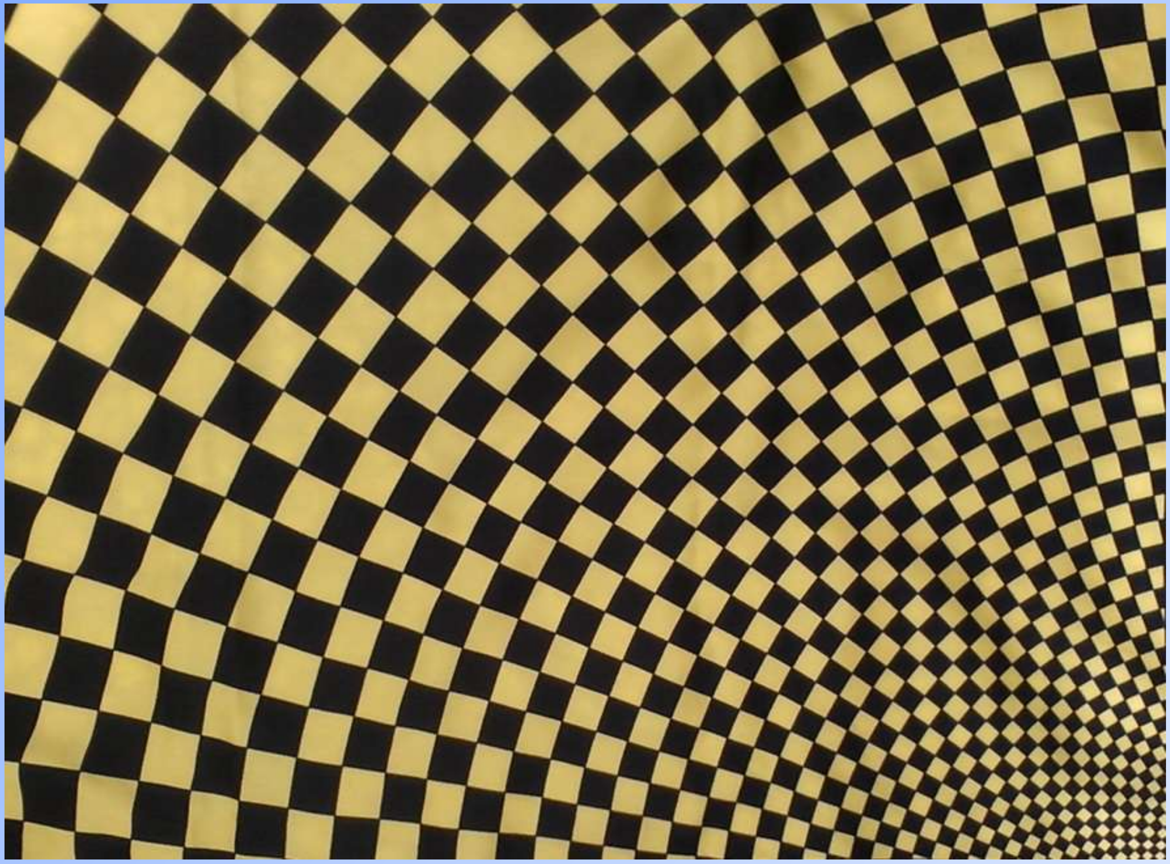
Informal learning

inside and outside school



Les Diablerets,
October 2022
Maria Dedò

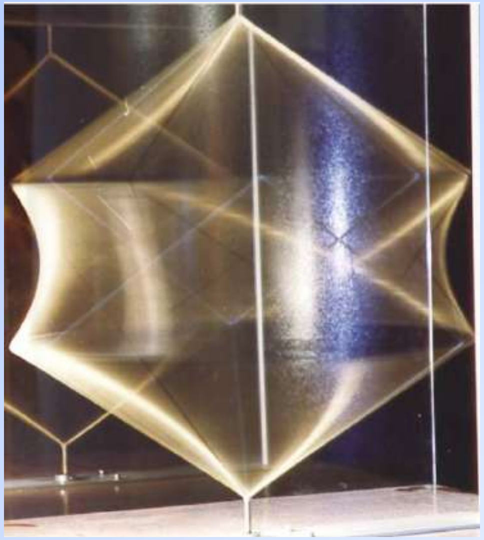
Origins…

Oltre il compasso, Pisa, 1992

Franco Conti (1943? - 2003)

30 years ago, a maths exhibition was something **very strange**.

A home-made exhibition…

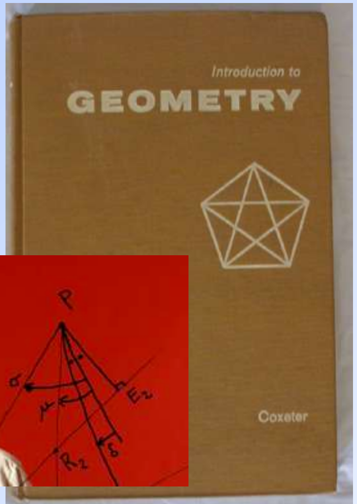


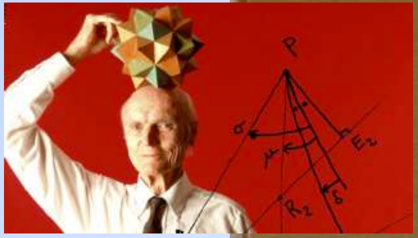
e.g. an old lp record player

and a bicycle wheel

From a personal point of view

Even in periods where exhibitions or other kinds of informal communication were quite rare (or didn't exist at all), I was lucky to interact with people who had very clear ideas about informal learning (and its **connection** with formal learning).





From a personal point of view

I attended *different activities*: I have been a professor at university; since the 80's I was also concerned with courses for pre-university teachers; since the midst of 90's I began to create and organize exhibitions… **BUT…**

… but it hasn't been difficult to reconcile these *different activities*: they naturally combined, without contradictions.

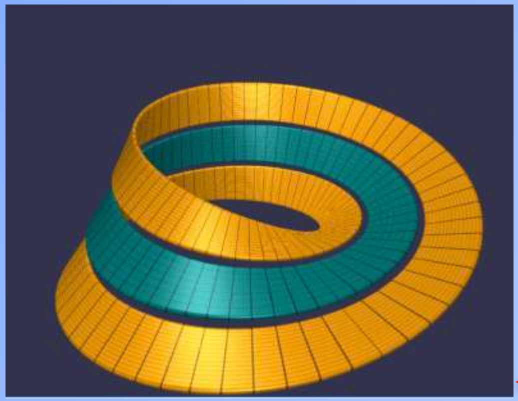
Simmetria, giochi di specchi, 2000

matemilano, Milano, Museo NST, 2003

mateinitaly, Milano, Triennale, 2014

From a personal point of view

I could benefit from a continuous **exchange of experiences and examples** between material for exhibitions and material for courses (courses for university students, courses for pre-university teachers…).

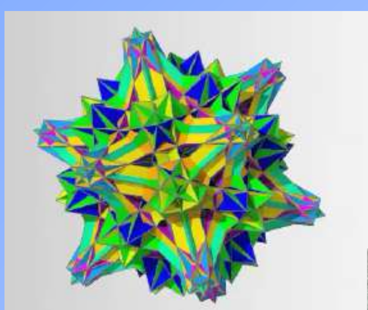


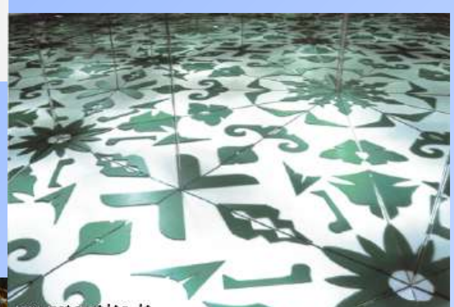
Sometimes, the same example can be used with general public just to raise curiosity about *strange* geometrical shapes, and with university students who are formally learning topology (here: the example of a cylinder which double covers a Moebius band).

