Exceptional finite groups of Lie type and their primitive actions

Young Group Theorists Workshop

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Previously on Young Group Theorists Workshop...

- Almost simple groups
- Base size of a primitive permutation group
- Groups of Lie type
- Intersections of subgroups
- ...

Subdegrees and intersections

Let G be a group acting transitively on a set Ω.

- Denote $\mathsf{Stab}_G(x)$ for the *point stabiliser* of $x \in \Omega$.
- A *suborbit* is any orbit of $\mathsf{Stab}_G(x)$ for any $x \in \Omega$.
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Example

Let H be a subgroup of a finite group G.

- Then G acts on the set of cosets $(G : H)$ transitively.
- Such an action is primitive if and only if H is maximal in G.
- $\mathrm{Stab_G([1]) = H}$ and $\mathrm{Stab_G([g]) = H^g}$.
- $\mathrm{|Orb_H([g])| = \frac{|H|}{|H \cap H^g|}}$.

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General setup:

- Let $k = \overline{\mathbb{F}_p}$, for some prime p.
- Let G be a linear algebraic group over k.
- Let $\sigma \colon G \to G$ be a Steinberg endomorphism.
- G_σ is a finite subgroup of G.

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- Determine the subdegrees of primitive actions of G on $(G : H)$.

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A: By calculating the intersections $H \cap H^g$.

But first, there are some known results

- Lawther & Saxl, 1988: $B_2(q)$ on the cosets of ${}^2B_2(q)$, $q = 2^m$.
- Lawther & Saxl, 1988: $B_2(q^2)$ on the cosets of $B_2(q)$, $q = 2^m$.
- Lawther, 1989: $G_2(q)$ on the cosets of ${}^2G_2(q)$, $q = 3^m$.
- Lawther, 1989: $G_2(q^2)$ on the cosets of $G_2(q)$, q any prime power.
- Lawther, 1999: $F_4(q)$ on the cosets of $B_4(q)$, q any prime power.
- Bannai, Song & Yamada, 2008: $G_2(q)$ on the cosets of $A_2(q).2$, q odd.

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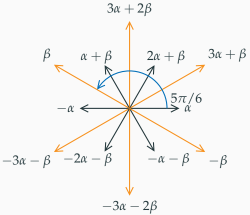
Abstract The character tables of the commutative association schemes coming from the action of the Chevalley group $G_2(q)$ on the set Ω_ϵ of hyperplanes of type $O_6^\epsilon(q)$ in the seven dimensional orthogonal geometry over $GF(q)$ related to the orthogonal group $O_7(q)$ are constructed by modifying the character tables of the association schemes obtained from the action of $O_7(q)$ on Ω_ϵ.

Keywords Multiplicity-free permutation characters

Example: subdegrees of $G_2(q)$-action on $(G_2(q) : \mathrm{SL}_3(q).2)$



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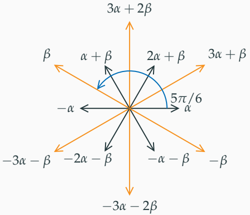
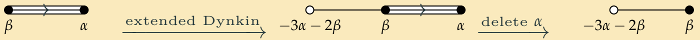


The algebraic group $\overline{G} = \langle T, X_r \mid r \in \Phi \rangle$ has a (subsystem) subgroup $\overline{H} = \langle T, X_{\pm\beta}, X_{\pm(3\alpha+2\beta)} \rangle$ of type A_2.

By the Borel–de Siebenthal algorithm



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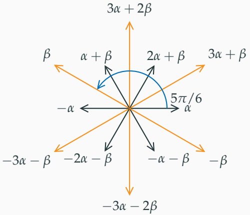

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- $G = \overline{G}_\sigma \cong G_2(q)$.
- $\mathrm{SL}_3(q) \cong H_0 = \overline{H}_\sigma \leq G$.
- $H = N_G(H_0) \cong \mathrm{SL}_3(q).2$ is maximal in G.

By the Borel–de Siebenthal algorithm

Recall that the subdegrees of the G-action on $(G : H)$ are closely related to double cosets and intersections:

$$|\mathrm{Orb}_{\mathrm{H}}([\mathrm{g}])| = \frac{|\mathrm{H}|}{|\mathrm{H} \cap \mathrm{H}^{\mathrm{g}}|} = \frac{|\mathrm{HgH}|}{|\mathrm{H}|}.$$

Goal: To find a complete and irredundant set of representatives $\Gamma \subset G$ such that

$$G = \bigsqcup_{g \in \Gamma} HgH.$$

Then we calculate $H \cap H^g$ to find the subdegrees $\frac{|H|}{|H \cap H^g|}$.

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Table 1: Subdegrees of $G_2(q)$ on $(G_2(q) : \mathrm{SL}_3(q).2)$, q odd

Double coset representative	Number of suborbits	subdegree
1	1	1
$x_\alpha(1)$	1	$2(q^3-1)$
$x_{\alpha+\beta}(1)x_{2\alpha+\beta}(1)$	1	$(q^2-1)(q^3-1)$
$x_\alpha(\frac{1}{2}(p+1))n_\alpha^{-1}x_\alpha(1)$	1	$\frac{1}{2}q^2(q^3-1)$
$x_\alpha(\lambda)n_\alpha^{-1}x_\alpha(1)$, $\lambda \in \mathbb{F}_q \backslash \{0, 1, \frac{1}{2}(p+1)\}$	$\frac{1}{2}(q-3)$	$q^2(q^3-1)$

Outlook

- Explicitly determine the isomorphism types of $H \cap H^g$.
- Determine the subdegrees of $G_2(q)$ acting on the cosets of other maximal subgroups.
- Consider the coset actions of other finite groups of exceptional Lie type.
- Computational implementations.

Thank you!