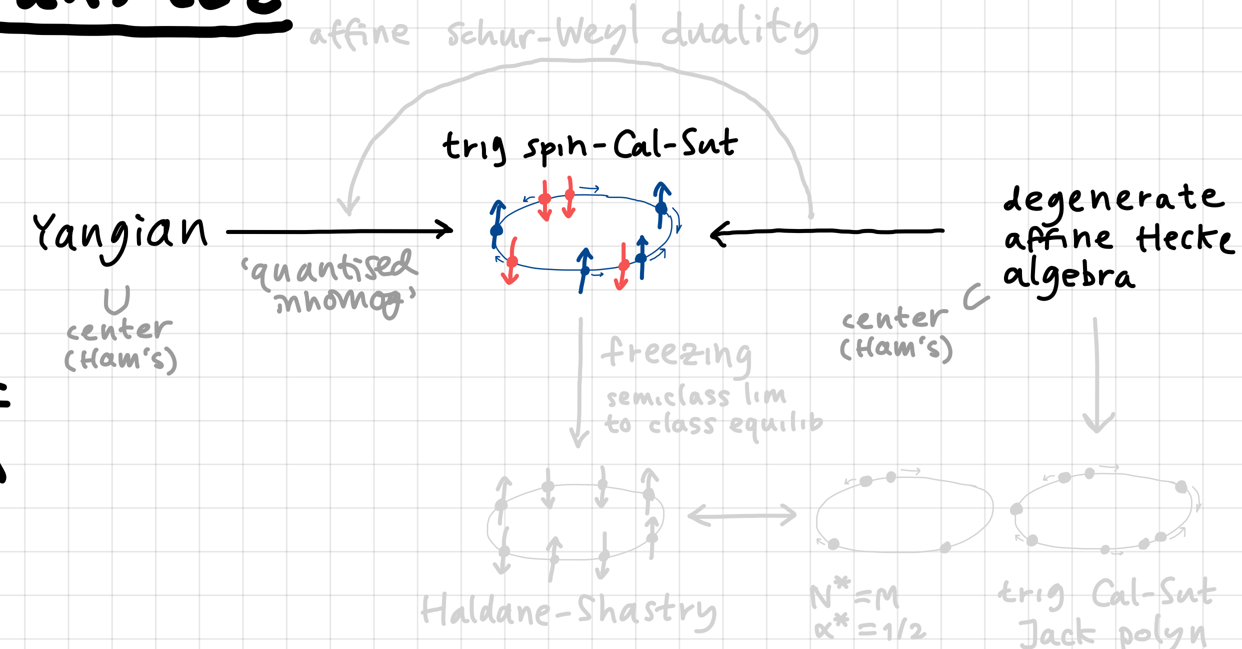
ex $\lambda = (\lambda_1 > \dots > \lambda_N)$ strict: $\quad (\mathbb{C}^2)^{\otimes N} \xrightarrow{\sim} \tilde{\mathcal{H}}^{\pm}_{\lambda}$

$\tilde{z}_j = \mp\delta_j(\lambda)$ generic $\quad |\uparrow\dots\uparrow\rangle \longmapsto P^{\pm}_{\lambda}(\underline{z})\,|\uparrow\dots\uparrow\rangle$ $\quad P^{\pm}_{\lambda} = \begin{cases} P_\lambda^{(\frac{1}{\beta})} \\ \mathrm{Vand}\cdot P_\lambda^{(\frac{\beta}{\beta+1})} \end{cases}$

ex fermions: $\quad (\mathbb{C}^2)^{\otimes N_\lambda} \xrightarrow{\sim} \tilde{\mathcal{H}}^{-}_{\lambda}$

$N_\lambda = N - n_{\text{repeats}}(\lambda)$ $\quad |0_\lambda\rangle \longmapsto |\tilde{0}_\lambda\rangle = \sum_{\mu\in S_N\lambda} E_\mu(\underline{z})$ "$\prod_{i<j:\,\mu_i<\mu_j} (-1)\check{R}(\delta's)$" $|0_\lambda\rangle$