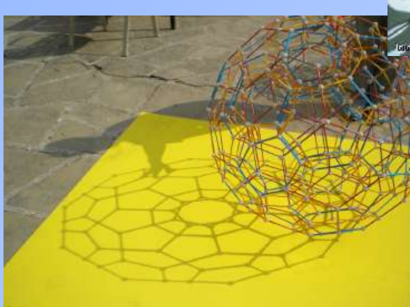
Of course the **accompanying message** will be different in the two cases, but the second one can simply be an extension of the first one.

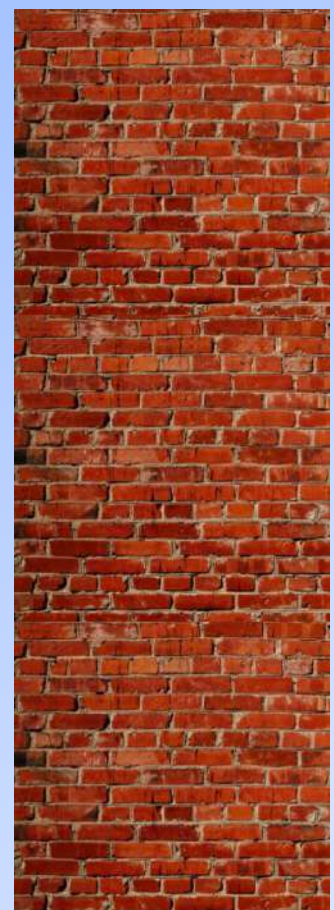
However…

Often, however, it does not seem to work in this smooth way: apparently, there seems to be a *gap* between an informal way of communicating mathematics and the usual, formalized, way of teaching and learning mathematics at school and university. And sometimes the gap becomes *an insurmountable wall.*









$$\left\{\left[\left(\frac{7}{2}\right)^2-\left(\frac{5}{2}\right)^2-\frac{9}{2}\right]:\frac{1}{2}-\left(\frac{5}{3}\right)^2:\left(\frac{5}{3}\right)^2\right\}\times\left\{\left[\left(\frac{2}{3}\right)^3-\frac{1}{8}\right]:\left[\frac{7}{9}+\left(\frac{1}{2}\right)^2\right]\times\left(4-\frac{7}{3}\right)^2\right\}$$

$$\left\{\left[\left(\frac{1}{2}\right)^2+\left(\frac{5}{3}\right)^2-\frac{1}{36}\right]^3\times\left(2-\frac{4}{3}\right)^3+\frac{1}{2}\right\}:\left\{\left[\left(1+\frac{1}{2}\right)^2-\left(2-\frac{7}{8}\right)\right]-\left(1-\frac{3}{4}\right)^2\right\}$$

$$\left\{\frac{2}{3}\times\left[\left(1+\frac{1}{2}-\frac{1}{4}\right)^2-\left(\frac{1}{2}-\frac{1}{4}\right)^2\right]\right\}:\left\{8\times\left[\left(\frac{3}{2}\right)^2\times\left(1-\frac{2}{3}:\frac{4}{3}\right)-1\right]\right\}$$

$$\left\{\left[\left(\frac{2}{3}+\frac{3}{4}-\frac{4}{5}\right):\frac{74}{15}+\frac{7}{8}\right]^5:\frac{5}{2}+\frac{3}{5}\right\}:\left[\left(\frac{3}{11}+\frac{5}{8}-\frac{35}{44}\right)\times\left(1+\frac{17}{27}\right)+\frac{1}{4}\right]$$

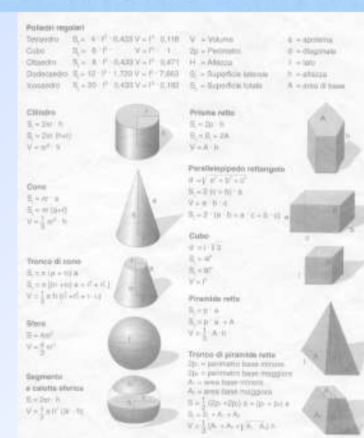
$$\frac{5}{8}+\left(\frac{1}{2}\right)^4-\left\{\left[3+\left(\frac{5}{3}-\frac{5}{6}\right):5\right]:\frac{19}{3}\right\}^2+\left(\frac{1}{2}\right)^5\times\left(\frac{1}{2}\right)^3:\left(\frac{1}{2}\right)^6+\left(1-\frac{1}{16}\right)$$

13. $\sqrt[m]{a^p}\cdot\sqrt[n]{b^q}=\sqrt[mn]{a^{pn}\cdot b^{qm}},(a,b\geq 0);$

14. $\sqrt[m]{\sqrt[n]{a}}=\sqrt[mn]{a}=\sqrt[n]{\sqrt[m]{a}},(a\geq 0);$

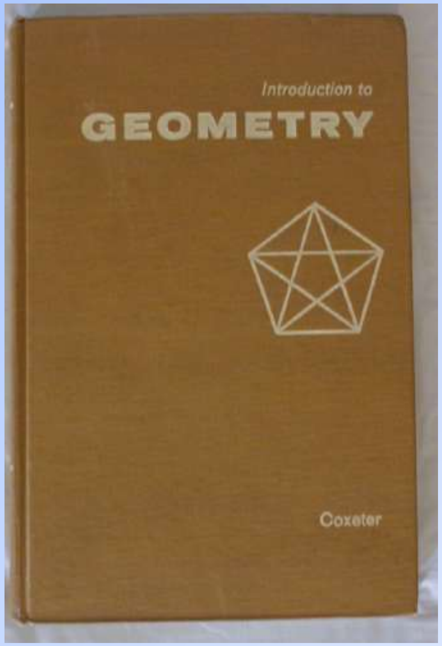
15. $\sqrt[m]{a^p}:\sqrt[n]{b^q}=\sqrt[mn]{a^{pn}:b^{qm}},(a\geq,b>0);$

16. $\sqrt{a^2}=|a|,(a\in\mathbb{R});$



An example

An example taken from this book.
It is a standard maths book, formal and systematic; however, it is not rare to find some comments **linking** the formal mathematical situation to one which could be used to describe it in an informal context. For example:



$$(RS)^{-1} = S^{-1}R^{-1},$$

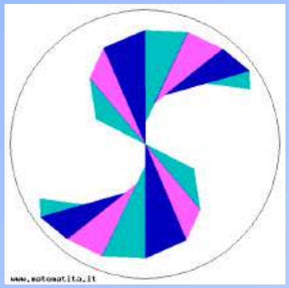
not $R^{-1}S^{-1}$. (This becomes clear when we think of R and S as the operations of putting on our socks and shoes, respectively.)

I found it illuminating. But, when I used it with students, quite often I had to admit that some of them were switching off the audio: apparently, they thought I was joking and they were waiting for me to become serious again.