Spin-Calogero-Sutherland: internal Bethe ansatz



deg AHA: $1\otimes s_{ij}$ $\quad z_i \leftrightarrow z_j$

$1\otimes d_j$ $\quad$ Dunkl $\quad d_j = \frac{1}{\beta} z_j \partial_{z_j} + \mathrm{rat}(\underline{z})\cdot \mathrm{perm}$

$d_i\, s_{i\,i+1} = s_{i\,i+1}\, d_{i+1} + 1$

$[d_i, d_j] = 0$

'physical subspaces'
bosons/fermions

$$\widetilde{\mathcal{H}}^{\pm} := \bigcap_{i=1}^{N-1} \ker(s_{i\,i+1} P_{i\,i+1} \mp 1) \cong \bigoplus_{\lambda} \mathcal{H}^{\pm}_{\lambda}$$

$$\widetilde{L}_0^{\pm}(u) := L_0(u; \mp \underline{d})$$

spin-CS eigenspaces $\qquad \mathcal{H}^{\pm}_{\lambda} \cong$ 'effective inh Heis spin chain' $\subset (\mathbb{C}^2)^{\otimes N}$ $\qquad$ fusion iff $\lambda_j = \lambda_{j+1}$

$$L_0(u; \{\mp \delta_j(\lambda)\}) \qquad \delta_j(\lambda) = \frac{1}{\beta}\lambda_j + \frac{N-2j+1}{2}$$

Ferrando, JL, Serban, Levkovich-Maslyuk '23

Heisenberg-like symm^s

G_i, H_i don't act on $\widetilde{\mathcal{H}}^{\pm}$

$$\widetilde{t}_2^{\pm} = \underbrace{\sum_{i<j} q^{\sigma_j^z} P_{ij}}_{\text{'twisted Casimir'}} \pm \underbrace{\sum q^{\sigma_j^z} d_j}_{\text{'twisted } \widetilde{P}_{CS}\text{'}}$$

$$t_3 = \sum_{i<j<k} q^{\sigma_k^z} P_{ij} P_{ik} \underbrace{\pm \sum_{i<j} q^{\sigma_j^z} P_{ij}(d_i + d_j)}_{\text{beyond qdet: not Yangian invariant}} + \underbrace{\sum q^{\sigma_j^z} d_j^2}_{\text{'twisted } \widetilde{H}^{\pm}_{CS}\text{'}}$$

$\vdots$

$$\widetilde{t}^{\pm}(u) = \sum_{n=0}^{N} u^{-n}\, \widetilde{t}_n^{\pm}(q)$$

diagonalised per $\widetilde{\mathcal{H}}^{\pm}_{\lambda}$ $\qquad \prod B(u_m; \{\mp\delta_j\})\,|0_\lambda\rangle \longmapsto \widetilde{B}(u_1)\cdots\widetilde{B}(u_M)\,|\widetilde{0}_\lambda\rangle$