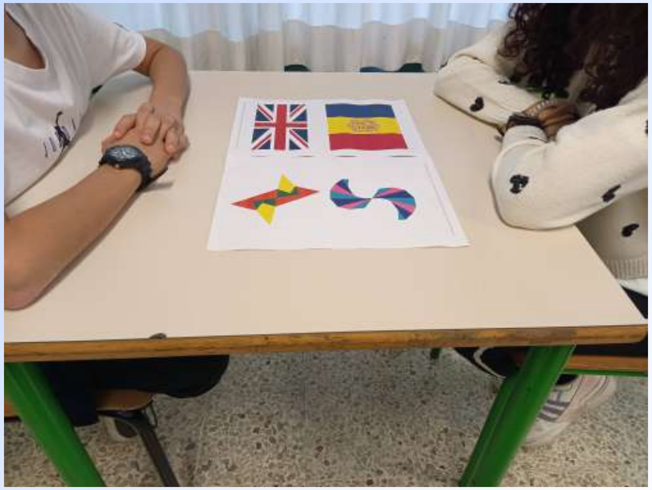
Formal *versus* informal

It is useful for a teacher (at any level) to collect many examples of seeing an informal situation into a mathematical formula (or, the other way round, seeing a mathematical fact in an informal context).

Surely a difficulty is given by *mathematical language*: it may be disconcerting to recognize the same problem in a page with formulas and symbols and in a situation of everyday life.



An example. Deciding whether a rotation of 180° belongs to the symmetry group of a given image is **the same as** putting the image on a table and asking if two persons, sitting at the table one in front of the other, see the image in the same way.



Formal *versus* informal communication

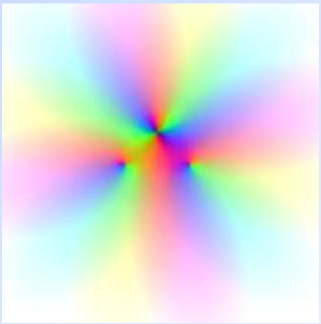
Formal communication in classrooms:

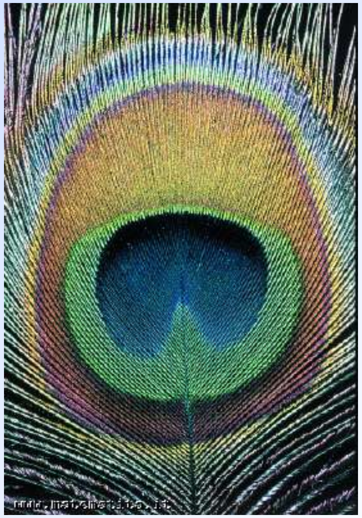
- **passive** attitude of pupils;
- pupils (and teachers) are afraid of **mistakes**;
- ideas hidden by **techniques**;
- we **lose the meaning** of what we are doing;
- mathematics looks **rigid** and **boring**;
- mathematics has **nothing to do** with everyday life;
- mathematics has **nothing to do** with imagination;
- the **answers** are given before the questions have been formulated.

Informal communication outside:

- **active** attitude of visitors;
- positive role of **mistakes**;
- **ideas** more than techniques;
- **give meaning**;
- mathematics can be **challenging**;
- mathematics can be seen **everywhere**;
- **imagination** plays an important role;
- problems provoke **questions.**

Of course good teachers (and there are a lot of them!) don't behave in this way, but they often seem to feel cooped up… (at least in Italy).

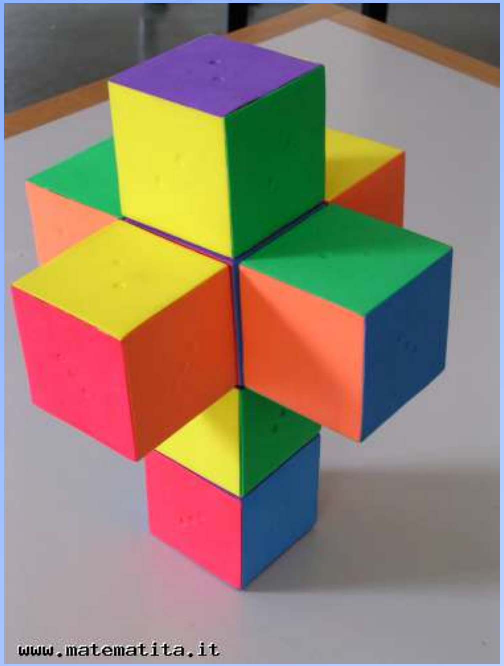




Informal communication

A public event like an exhibition, aiming at communicating mathematics to general public, necessarily has to use an informal way of communicating, with a large use of images, models, interactive virtual animations,…

… which could be useful in a classroom too!



Making some use of informal learning and filling the gap between formal and informal could be precious for improving the **effectiveness** of school learning.

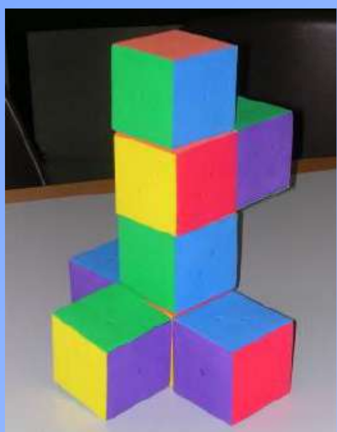
Advantages of informal communication

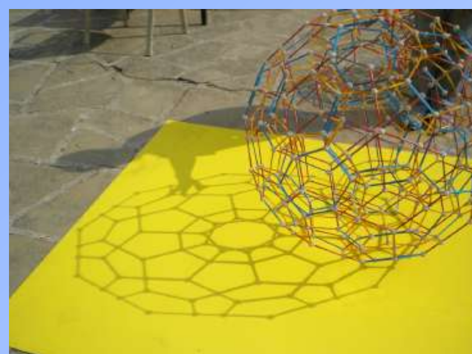
- It is captivating, **challenging**, not boring;
- it naturally underlines the **ideas** which are behind a mathematical fact: the aim is to give **meaning**, without getting lost in technical details;
- it makes it easy to get a collaborative atmosphere, where a **mistake** can be seen for what it is: a step towards a better comprehension.

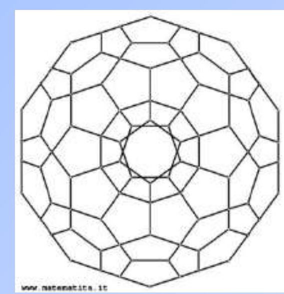
... which could be useful in a classroom too!

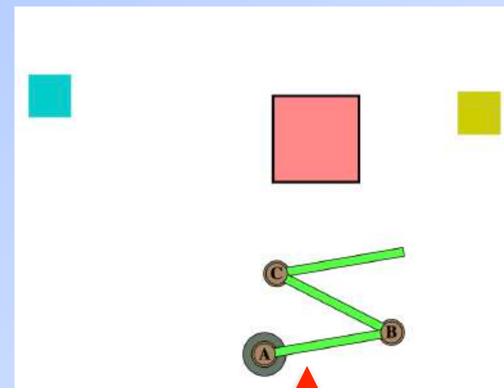
Advantages of informal communication

An informal communication does not pretend to *teach*, at least not always; it may just *show* something: and this may have the effect to make (some of) the visitors willing to know more…









Of course just watching a video, or moving a robotics arm in an interactive animation, is not enough to learn 4d geometry, but it raises curiosity and, most of all, it **provokes questions**.
And a good question is much better than the right solution of an exercise!

… which is precious (and necessary!) in a classroom too!

Advantages of informal communication

In an informal communication, it is often possible to push people to **do** something, to have an **active** role.