extreme twist $q \to 0, \infty$: Gelfand-Tsetlin basis of Takemura Uglov '97

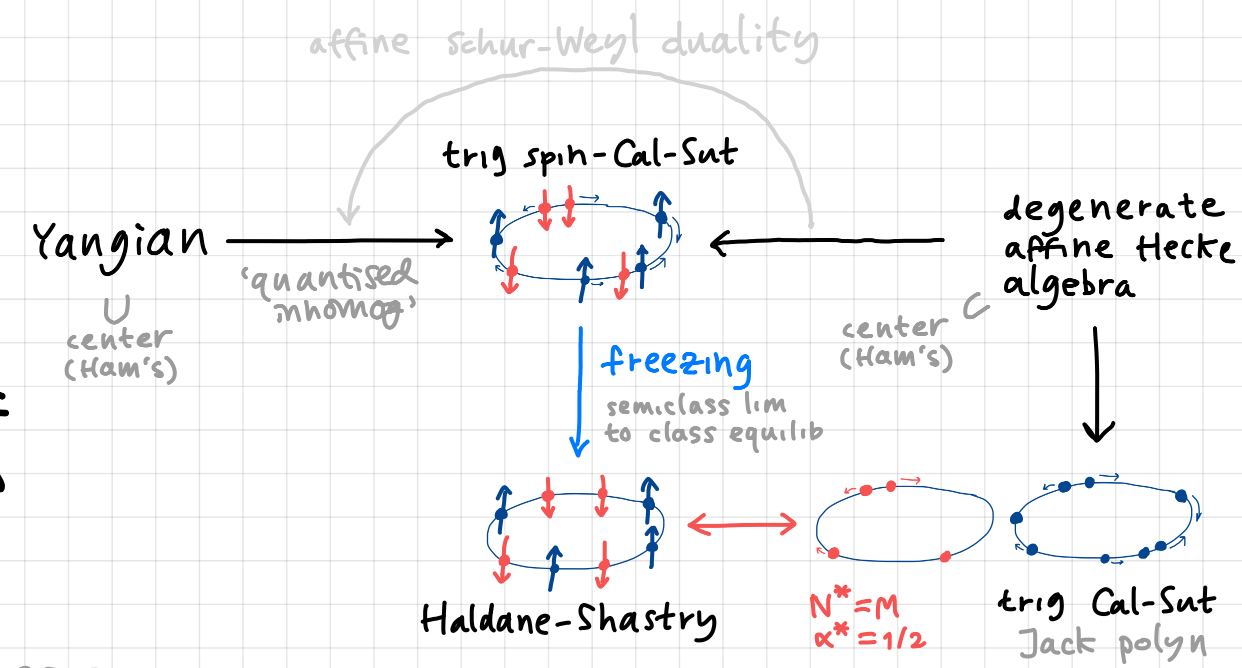
Spin-Calogero-Sutherland: freezing

deg AHA: $1\otimes s_{ij}$ $\quad z_i \leftrightarrow z_j$

$1\otimes d_j$ Dunkl $\quad d_j = \frac{1}{\beta} z_j \partial_{z_j} + \mathrm{rat}(\underline{z})\cdot \mathrm{perm}$

$d_i\, s_{i\,i+1} = s_{i\,i+1}\, d_{i+1} + 1$

$[d_i, d_j] = 0$



'physical subspaces'
bosons/fermions

$$\tilde{\mathcal{H}}^{\pm} := \bigcap_{i=1}^{N-1} \ker(s_{i\,i+1} P_{i\,i+1} \mp 1) \cong \bigoplus_{\lambda} \mathcal{H}^{\pm}_{\lambda}$$

$$\tilde{L}^{\pm}_0(u) := L_0(u; \mp \underline{d})$$

spin-CS eigenspaces $\quad \mathcal{H}^{\pm}_{\lambda} \cong$ 'effective inh Heis spin chain' $\subset (\mathbb{C}^2)^{\otimes N}$ $\quad$ fusion iff $\lambda_j = \lambda_{j+1}$

$$L_0(u; \{\mp\delta_j(\lambda)\}) \qquad \delta_j(\lambda) = \frac{1}{\beta}\lambda_j + \frac{N-2j+1}{2}$$

freezing: $\beta \to \infty$ (semi)classical limit

$$\tilde{H}^{\pm}_{CS} = -\frac{1}{2}\sum_{j=1}^{N} \partial_{x_j}^2 + \sum_{i<j}^{N} \frac{\beta(\beta \mp P_{ij})}{4\sin((x_i - x_j)/2)^2} = \beta^2 \underbrace{\sum_{i<j}^{N} \frac{1}{4\sin^2((x_i - x_j)/2)}}_{\text{minimize}} + \beta \underbrace{\sum_{i<j}^{N} \frac{\mp P_{ij}}{4\sin((x_i - x_j)/2)^2}}_{\text{cst} \mp H^{HS}} + \cdots$$