Of course, again, using an animation like this one does not teach us formal details about classification of wallpaper patterns, but…

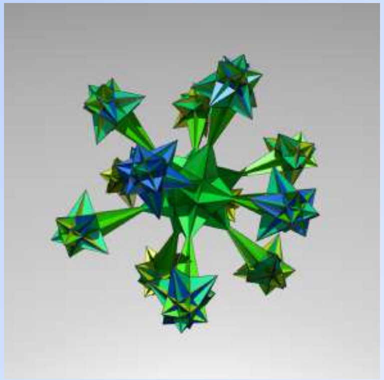
…but we get a sort of operative comprehension: *our hands learn* something before our brain. And this may become a concrete help when, later on, we meet the same subject in a systematic way.

… and this could be useful in a classroom too!

Risks of informal communication

A verbal message is much easier to **check** when it is written in formal language. To check the correctness of an informal verbal message should include the checking of how the message *can be interpreted* by laymen: which is impossible (or possible only to a small extent).

The risk to pass a message which is contrary to what we want to say is always there…!

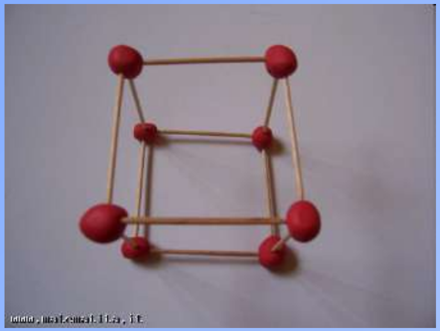


An "experiment".

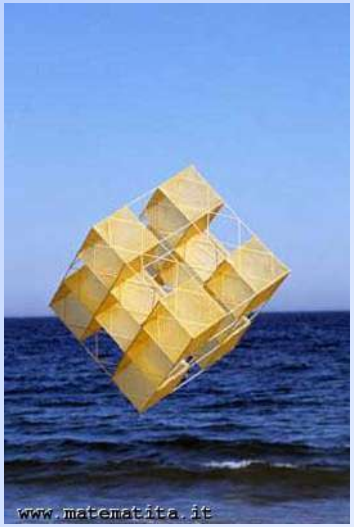
Write a formal definition (e.g. the definition of *group*) in usual mathematical language; if you let it check by n (>1) different mathematicians, you will get the same answer by each one. Then write an informal explanation of what is a group and ask them to check it: it is very likely you get n different answers!

Risks of informal communication

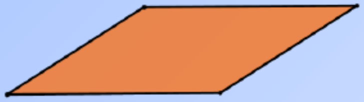
When we use images, or models, this is even more difficult: **any** image and **any** model is, necessarily, *a bit wrong*; which should **not** prevent us from using them.







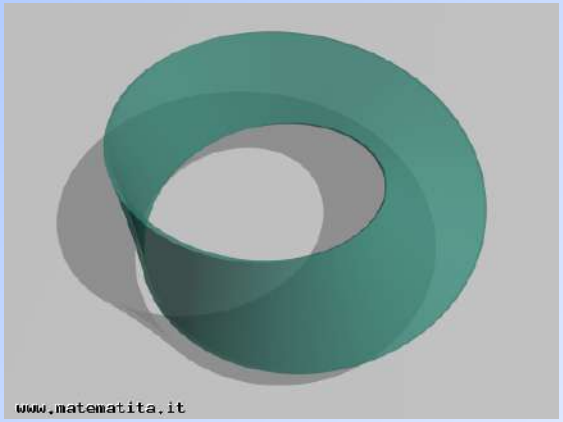
We all see cubes in these images, while we perfectly know they are not cubes. This overlapping between a mental image and a physical image can create problems in communicating with someone who does not yet have a clear mental image of a cube.



…è uno di quei quadrati del cubo.
… it is one of those squares of the cube.

Images and models: a long example

How thick should a Moebius band be?



We met this apparently meaningless question while working for the exhibition ***matemilano***, where we had some physical models and some virtual models of Moebius bands.

This work was involving mathematicians (who would have liked a thickness 0), but also graphic designers and architects (who need of course some thickness >0 for the, real or virtual, object).

A long example

The question opened a lot of interesting considerations, for example:

- Mathematicians, or mathematics students (those who were fighting for having thickness 0 in the virtual images of a Moebius band) immediately saw Moebius bands in these sculptures, without even noticing the thickness (evidently, mental images are very powerful!).

- Once noticing the thickness, they may come to the conclusion that the 3d object is the product of a Moebius band and an interval: **but this is false** (it is a solid torus)!

As for the example of the cube, this points out the problems in a communication between someone who already has a (strong) mental image about a certain math concept and someone who doesn't.

Risks of informal communication

In an informal communication we deliberately decide to give up formal rigour, statements and definitions. So we are moving on a slippy territory…

We need to keep *a certain kind* of rigour. But what does it mean and how can we decide if something is *rigorous enough* in an informal communication?

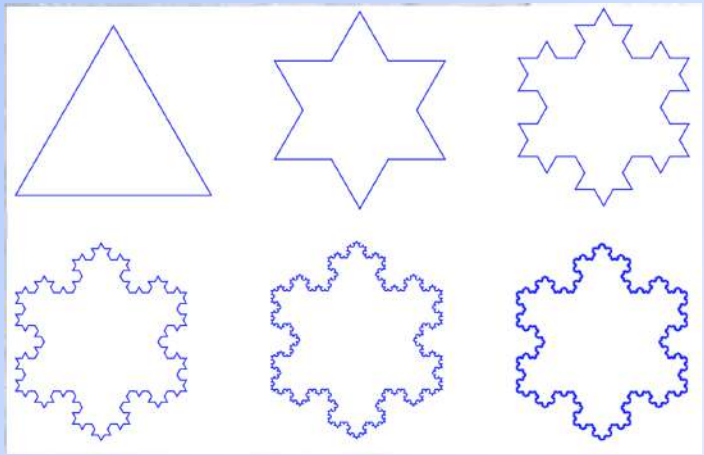
A second "experiment".
Start with an informal message, addressed to general public, and imagine to use it with a different target; e.g. a university mathematics student listened to it and asks you some more info, maybe in order to pass to a formal level.
There are two possibilities:

- you keep the info in the original message and add something else: then the original message was ok.
- you tell the student to forget the original message and begin over again: then something in the original message was wrong (not only for the student, but also for general public!).

Is it rigorous enough?

Sometimes we feel an image is not *rigorous enough* and a verbal message must be added. There is an obvious risk: the message related to the image dominates the verbal one!

Even if we write a message like the one here below, it is likely that the effect of the image will be to think that the last one **is** a fractal.



The n-th level of a process leading to a fractal is **NOT** a fractal when $n < \infty$.

Often **examples** are precious in an informal communication: through examples we can give an idea of a mathematical concept, avoiding formal definitions and statements.

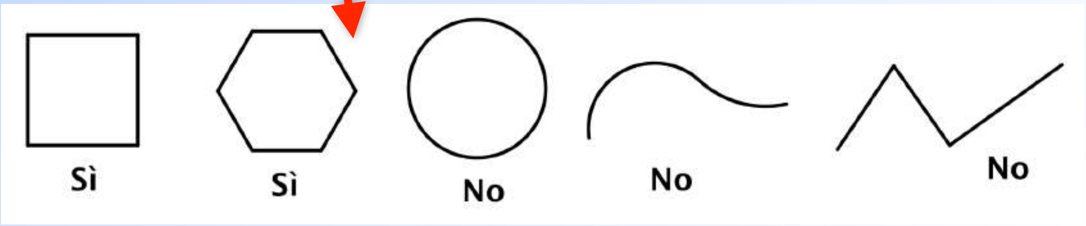
But: is this *rigorous enough*? Sometimes the answer is given by the *quantity* (and the *choice*) *of examples*.

The role of examples

A first example: the definition of polygon.

The definition of polygon on the great majority of (italian) school textbooks is either wrong or incomprehensible by pupils, or both.

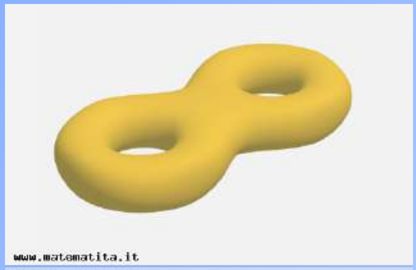
One could simply make some examples and say “These are polygons, these are not”. This **can** be a good choice **if** the *quantity* and the *variety* of shapes is big enough (**not** like this…!)

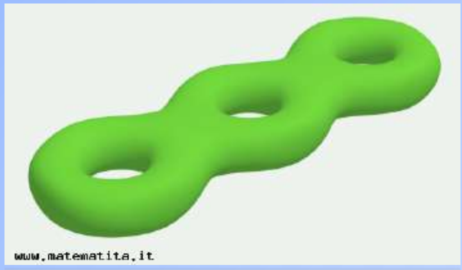


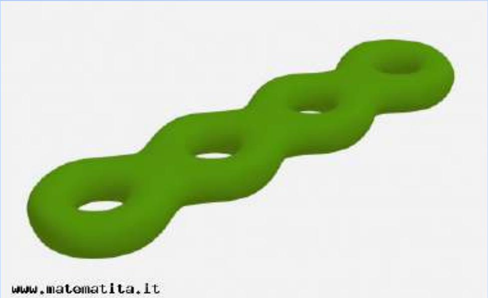
The role of examples

A second example: the genus of a surface.



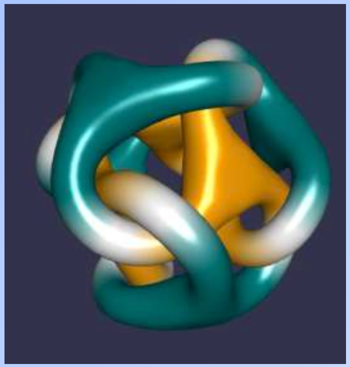


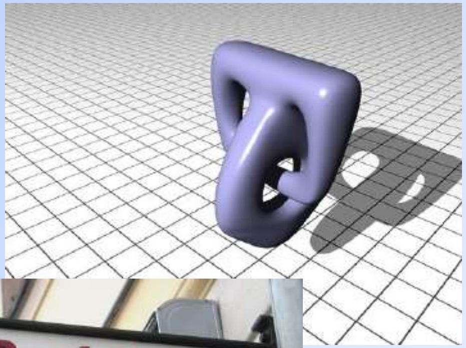




In order to tell what is the genus of a surface, it is very different if we use only the examples here above or if we also use some of the examples below…

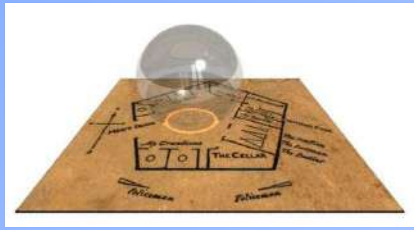







Analogy (and ambiguity)

Analogy can be an extremely powerful tool to tell some mathematical stories, even going beyond what are the knowledges of the people we are speaking with.



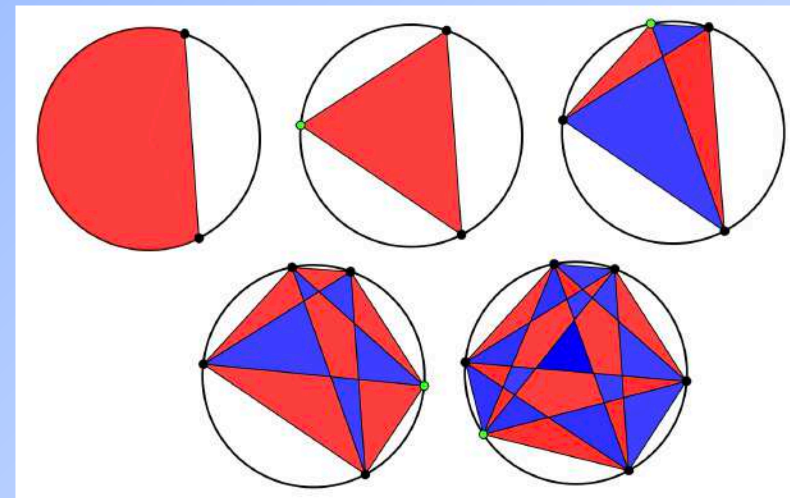
For example, playing with analogy can be a very effective way to tell something about 4-dimensional geometry.

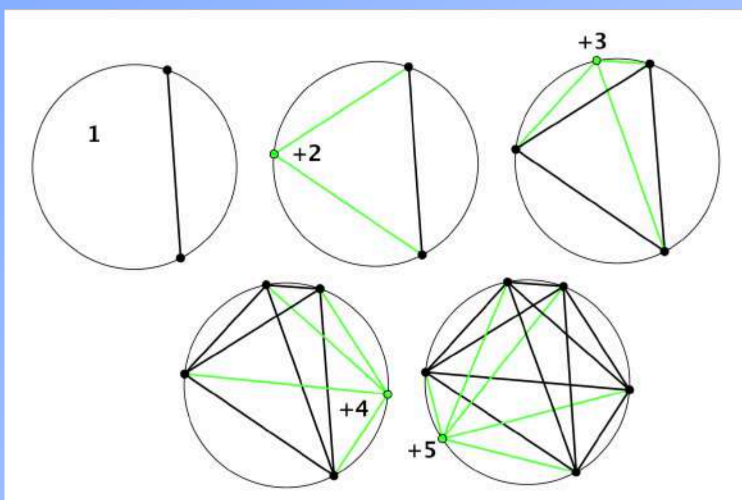
Playing with analogy can create a favourable context for the genesis of mental associations, giving very good results, but we should not forget we are in a slippery situation…

… analogy can also lead to mistakes, if we don’t learn to distinguish between analogy supported by some evidence in the structure of what we describe and analogy which could be a simple coincidence.

Two examples from elementary geometry

points	2	3	4	5	…	n
chords	1	3	6	10	…	$n(n-1)/2$??
regions	2	4	8	16	…	2^{n-1} ??





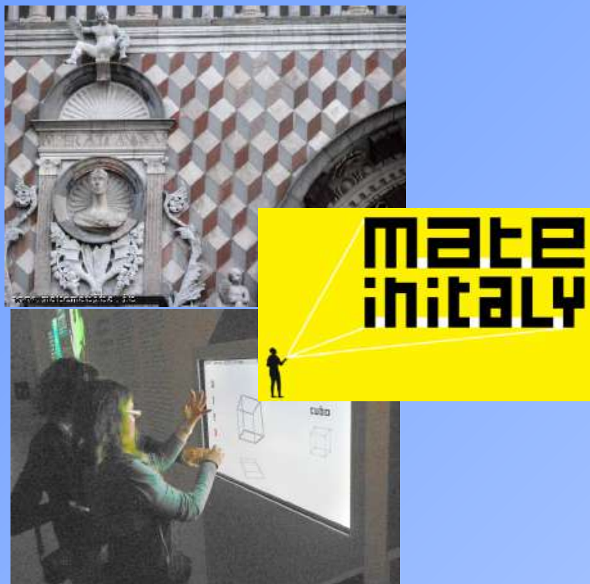
The *structure* of the construction gives a good reason for going on in the second row: it is not only analogy!

points	2	3	4	5	6	n
chords	1	3	6	10	15	$n(n-1)/2$
regions	2	4	8	16	31	

But we find no such reason for the third row: this was just a coincidence!