$z_j \to e^{2\pi i j/N}$: $\qquad \sum_{i<j}^{N} \frac{1-P_{ij}}{\sin^2 \frac{\pi}{N}(i-j)}$

$/\langle p_1, \ldots, p_{N-1}, p_N - N\rangle$

$$L^{HS}_0(u) = \lim_{\beta\to\infty} \tilde{L}^{\pm}_0(u)$$

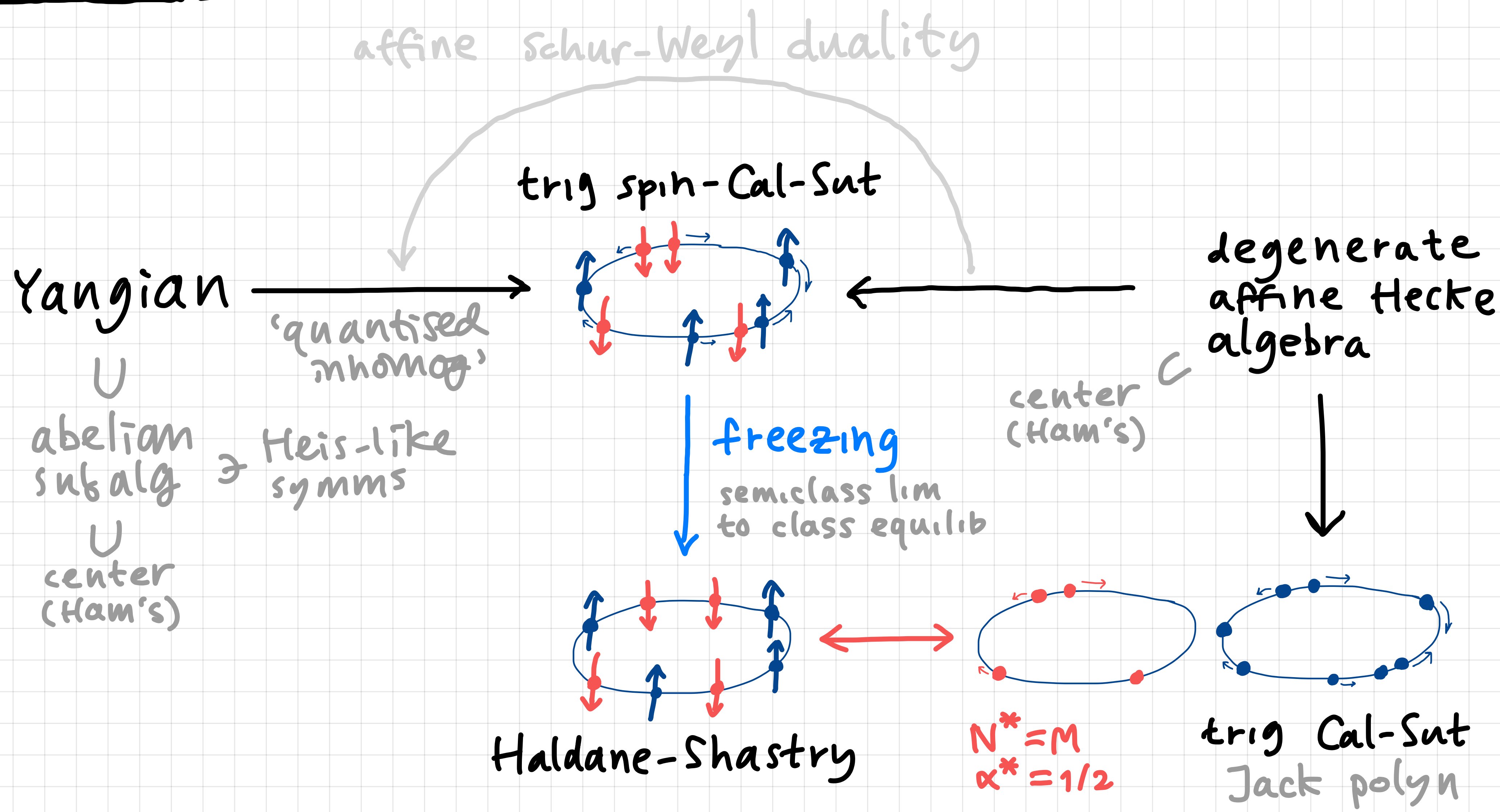
internal Bethe ansatz

$$|0^{HS}\rangle = \sum_{i_1 < \cdots < i_M}^{N} \underbrace{\mathrm{Vand}(z_{i_1}, \ldots, z_{i_M})^2 \times P_{\nu}^{(\frac{1}{2})}(z_{i_1}, \ldots, z_{i_M})}_{\text{scalar CS eigenf}^n,\ \alpha^* = \frac{1}{2}} \sigma^-_{i_1} \cdots \sigma^-_{i_M} |\uparrow\cdots\uparrow\rangle$$

(generalises) JL Serban '22

Haldane '91

Summary



upshot for each* partition λ get 'effective inhomog Heis XXX spin chain'

multiplicities $\lambda_j = \lambda_{j+1}$: fusion invariant subspace

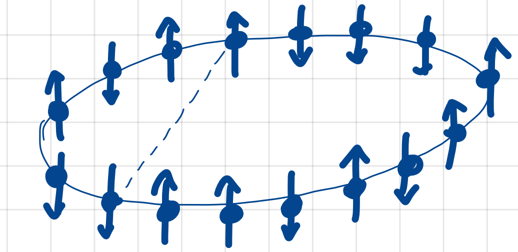
ABA works fine $\tilde{B}(u_1) \cdots \tilde{B}(u_M) |\tilde{0}_\lambda\rangle$

new eigenbasis for spin CS (and HS by freezing)

towards separation of variables (SoV) for long-range models

cf AdS/CFT integrability