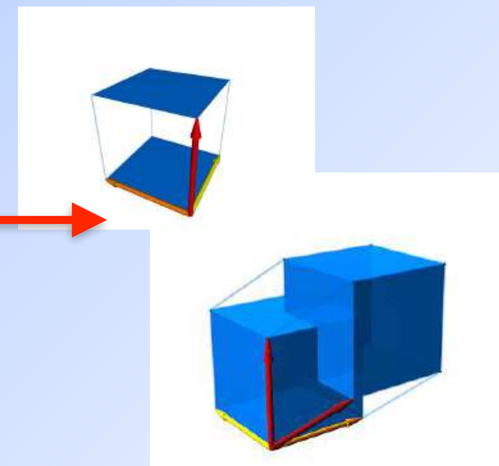
$$(n^4-6n^3+23n^2-18n+24)/24 = \binom{n}{0}+\binom{n}{2}+\binom{n}{4}$$

Another double example (4d geometry)

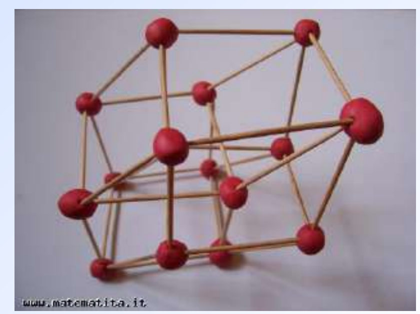


A square has 4 vertices; a cube has 8 vertices… will this go on?
Can we say that a 4-hypercube has 16 vertices, a 5-hypercube has 32, a *n*-hypercube has 2^n vertices?

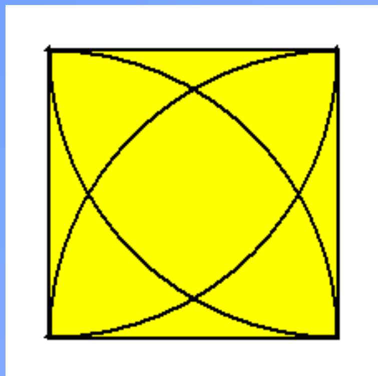
Yes! Prismatic construction gives us a good reason.



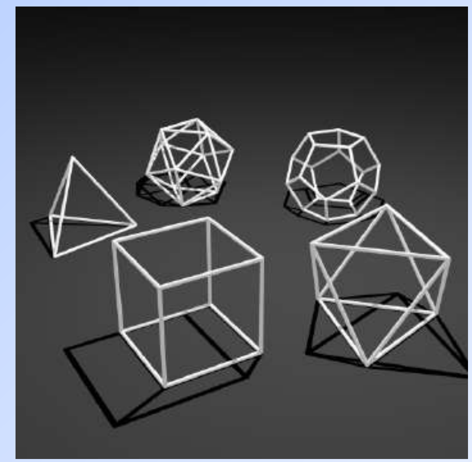
Coordinates may give us another good reason:
(±1,±1) are the coordinates of the 4 vertices of a square,
(±1,±1,±1) … of the 8 vertices of a cube,
(±1,±1,±1,±1) … of the 16 vertices of a 4d-hypercube, …



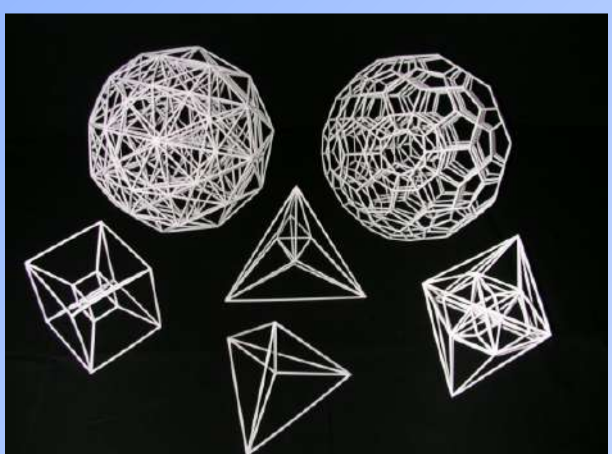
Another double example (4d geometry)



A square of side 2 is fully covered by 4 circles of radius 2 with centers in the vertices of the square.



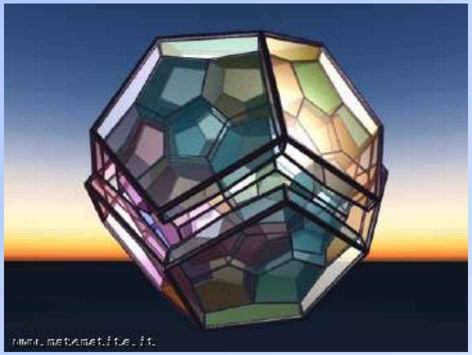
A cube with edge 2 is fully covered by 8 spheres of radius 2 with centers in the vertices of the cube.
Will this go on?



NO! $\sqrt{2}<2$ $\sqrt{3}<2$ $\sqrt{4}=2$ $\sqrt{5}>2$ $\sqrt{6}>2$ ...

Exercises in imagining

Imagination can be a powerful tool in the study of mathematics (and also elsewhere!).



Geometry and the Imagination in Minneapolis

John H. Conway Peter G. Doyle Jane Gilman
William P. Thurston

It is not a *natural gift*: it can and it should be trained systematically.

https://arxiv.org/pdf/1804.03055.pdf

Training imagination can help us to keep control of the gaps between physical and mental images!

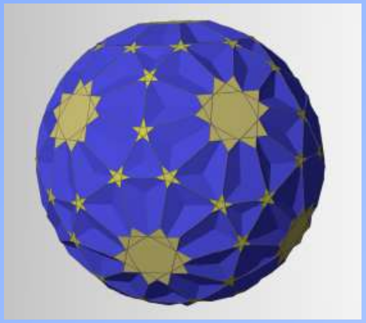
19 Exercises in imagining

How do you imagine geometric figures in your head? Most people talk about their three-dimensional imagination as 'visualization', but that isn't exactly right. A visual image is a kind of picture, and it is really two-dimensional. The image you form in your head is more conceptual than a picture—you locate things in more of a three-dimensional model than in a picture. In fact, it is quite hard to go from a mental image to a two-dimensional visual picture. Children struggle long and hard to learn to draw because of the real conceptual difficulty of translating three-dimensional mental images into two-dimensional images.

And in classrooms?

Informal learning could be precious in classrooms also (at any level, not only with young children!) and school teaching could benefit from all the advantages of informal communication.

Moments of informal learning (through problems, games, cooperative learning,…) should always **precede** a systematic learning.