Outlook landscape of long-range spin chains



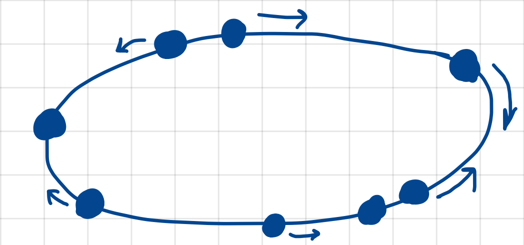
interaction range →

↓ degree of spin symmetry

	nearest neighbour	intermediate range	long range
anisotropic	**Heisenberg XYZ** Sutherland '70 Baxter '73	?	?
partially isotropic	**Heisenberg XXZ** Orbach '58 Yang-Yang '66	Klabbers JL '23	**q-deformed HS** Uglov '95, JL '18 JL Pasquier Serban '22
isotropic	**Heisenberg XXX** Heisenberg '28 Bethe '31	**Inozemtsev** Inozemtsev '90 Inozemtsev '90–'00, Klabbers JL '22	**Haldane-Shastry** Haldane '88, Shastry '88 Haldane '91, Bernard et al '93

Outlook landscape of quantum many-body systems

interaction range



	nearest neighbour contact (positions)	**intermediate range** elliptic (positions)	**long range** trig (positions)
(?) elliptic (momenta)	?	'DELL'	?
relativistic trig (momenta) affine Hecke alg	?	ell Ruijsenaars	trig Ruijsenaars-Macdonald
non-rlt rational (momenta) degenerate AHA	~Lieb-Liniger?	ell Cal-Sut	trig Cal-Sut

towards **grand unified theory** for quantum-integrable long-range spin chains?