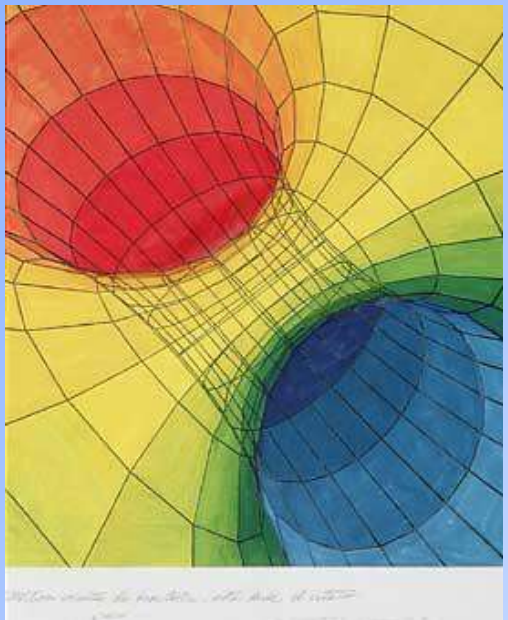
Through these activities the pupils can catch a glimpse of a new math concept **before** learning it in a standard way.

A glimpse of course is not enough to acquire a new tool. But it can be precious: when the new concept arrives, it will no more be an alien, far from their experiences and far from any meaning, but it can be felt by pupils as **their** idea, exactly that idea which was suitable to get over in that intriguing problem…

And in classrooms?

Also in classrooms a first step on an informal level can be precious (necessary?!) if we want to keep the focus on the **meaning** of what we teach.

In school of course we aim to some systematic kind of learning (and teaching), we must give the pupils some tools which they are able to use, we can not stop to the level of a rough idea, we must **go beyond**.



Teachers should pay attention (in avoiding the mentioned *risks* but especially) in the passage **connecting formal and informal**: pupils should recognize the same idea at the informal and formal level, otherwise they lose all the benefits of the informal one.

Not only schools: public awareness

Mathematics is what we do in school.

Mathematics has nothing to do with imagination and creativity.

Mathematics has nothing to do with everyday life.

Mathematics has nothing to do with beauty.

Mathematics has nothing to do with culture.

Mathematics has nothing to do with reasoning.

…

8077. IL NUMERO ESATTO

1525ª PROVA D'INTELLIGENZA

Si tratta di determinare esattamente un certo numero, basandosi esclusivamente sul ragionamento, ossia senza alcun ricorso alla matematica. Esso è formato da dieci cifre tutte diverse fra loro, e quindi da 0 a 9. Con

You have to find a certain number using exclusively the reasoning, **that is** without any use of mathematics.

La Settimana Enigmistica,
a very popular puzzles journal in Italy (and in the italian speaking Switzerland), which sometimes proposes rather beautiful mathematical problems.

Public awareness about maths

A better attitude towards maths would be precious not only to learn maths but also in order to be a responsible citizen (have a critical attitude towards numbers drawn to our attention, from news to advertisement)!

During the pandemics in the last 3 years (at least in Italy), it has been dramatically evident how a bad common perception about mathematics could have dangerous consequences.

We have been bombarded by numbers and graphs, but most people were not equipped to interpret them; so the reactions were often not rational, going from a blind trust to a total refusal.

Schools *versus* exhibitions

In the last 10 years we saw many maths exhibitions which often had a big success: visitors - young and old - have enjoyed the exhibition and have been realizing:

- how much of maths is hidden in everyday life;
- how much maths is part of the history of human culture;
- how challenging and amusing and beautiful maths can be;
- …

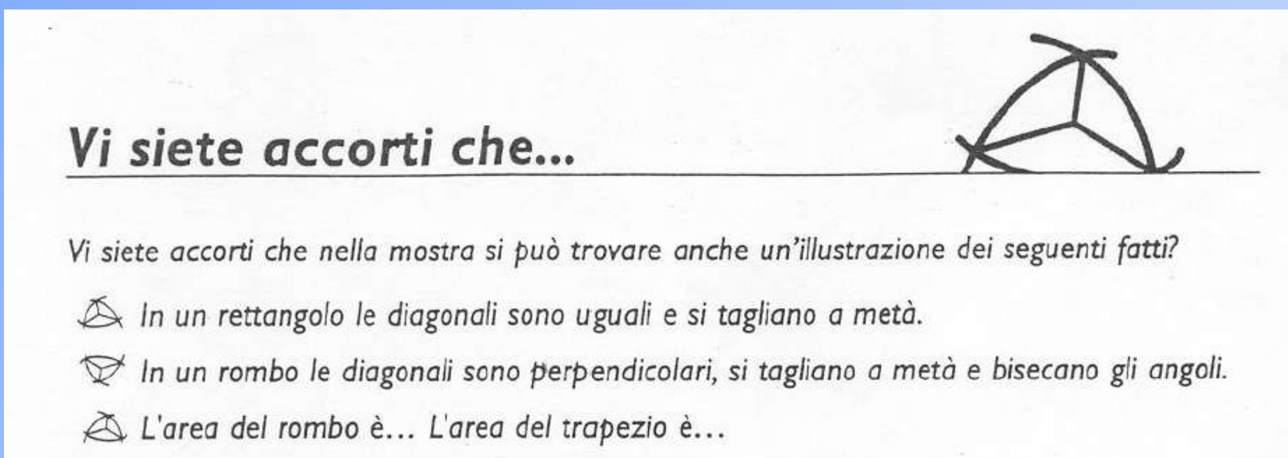
… so in principle this should have changed their perception about *what is mathematics.*

Often this does not happen. A common reaction seems to be: *it is true that **this** is very beautiful, and very useful, and very challenging… **BUT this is not mathematics!** Mathematics is what we do (or we did) at school.*

Schools *versus* exhibitions

This is not mathematics!

This is a very exasperating reaction. Moreover, sometimes, such a reaction may arrive… by mathematics teachers!



Vi siete accorti che...

Vi siete accorti che nella mostra si può trovare anche un'illustrazione dei seguenti fatti?

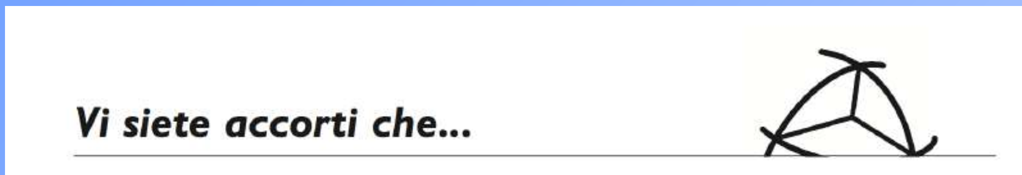
- *In un rettangolo le diagonali sono uguali e si tagliano a metà.*
- *In un rombo le diagonali sono perpendicolari, si tagliano a metà e bisecano gli angoli.*
- *L'area del rombo è... L'area del trapezio è...*

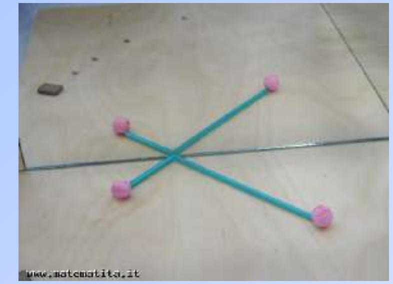
… and, to follow, other 10-15 mathematical facts, + an invitation to find more.

Did you notice that among the exhibits you can find an illustration of the following facts?

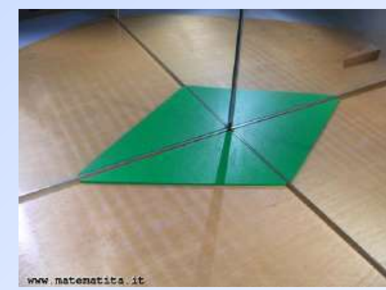
- The diagonals of a rectangle are equal and cut each other in their midpoint.
- The diagonals of a rhombus…
- …

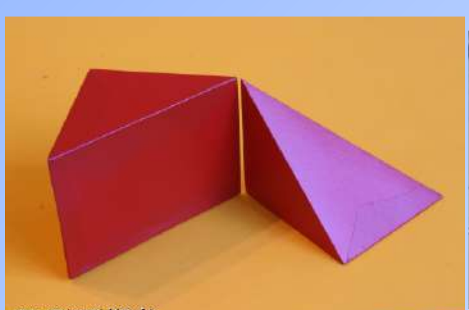
Symmetry - playing with mirrors





This was a leaflet which we used, in the exhibition *Symmetry playing with mirrors*, as an answer to those teachers who were telling us they found everything very beautiful, but*… I can't use it, I have to finish the program…*



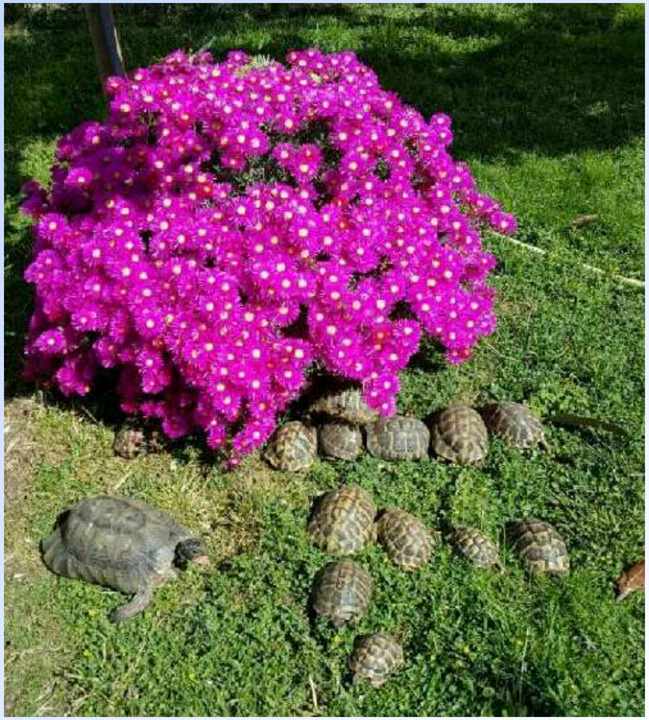


Maths and “true” maths

Quite often, in an informal communication, maths is not recognized as “true” maths. Why does this happen?

A first hypothesis.
This can be an effect of a lack of attention in the critical point of the link between informal and formal learning.

A second hypothesis.
A not trivial role may be played by the lack of geometry in schools.



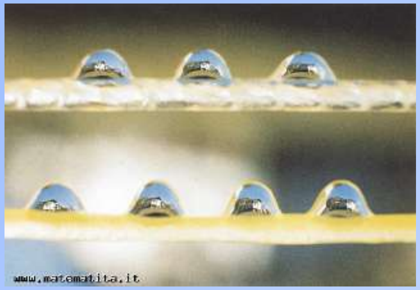
A third hypothesis.
We are (always!) too much in a hurry.
Teachers are often obsessed by the idea of *losing time*. Learning **requires time.**

About the first hypothesis

In a classroom, this amounts to give great importance to the moment when, after a laboratory activity in cooperative learning (or after attending a public event outside classroom), the teacher finally resumes what has been done.

This is a very difficult moment.
It requires a lot of **listening**.
And it requires **time**.

Teachers need to give the exact answer to the questions which have been discussed, but must be careful in making it **arise from** what the pupils were discussing in their group works.



Inside a public event like an exhibition, this problem is usually skipped and left over to teachers (an exhibition shows something and does not pretend to teach).

About the second one

In Italian schools, geometry seems to have been abandoned: I don't refer here to geometrical content, but rather to what I would call *a geometric way* to look at problems, which is one of the many possible ways, and which can be precious not only for learning geometry.
It is related to intuition and imagination; and naturally arises in outreach events (not only events aiming to tell about geometry).

Just to make an example: this is meant to be geometry, but it has **nothing to do** with *geometry*!

La misura del lato di un quadrato è data, in centimetri, dal valore della seguente espressione:

$$36\times\left\{\left[\frac{4}{3}\times\left(\frac{3}{7}+\frac{1}{4}+\frac{1}{14}\right)^2\right]:\left[\frac{3}{5}\times\left(\frac{26}{15}-\frac{9}{10}\right)+\left(\frac{5}{2}-\frac{9}{5}\right):\frac{7}{5}\right]-\frac{1}{3}\right\}$$

Calcola le misure dei lati di un parallelogramma isoperimetrico al quadrato, sapendo che due lati consecutivi del parallelogramma sono uno il quintuplo dell'altro. [5 cm; 25 cm]

About the third one

Quest'anno ho meno timore di non "terminare il programma" perché mi accorgo che ***i concetti da affrontare emergono da soli*** *durante le discussioni in classe. Dovrò stare molto attenta ad aiutarli nel fare ordine per non cadere nella dispersività.*

This is the comment of a teacher who tried laboratory activities with problems in her classes.

This year I am less worried "not to end the program", because I realized that the concepts which we should deal with naturally arise during the collective discussion in class after a laboratory session. I need to be very careful in helping them to put in order their ideas otherwise we risk to get into dispersiveness.

And this was the comment of a group of pupils!

Lasciateci il tempo di pensare, però!

You must leave us the time to think!

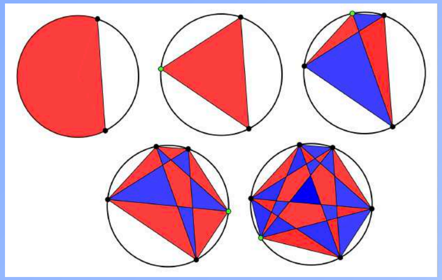
(Provisional) conclusions

Quite a lot of what we discussed heavily **depends on teachers**. The job of teaching has become more and more difficult and often (in Italy) teachers have been left alone to cope with these difficulties.

This is *in primis* a concern of ministers and governments of course, but it should be also **our** concern.

Workshops!
Theatre!
Comics!
Exhibitions!

The first channel for maths communication is **through schools**; and we should find more and more ways to help teachers with the new difficulties they have to face!

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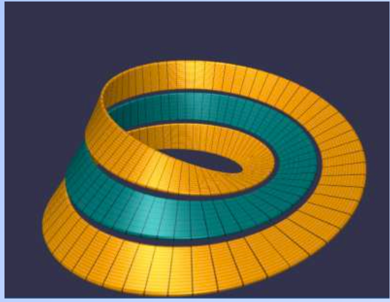

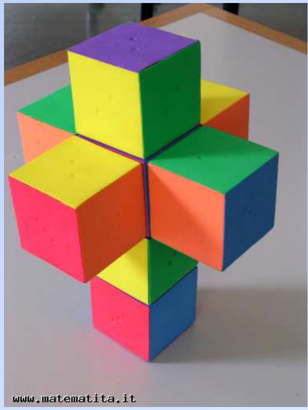

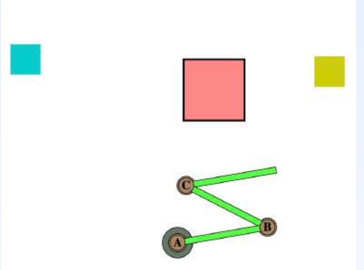

Thank you for your attention!

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